$\boxed{\text{Ch. 1}}$

#1 30 Eylül 2017

BASIC CONCEPTS

Charge: A fundamental conserved property of subatomic particles. Measured in Coulombs (C) For instance, the electron has -1e charge, the proton +1e. (e : elementary charge unit. $1e \cong 1.602 \times 10^{-19}$ C) We can talk about the net charge of arbitrary matter.

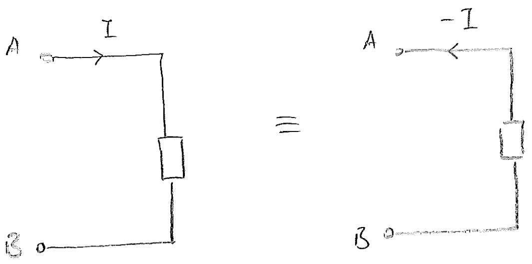
Ex Block $\boxed{\begin{matrix} n^\circ\ p^+\ e^- \\ n^\circ\ p^+ \end{matrix}}$ has net charge of 1e.

Charge is denoted by q or Q.

Current: Rate of flow of (net) charge through a given area. Measured in Amperes (A). (1A = 1C/s) Denoted by i or I.

$$\boxed{i(t) = \frac{dq(t)}{dt}} \Rightarrow q(t) = q(0) + \int_0^t i(\tau)\,d\tau$$

To be able to talk about current we need a specified direction.



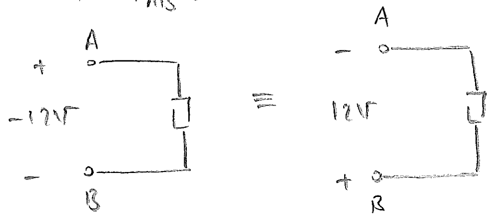
Voltage Consider two points x & y. The work done against the electrical field to move a unit charge (1C) from y to x is the voltage V_{xy}. This work is independent of the path through which the charge travels and depends only on the endpoints x & y. Therefore $V_{xy} = -V_{yx}$.

Voltage is measured in volts (V). (1V = 1J/C)

$$\boxed{V = \frac{dW}{dq}}$$ (W: work, energy)

To be able to talk about voltage we need a specified polarity.

Ex Let $V_{AB} = -12V$



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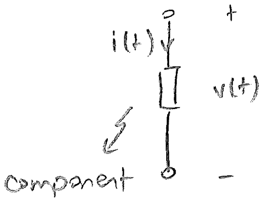
Ground : common reference point for voltage measurements in a circuit. Given some point x in a circuit, by V_x we mean the voltage between point x and ground, i.e. V_{xg}. Symbol for ground : ⏚

Power Rate of change of energy. Denoted by P or p. Measured in watts (W). (1W = 1J/s)

$$p = \frac{dW}{dt} = \underbrace{\frac{\partial W}{\partial q}}_{v(t)} \cdot \underbrace{\frac{dq}{dt}}_{i(t)} \Rightarrow \boxed{p(t) = v(t)\, i(t)}$$

Passive sign convention we adopt the convention that, given an electrical component, the polarity of voltage and the direction of current are chosen such that the current enters (the component) from the "+" labeled terminal and leaves from the "−" labeled terminal.



According to our convention:

if $p(t) = i(t)v(t) > 0$ then the component is absorbing electrical power. E.g. heater.

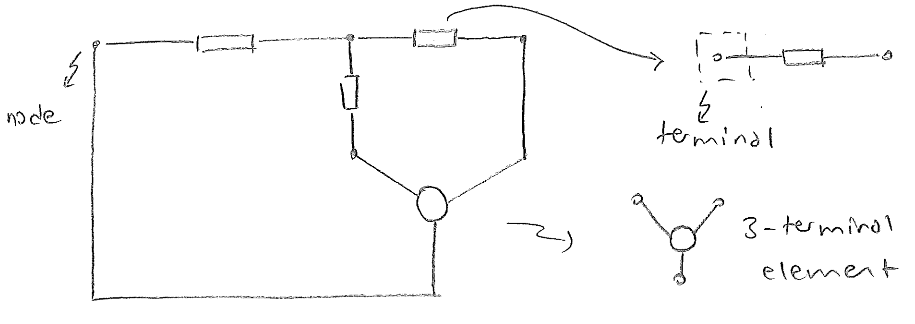
if $p(t) = i(t)v(t) < 0$ then the component is delivering power. E.g. battery.

Assumptions In this course we make the following assumptions

(A1) Electrical effects happen instantenously throughout the circuit.

(A2) The net charge inside any closed surface is always zero.

Electric circuit Interconnection of (basic) electrical components. Circuits that let us make (A1) are called lumped circuits. Lumped circuit generally means a small circuit.



2-terminal element (single branch element)

node : point where two or more terminals meet.

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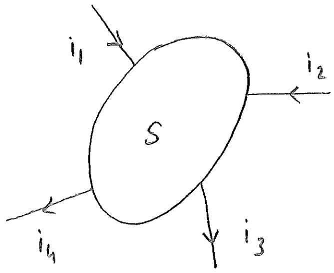
KIRCHHOFF'S LAWS

Kirchhoff's Current Law (KCL) (A direct implication of our assumption (A2) " The net charge inside any closed surface is always zero.") for any closed surface the sum of incoming currents equals the sum of outgoing currents.

Ex



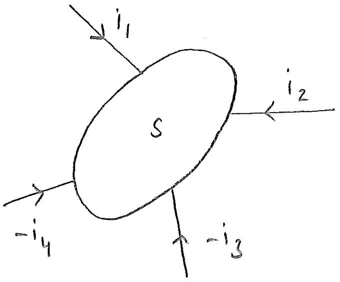
incoming : i_1, i_2

outgoing : i_3, i_4

KCL : Σ incoming = Σ outgoing

$\Rightarrow i_1+i_2 = i_3+i_4$

$\equiv$

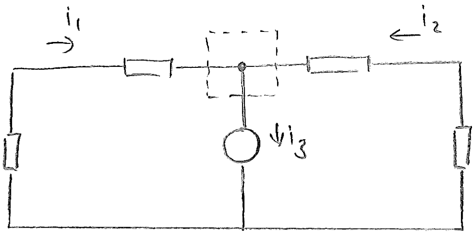


incoming : $i_1, i_2, -i_3, -i_4$

outgoing : none

KCL $\Rightarrow$ $i_1+i_2+(-i_3)+(-i_4)=0$

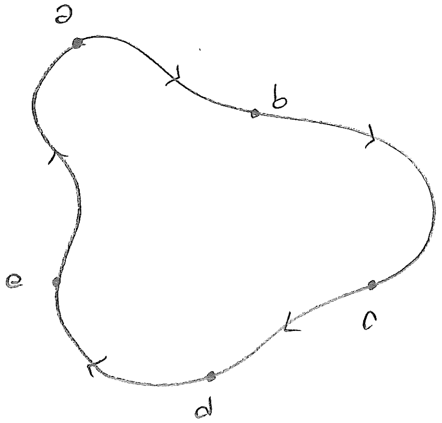
Ex :



KCL $\Rightarrow$ $i_1+i_2=i_3$

Kirchhoff's Voltage Law (KVL) (A direct implication of path independence for the potential energy difference between two points.) Sum of voltages along any loop (closed path) is zero.

Ex :



$V_{ab}+V_{bc}+V_{cd}+V_{de}+V_{ea} = V_{aa} = 0$

Likewise, $V_{ae}+V_{ed}+V_{dc}+V_{cb}+V_{ba}=0$

Also, $V_{ab}+V_{bc} = V_{ae}+V_{ed}+V_{dc}$

and so on.

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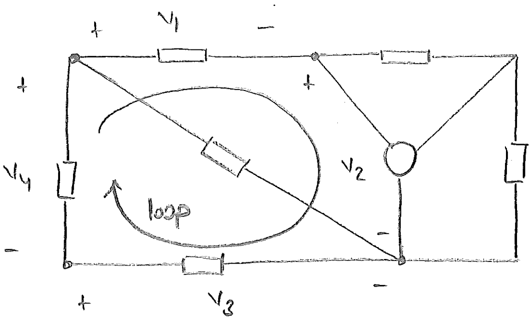
Ex



KVL $\Rightarrow$ $v_1 + v_2 - v_3 - v_4 = 0$

let $v_1(t) = 2\cos(10t + \frac{\pi}{4})$ V
$v_2(t) = 12$ V
$v_3(t) = 5\sin(5t)$ V
$\Bigg\}$ $v_4(t) = ?$

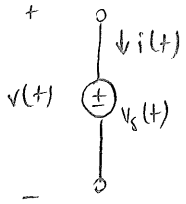
KVL $\Rightarrow$ $v_4(t) = v_1(t) + v_2(t) - v_3(t)$

$\Rightarrow v_4(t) = 12 + 2\cos(10t + \frac{\pi}{4}) - 5\sin(5t)$ V

TERMINAL EQUATIONS

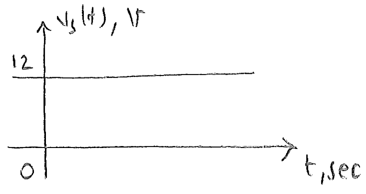
A circuit element is characterized by well-defined relations satisfied by the current(s) through the element and voltage(s) across its terminals. Such relations are called terminal equations. Terminal equations depend only on the element and not on the rest of the circuit that the element is part of.

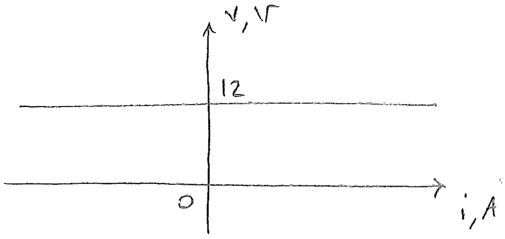
Independent voltage source (IVS) A two-terminal (single branch) element with branch relation $v(t) = v_s(t)$, i.e., the voltage of IVS is independent of the current $i(t)$ passing through it.



schematic representation

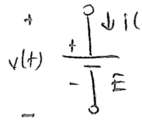
Ex





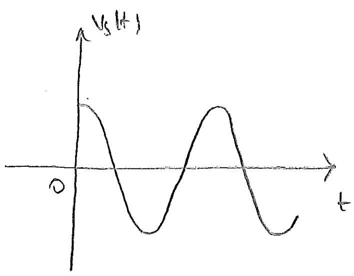
A constant voltage source (sometimes called battery)

A battery is sometimes depicted by

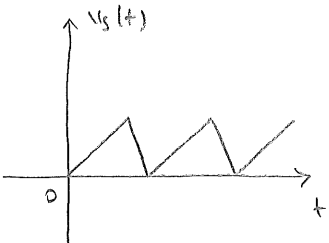


$v(t) = E$. (constant)

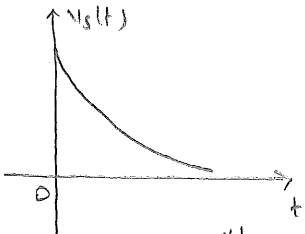
Ex



sinusoidal voltage

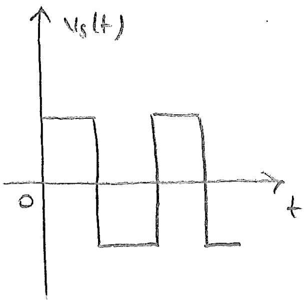


sawtooth



$v_s(t) = Ke^{-\alpha t}$

exponential



square

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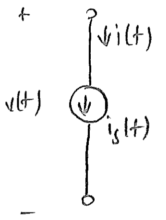
Independent current source (ICS) A single branch element with branch relation $i(t) = i_s(t)$, i.e., the current of ICS is independent of the voltage across its terminals.



Ex $i_s(t) = I_m \cos(\omega t + \phi)$ sinusoidal current source (AC source)

$i_s(t) = I_0$ constant current source (DC source)

Remark IVS & ICS are called active elements because they are capable of delivering arbitrary amount of power to the circuit they are connected to throughout intervals of arbitrary length.

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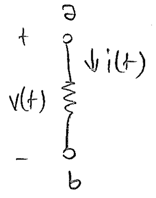
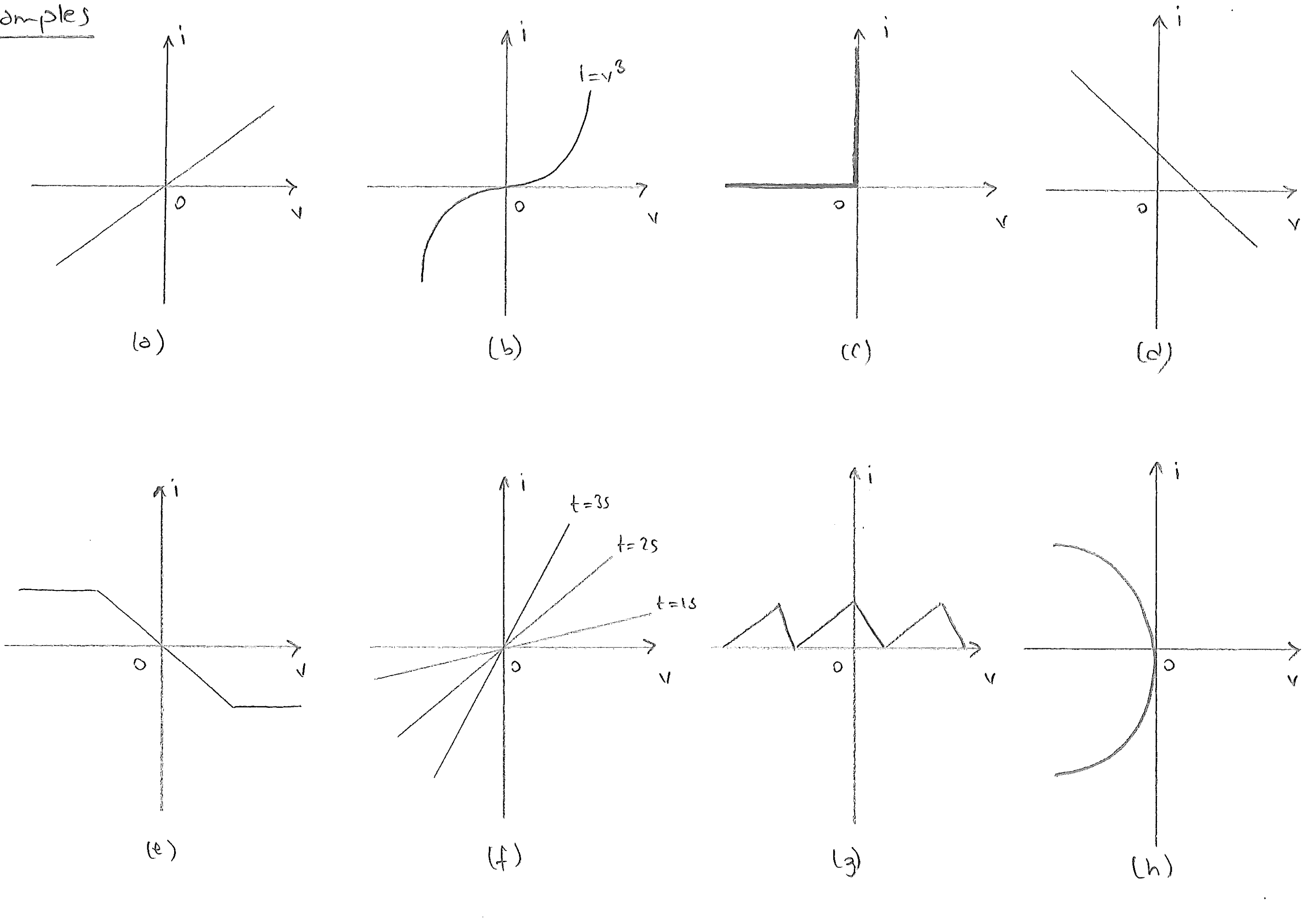
Resistor General name for a 2-terminal element whose branch current & branch voltage satisfy a relation described by a curve on the i-v plane.



Examples



(a) (b) (c) (d)

(e) (f) (g) (h)

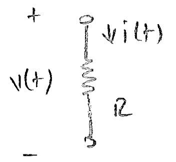
Linear: Variables x, y are said to satisfy a linear relation if there exist constants a, b (not both a and b are zero) such that $ax + by = 0$.

Bilateral Bilateral elements have i-v curves that are symmetric w.r.t. the origin. When we are to connect a bilateral element between two nodes it is immaterial which terminal is connected to which node. For nonbilateral elements this is not so.

Active A resistor is said to be active if its i-v curve contains a point (v_0, i_0) satisfying $v_0 i_0 < 0$. In other words, active ⇔ i-v curve visits 2nd or 4th quadrant.

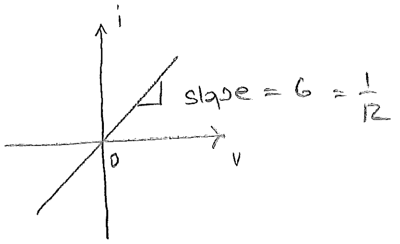
	linear / nonlinear	voltage-controlled / current-controlled	time-varying / time-invariant	bilateral / nonbilateral	active / passive
a	L	VC & CC	TI	B	P
b	N	VC & CC	TI	B	P
c	N	neither	TI	N	P
d	N	VC & CC	TI	N	A
e	N	VC	TI	B	A
f	L	VC & CC	TV	B	P
g	N	VC	TI	N	A
h	N	CC	TI	N	A

LTI (linear time-invariant) resistor



$$\boxed{v(t) = Ri(t)}$$ (Ohm's Law)

(R can be positive or negative.)



R : resistance, measured in Ohms (Ω)

G : conductance, measured in Siemens (S) or Mhos (℧)

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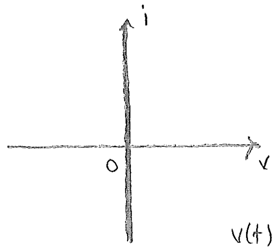
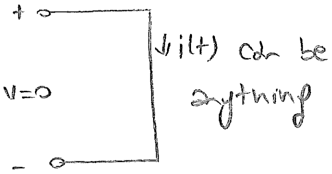
An LTI resistor is both voltage & current controlled for all R except $R = 0$ & $R = \infty$ $(G = 0)$

$R = 0$



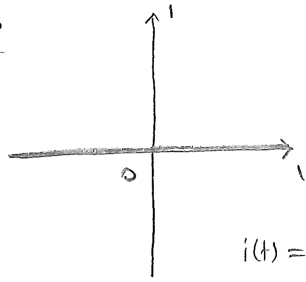
$v(t) = 0$

Symbol:



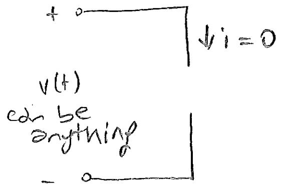
this case is called "short circuit"

$R = \infty$



$i(t) = 0$

Symbol:



this case is called "open circuit"

power:

$$p = vi = v \cdot \frac{v}{R} = \frac{1}{R} v^2$$
$$= Ri \cdot i = Ri^2$$

$$\boxed{p = \frac{1}{R} v^2 = Ri^2}$$ for LTI resistor

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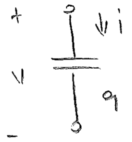
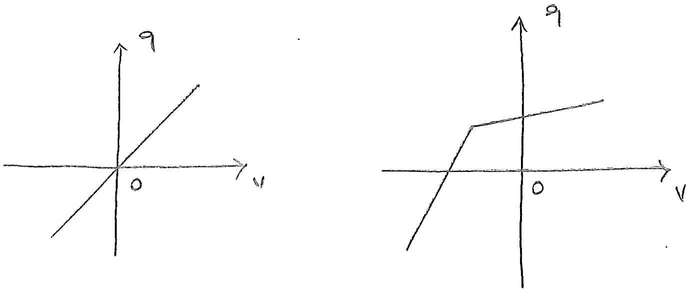
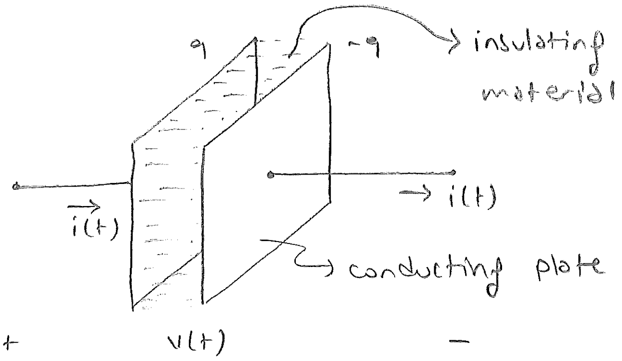
Capacitor A 2-terminal element that can store charge. The charge q it stores and the voltage v across its terminals satisfy a relation described by a curve on the q-v plane.



Ex





$i(t) = \frac{d}{dt} q(t)$ (def. of current)

$\Rightarrow q(t) = q(t_0) + \int_{t_0}^{t} i(\tau) d\tau$

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LTI Capacitor

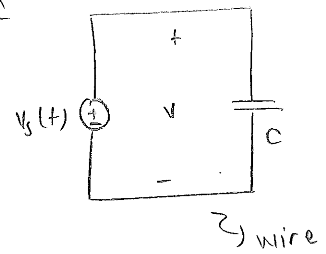
$\boxed{q(t) = Cv(t)}$ C: constant named capacitance, measured in Farads (F)

i-v relation?
$$i(t) = \frac{d}{dt}q(t) = \frac{d}{dt}\{Cv(t)\} = C\frac{dv(t)}{dt} \Rightarrow \boxed{i(t) = C\frac{dv(t)}{dt}}$$ for LTI capacitor

Equivalently,
$$v(t) = \frac{1}{C}q(t) = \frac{1}{C}\left\{q(t_0) + \int_{t_0}^{t} i(\tau)d\tau\right\} \Rightarrow \boxed{v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau}$$

($v(t_0)$: initial voltage of the capacitor.)

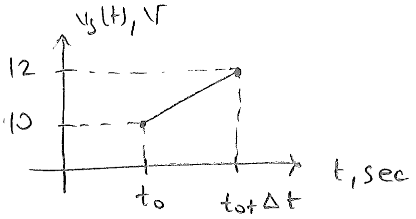
Ex



$v_s(t_0) = 10V$

$C = 1\mu F$

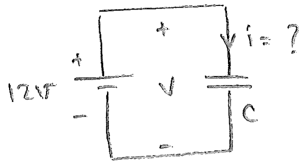
Suppose the max. current the wire can carry is 40mA. We want to raise the voltage from 10V to 12V without breaking the wire. What is the minimum time required for voltage raise?



$$\underbrace{v(t_0+\Delta t)}_{12V} = \underbrace{v(t_0)}_{10V} + \underbrace{\frac{1}{C}}_{\frac{1}{10^{-6}F}}\int_{t_0}^{t_0+\Delta t} \underbrace{i_{max}}_{40\times10^{-3}A}\, dt$$

$$\Rightarrow 12 = 10 + \frac{40\times10^{-3}\Delta t}{10^{-6}} \Rightarrow \Delta t = 5\times10^{-5} = \boxed{50\mu s}$$

Ex



$$i = C\cancelto{0}{\frac{dv}{dt}} = 0$$

Therefore, the capacitor behaves as open circuit under DC voltage.

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power $p(t) = i(t)v(t)$
$= C\frac{dv(t)}{dt}v(t)$ $\Big\}$ $\boxed{p(t) = Cv(t)\frac{dv(t)}{dt}}$ for LTI capacitor.

stored energy $(C>0)$ $$w(t) = \int_{-\infty}^{t} p(\tau)d\tau = \int_{\underset{=0}{v(-\infty)}}^{v(t)} Cv\,dv = \frac{1}{2}Cv(t)^2$$

Hence, energy stored in an LTI capacitor : $\boxed{w = \frac{1}{2}Cv^2 = \frac{1}{2}\frac{1}{C}q^2}$

$\{$Energy transfer during the interval $[t_0, t]$: $\Delta w = w(t) - w(t_0) = \frac{1}{2}C(v(t)^2 - v(t_0)^2)$ $\}$

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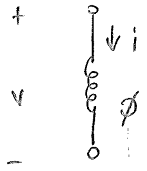
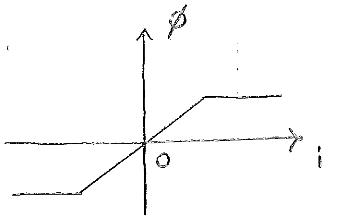
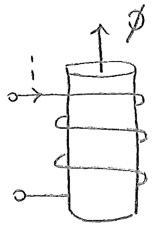
Inductor 2-terminal element whose magnetic flux ϕ and current i satisfy a relation described by a curve on the ϕ-i plane.



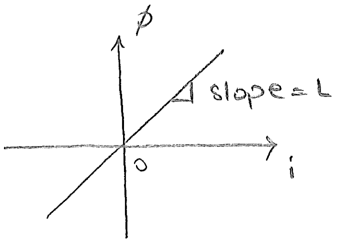
Ex





Faraday's Law : $\boxed{v(t) = \frac{d}{dt}\phi(t)}$

LTI inductor



$\boxed{\phi(t) = Li(t)}$ L : inductance, measured in Henries (H)

i-v relation? $v(t) = \frac{d}{dt}\phi(t) = \frac{d}{dt}\{Li(t)\} = L\frac{di(t)}{dt}$ $\Rightarrow$ $\boxed{v(t) = L\frac{di(t)}{dt}}$ (LTI ind.)

Equivalently, $\boxed{i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau}$

power $\quad p(t) = v(t)\, i(t) = L\frac{di(t)}{dt}\, i(t) \Big\} \boxed{p(t) = L\, i(t)\frac{di(t)}{dt}}$

stored energy $(L>0)$ $\quad w(t) = \int_{-\infty}^{t} p(\tau)d\tau = \int_{\underset{\approx 0}{i(-\infty)}}^{i(t)} L\, i\, di = \frac{1}{2}L\, i(t)^2$

$$\Rightarrow \boxed{w = \frac{1}{2}L i^2}$$

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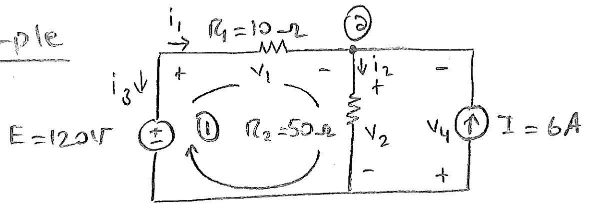
Summary of LTI R, L, C elements' properties

	Resistor (R)	Capacitor (C)	Inductor (L)
Algebraic equation	$v(t) = R\, i(t)$	$q(t) = C\, v(t)$	$\phi(t) = L\, i(t)$
i-v char.	$v(t) = R\, i(t)$ $i(t) = \frac{1}{R} v(t)$	$i(t) = C\frac{dv(t)}{dt}$ $v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau$	$v(t) = L\frac{di(t)}{dt}$ $i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau$
energy stored at time t	———	$w(t) = \frac{1}{2}C\, v(t)^2$	$w(t) = \frac{1}{2}L\, i(t)^2$

Example



a) Find i_1, i_2, v_1, v_2

b) Verify total generated power equals total dissipated power.

Sol'n $\quad$ a) Ohm's law : $v_1 = R_1 i_1 = 10 i_1$ & $v_2 = R_2 i_2 = 50 i_2$

KCL at node ⓐ : $i_1 - i_2 + I = 0 \Rightarrow i_1 - i_2 = -6$ (1)

KVL at loop ① : $v_1 + v_2 - E = 0 \Rightarrow v_1 + v_2 = 120 \Rightarrow 10 i_1 + 50 i_2 = 120$ (2)

(1) & (2) $\Rightarrow \boxed{i_1 = -3A,\ i_2 = 3A} \Rightarrow \boxed{v_1 = -30V,\ v_2 = 150V}$

b) voltage source : $i_3 = -i_1 = 3A \Rightarrow P_E = 3 \cdot 120 = 360W$ (absorbing)

current source : $v_4 = -v_2 = -150V \Rightarrow P_I = 6 \cdot (-150) = -900W$ (delivering)

R_1 : $P_{R_1} = i_1 v_1 = 90W$ (absorbing)

R_2 : $P_{R_2} = R_2 i_2^2 = 450W$ (absorbing)

Hence,

power generated $= |-900| = \underline{\underline{900W}}$ (by ICS)

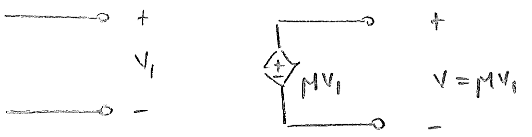
power dissipated :

$= 360 + 90 + 450 = \underline{\underline{900W}}$

#11 30 Eylül 2017

Dependent source: A linear dependent source has terminal characteristics controlled by a current or a voltage of some other branch in the circuit.

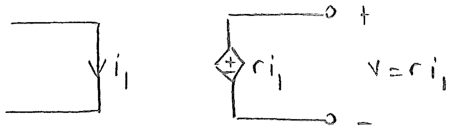
1| voltage controlled voltage source (VCVS)



v_1 : control variable

μ : voltage gain

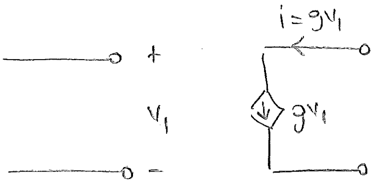
2| Current controlled voltage source (CCVS)



i_1 : control var.

r : transresistance

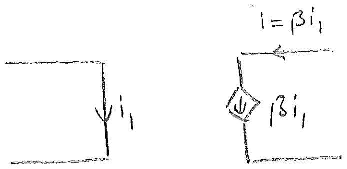
3| VCCS



v_1 : control var.

g : transconductance

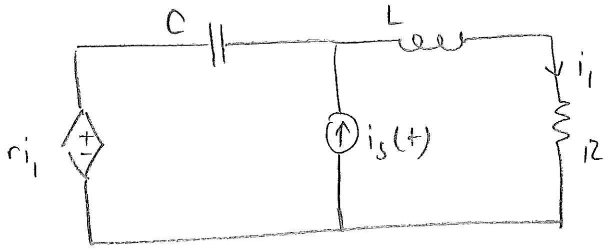
4| CCCS



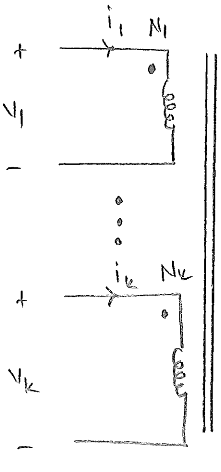
i_1 : control var.

β : current gain

Ex



Ideal transformer A k-branch IT is depicted by



The branch relations:
$$\boxed{\frac{v_1}{N_1} = \frac{v_2}{N_2} = \dots = \frac{v_k}{N_k}}$$

$N_j > 0$ is the number of turns for the j^{th} branch $j = 1, 2, \dots, k$

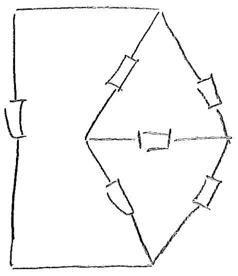
&
$$\boxed{N_1 i_1 + N_2 i_2 + \dots + N_k i_k = 0}$$

The total power of IT is $\sum_{j=1}^{k} i_j v_j = 0$ (Exercise: prove this.)

#12 30 Eylül 2017

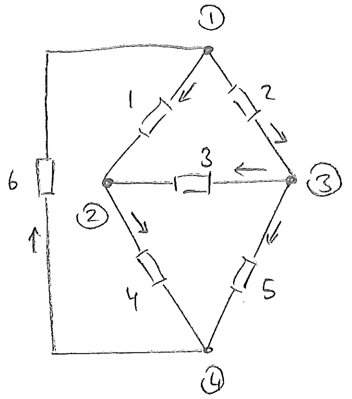
FROM CIRCUIT TO GRAPH

To systematically analyze circuits we study their graphs

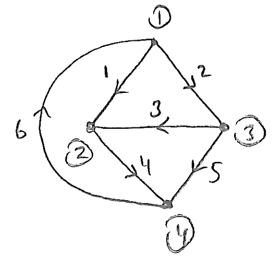


circuit N

1) label each branch
2) assign a (arbitrary) direction to each branch
3) label each node
4) obtain the graph



=> the graph of circuit N is



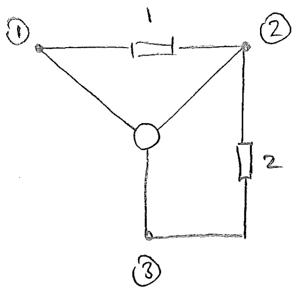
nodes = {①, ②, ③, ④}

branches = {1, 2, 3, 4, 5, 6}

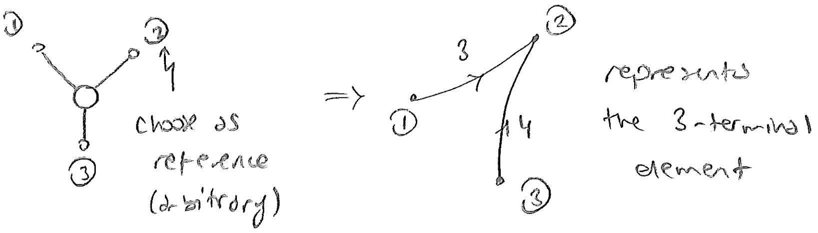
# of nodes $n=4$

# of branches $b=6$

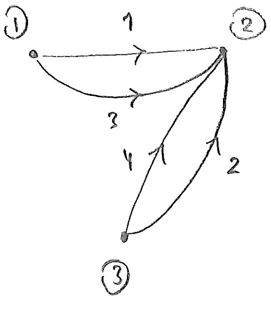
How about the below circuit?



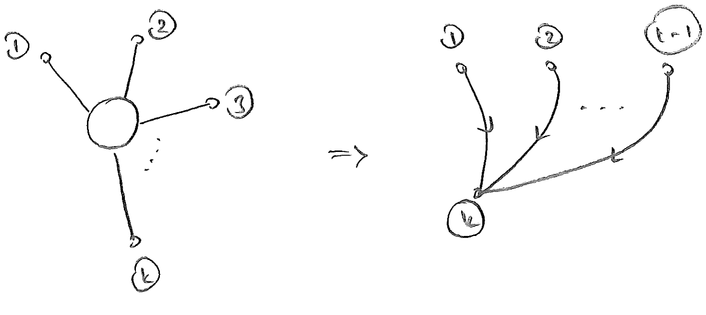
→ When we have a k-terminal ($k \geq 3$) element we assign one of its terminal nodes as reference and treat it as k-1 single branch elements as shown:



Then the graph of the original circuit becomes



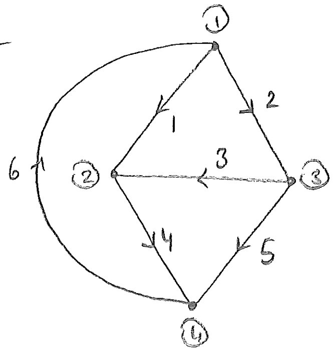
For k-terminal



#13 30 Eylül 2017

Matrix formulation of KCL

for



write KCL at each node :

$$\left.\begin{array}{ll} ① : & i_1+i_2-i_6=0 \\ ② : & -i_1-i_3+i_4=0 \\ ③ : & -i_2+i_3+i_5=0 \\ ④ : & -i_4-i_5+i_6=0 \end{array}\right\} (*)$$

Eqns (*) can be written as

$$\underbrace{\begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ ① & 1 & 1 & 0 & 0 & 0 & -1 \\ ② & -1 & 0 & -1 & 1 & 0 & 0 \\ ③ & 0 & -1 & 1 & 0 & 1 & 0 \\ ④ & 0 & 0 & 0 & -1 & -1 & 1 \end{array}}_{A_a} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}}_{i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A_a : incidence matrix ($n \times b$ matrix) obtained as

$$a_{ij} = \begin{cases} +1 & \text{if } j^{th} \text{ branch leaves } i^{th} \text{ node} \\ -1 & \text{if } j^{th} \text{ branch enters } i^{th} \text{ node} \\ 0 & \text{otherwise} \end{cases}$$

i : branch current vector $i=[i_1\ i_2 \dots i_b]^T$

— o —

<u>linear dependence</u> Let $v_1, v_2, \dots, v_k$ be k vectors (all same size). If we can find real numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ (not all zero) such that $\Sigma \alpha_i v_i = 0$ then the vectors $v_1, v_2, \dots, v_k$ are lin. dependent. Otherwise, they are linearly independent.

<u>Ex</u> $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}$ are lin. ind.

whereas $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$ are lin. dep. (because $3v_1-6v_2+3v_3=0$)

— o —

Rows (row vectors) of A_a sum up to zero. Therefore they're linearly dependent. Choose one node as reference node and remove the corresponding row from A_a. The new matrix of size $(n-1) \times b$ is called the <u>reduced incidence matrix</u> and denoted by A.

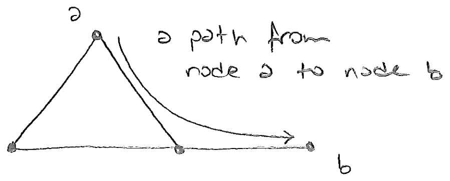
$$A_a = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \xrightarrow[\text{reference node (ground)}]{\text{let \textcircled{4} be the}} A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Rows of A are linearly independent if our graph is <u>connected</u>. Such A is called full row rank.

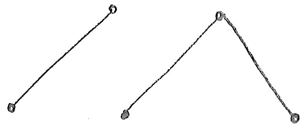
— o —

<u>Definition</u> A graph is connected if there is a path (disregarding the direction of branches) connecting any two nodes.

<u>Ex</u> :



Connected graph



not connected graph

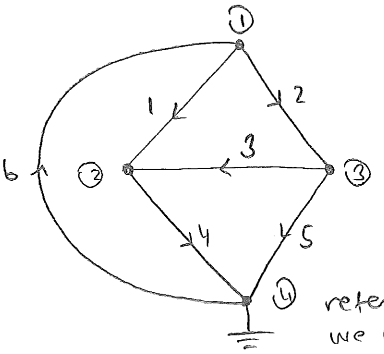
— o —

Hence, we obtained $\boxed{Ai=0}$ (KCL) $A \in \mathbb{R}^{(n-1)\times b}$: reduced incidence matrix

$i \in \mathbb{R}^b$: branch current vector

<u>Remark</u> $Ai = 0 \Rightarrow A_a i = 0$. (why?)

<u>Relation of incidence matrix to branch voltages</u>



reference node we chose to obtain A from A_a

branch voltages : $v_1, v_2, \dots, v_6$

$v_1 = v_{\textcircled{1}\textcircled{2}}$ (voltage between node ① (+) and node ② (−))

$v_2 = v_{\textcircled{1}\textcircled{3}}$

$\vdots$

$v_6 = v_{\textcircled{4}\textcircled{1}}$

node voltages : e_1, e_2, e_3

$e_1 = v_{\textcircled{1}\textcircled{4}}$, $e_2 = v_{\textcircled{2}\textcircled{4}}$, $e_3 = v_{\textcircled{3}\textcircled{4}}$

Thanks to KVL, each branch voltage can be written in terms of node vol.

<u>ex</u> $v_1 = v_{\textcircled{1}\textcircled{2}} = v_{\textcircled{1}\textcircled{4}} + v_{\textcircled{4}\textcircled{2}} = v_{\textcircled{1}\textcircled{4}} - v_{\textcircled{2}\textcircled{4}} = e_1 - e_2$

$$\left.\begin{aligned} v_1 &= e_1 - e_2 \\ v_2 &= e_1 - e_3 \\ v_3 &= e_3 - e_2 \\ v_4 &= e_2 \\ v_5 &= e_3 \\ v_6 &= -e_1 \end{aligned}\right\} \xrightarrow[\text{matrix form}]{\text{put in}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}}_{\substack{v \in \mathbb{R}^b \\ \text{(branch voltage vector)}}} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}}_{e \in \mathbb{R}^{n-1}} \text{ (node voltage vector)}$$

Hence, we obtained $\boxed{v = A^T e}$

Remark Once we know the node voltages we can determine all the branch voltages. Also, KVL puts no constraints on node voltages.

Tellegen's Theorem Consider two circuits that share the same graph. Let $i = [i_1\ i_2 \cdots i_b]^T$ be the current vector from circuit 1 and $v = [v_1\ v_2 \cdots v_b]^T$ be the voltage vector from circuit 2. Then $i^T v = 0$.

Proof : $i^T v = i^T(A^T e) = (i^T A^T) e = \underbrace{(Ai)}_{0}{}^T e = 0$ ▨

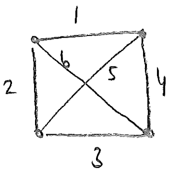
Remark If $i = i(t)$ & $v = v(t)$ belong to the same circuit. Then

$$0 = i(t)^T v(t) = \sum_{k=1}^{b} i_k(t) v_k(t) = \text{total power.} \quad \text{(conservation of power)}$$

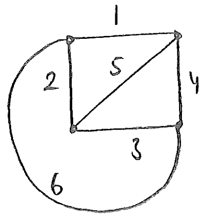
— o —

Planar graph Can be drawn on plane in such a way that no two branches intersect at a point that is not a node.

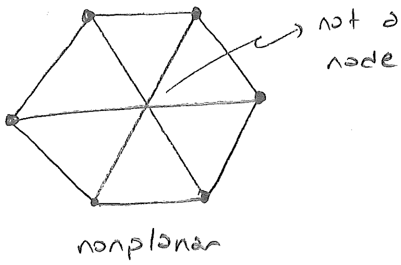
Ex :



planar since ≡



whereas



nonplanar

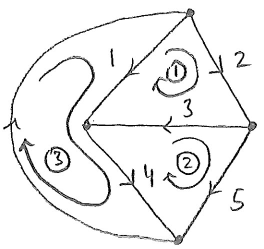
Nonseparable graph If a connected graph stays connected after removing any one node (and all the branches attached to that node) then it is nonseparable.

Matrix formulation of KVL

Let G be a planar, nonseparable graph. The mesh matrix M is an $\ell \times b$ matrix (where ℓ is the number of inner meshes and b is the number of branches) defined as follows:

$$m_{ij} = \begin{cases} +1 & \text{if } j^{th} \text{ branch belongs to } i^{th} \text{ mesh and their directions coincide} \\ -1 & \text{if } j^{th} \text{ branch belongs to } i^{th} \text{ mesh and their directions are opposite} \\ 0 & \text{otherwise} \end{cases}$$

Ex



inner meshes = {①, ②, ③} (directions are chosen cw by convention)

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \leftarrow \text{branches} \\ M = & \left[\begin{array}{cccccc} -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{c} ① \\ ② \\ ③ \end{array} \end{array}$$

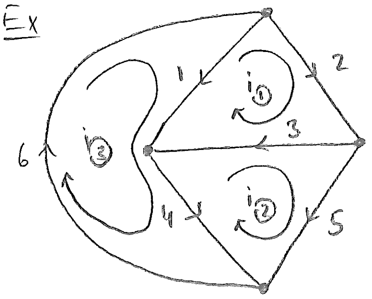
↑ inner meshes

let $v = [v_1\ v_2 \cdots v_b]^T$ be the branch voltage vector. then $\boxed{Mv = 0}$ by KVL.

— o —

To each (inner) mesh there corresponds a mesh current which is related to the branch currents as follows:

Ex



$$\left.\begin{array}{l} i_1 = i_③ - i_① \\ i_2 = i_① \\ i_3 = i_① - i_② \\ i_4 = i_③ - i_② \\ i_5 = i_② \\ i_6 = i_③ \end{array}\right\} \Rightarrow \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}}_{i} = \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M^T} \underbrace{\begin{bmatrix} i_① \\ i_② \\ i_③ \end{bmatrix}}_{i_M}$$

i = branch current vector

i_M = mesh current vector

$$\Rightarrow \boxed{i = M^T i_M}$$

Remark once we know the mesh currents we can determine all the branch currents. Also, KCL puts no constraints on mesh currents. Therefore:

$$0 = Ai = AM^T \underset{\substack{\downarrow \\ i_M \text{ can be anything}}}{i_M} \Rightarrow AM^T = 0.$$ That is, rows of A are orthogonal to rows of M.

Summary go to #18

#17 30 Eylül 2017

Formulations for nonplanar circuits

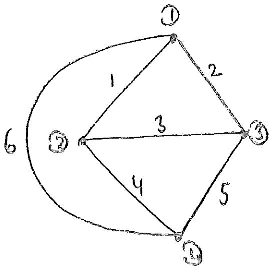
Tree Given a connected graph G, its subgraph T is called a tree if

1) T has all the nodes of G,

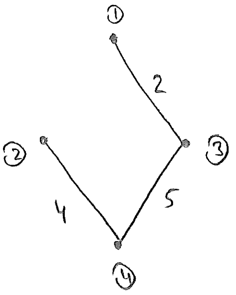
2) T is connected, and

3) T contains no loops.

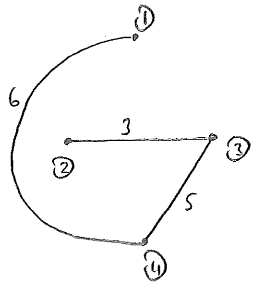
Ex



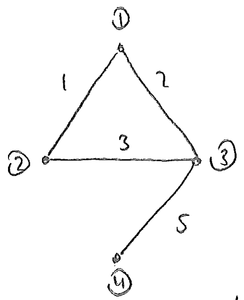
graph G



tree T_1



tree T_2



not a tree! (contains loop)

Exercise Show that a tree always has n-1 branches. (n = # of nodes)

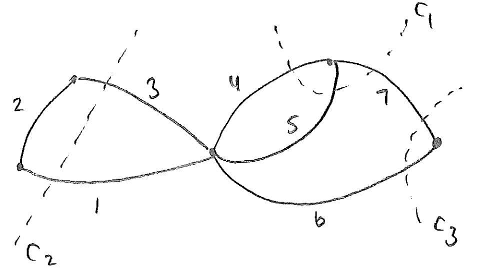
—o—

Cutset Let G be a connected graph. A set of branches C is called a cutset if

1) the removal of all the branches of C from G makes an unconnected graph and

2) if we remove all the branches of C except an arbitrary one, the graph G stays connected.

Ex



$c_1 = \{4, 5, 7\}$
$c_2 = \{1, 3\}$
$c_3 = \{6, 7\}$

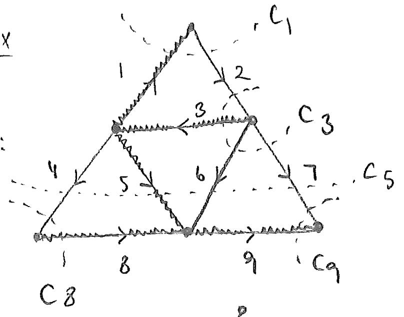
cutsets

Set $\{1, 3, 4, 5, 6\}$ is not a cutset!

Fact Given a connected graph G and a tree T (of G), let β be a branch of T. Then there exists a cutset of G that contains β and no other branches from T. That cutset is unique and is called a fundamental cutset.

Ex

G:

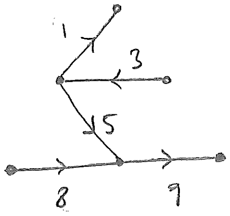


tree branches = $\{1, 3, 5, 8, 9\}$

cotree br. = $\{2, 4, 6, 7\}$

(non-tree)

a tree T



At each cutset we can obtain current equations by KCL:

$c_1: \; i_1 - i_2 = 0$

$c_3: \; i_3 - i_2 + i_6 + i_7 = 0$

$c_5: \; i_5 + i_4 + i_6 + i_7 = 0$

$c_8: \; i_8 - i_4 = 0$

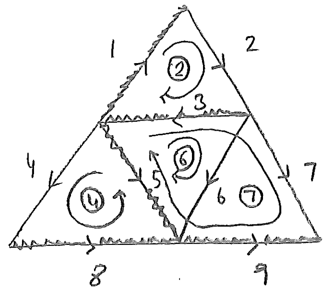
$c_9: \; i_9 + i_7 = 0$

__Remark__ Note that once all the cotree branch currents are known, the remaining (tree) branch currents can be computed.

— ◦ —

__Fundamental loop__ Remember that each tree branch defines a unique cutset (called a fundamental cutset). Likewise each cotree (non-tree) branch defines a unique loop (called a fundamental loop) such that it contains only one cotree branch.

__Ex__



cotree = $\{2, 4, 6, 7\}$

branch 2 defines loop ②, i.e., $\{2, 3, 1\}$

branch 4 defines loop ④, i.e., $\{4, 8, 5\}$

and so on

At each fund. loop we can obtain voltage equations by KVL:

② : $v_2 + v_3 + v_1 = 0$

④ : $v_4 + v_8 - v_5 = 0$

⑥ : $v_6 - v_5 - v_3 = 0$

⑦ : $v_7 - v_9 - v_5 - v_3 = 0$

__remark__ Note that once all the tree branch voltages are known, the remaining (cotree) branch voltages can be computed.

— ◦ —

__Summary__ Given a circuit with planar, nonseparable graph with n nodes & b branches

$$\boxed{\begin{array}{ll} Ai = 0 & Mv = 0 \\ v = A^T e & i = M^T i_M \end{array}}$$

A: reduced incidence matrix, $(n-1) \times b$

M: mesh matrix, $[b-(n-1)] \times b$

i: branch current vector

v: branch voltage vector

i_M: mesh current vector

e: node voltage vector

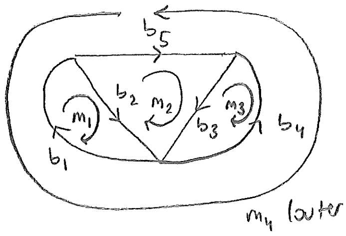
#19 30 Eylül 2017

Duality

Dual graph Given a planar, nonseparable graph G, we construct its dual $\hat{G}$ as follows:

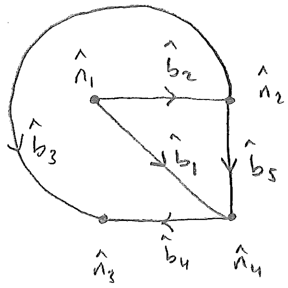
1) Assign clockwise direction to each mesh and ccw to outermesh.
2) Let branch b_j be touching meshes m_x & m_y. If the direction of b_j is same as m_x and opposite to m_y, then $\hat{b}_j$ leaves node $\hat{n}_x$ and enters $\hat{n}_y$.

Ex



G

→

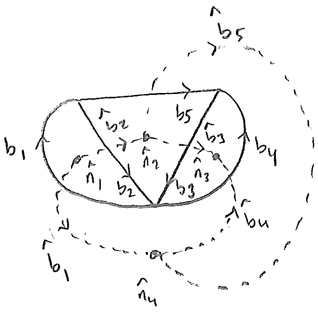


$\hat{G}$

direction of	same as	opposite to
b_1	m_1	m_4
b_2	m_1	m_2
b_3	m_2	m_3
b_4	m_4	m_3
b_5	m_2	m_4

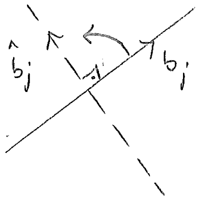
branch	leaves	enters
$\hat{b}_1$	$\hat{n}_1$	$\hat{n}_4$
$\hat{b}_2$	$\hat{n}_1$	$\hat{n}_2$
$\hat{b}_3$	$\hat{n}_2$	$\hat{n}_3$
$\hat{b}_4$	$\hat{n}_4$	$\hat{n}_3$
$\hat{b}_5$	$\hat{n}_2$	$\hat{n}_4$

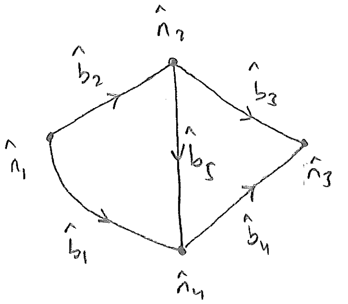
shortcut



G

rotate b_j by 90° ccw to obtain $\hat{b}_j$





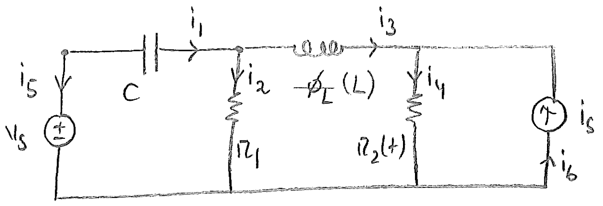
$\hat{G}$

Dual circuit Let N be a circuit with only two-terminal elements and planar, nonseparable graph G. Circuit $\hat{N}$ is the dual of circuit N if

1) Graph of $\hat{N}$ is $\hat{G}$ (dual of G)
2) The branch equation of branch $\hat{b}$ of $\hat{N}$ is obtained from its corresponding equation associated to branch b of N by performing the following substitutions:

variables of N		variables of $\hat{N}$
v_k	⟶	$\hat{i}_k$
i_k	⟶	$\hat{v}_k$
q_k	⟶	$\hat{\phi}_k$
ϕ_k	⟶	$\hat{q}_k$

Example Let us obtain the dual of the below circuit.



circuit N

$v_s(t) = 2\cos t$ V

$i_s(t) = 2$ A

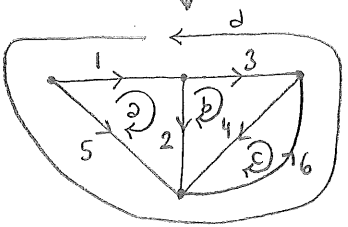
~~$\phi_L = \tanh(i_3)$ Wb~~

$(L = 3H)$

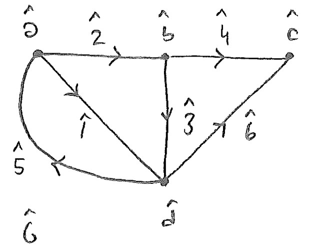
$C = 10\mu F$

$R_1 = 10\Omega$

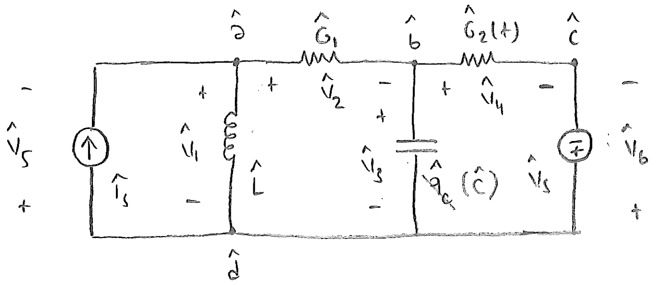
$R_2(t) = 2+\cos t\ \Omega$



graph G



graph $\hat{G}$



circuit $\hat{N}$

$\hat{i}_s(t) = 2\cos t$ A

$\hat{v}_s(t) = 2$ V

~~$\hat{q}_c = \tanh(\hat{v}_3)$ C~~

$(\hat{C} = 3F)$

$\hat{L} = 10\mu H$

$\hat{G}_1 = 10$ ℧

$\hat{G}_2(t) = 2+\cos t$ ℧

At any given time we would have $\hat{v}_k = i_k$ & $\hat{i}_k = v_k$ for $k = 1, 2, \ldots, 6$.

— o —

Principle of duality Let N and $\hat{N}$ be dual circuits. Let S be a true statement about N. Obtain statement $\hat{S}$ by replacing each graph-theoretic and electrical word / quantity by its dual. Then $\hat{S}$ is a correct statement about $\hat{N}$.

Some dual pairs:

node - mesh

fund. cutset - fund. loop

ref. node - outer mesh

tree - cotree

voltage - current

KCL - KVL

charge - flux

resistance (Ω) - conductance (℧)

inductor - capacitor

current source - volt. source

short circuit - open circuit

Exercise Think about the physical interpretation of mesh current. (Hint: use mesh current - node voltage duality.)

#1 13 Ekim 2017

Ch. II Linear Time-Invariant Resistive Circuits

Q: What is an LTI resistive circuit?

Def. A nonlinear circuit contains a nonlinear R or L or C or dependent source (DS).

A circuit is linear if it is NOT nonlinear.

Def. A time-varying circuit contains a TV R or L or C or DS.

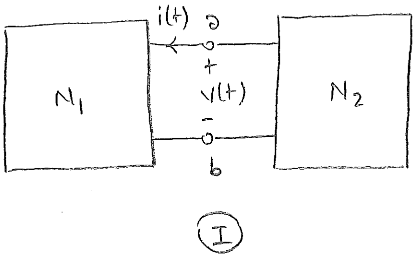
A circuit is time-invariant if it is NOT time-varying.

Def. A dynamic circuit contains an L or C.

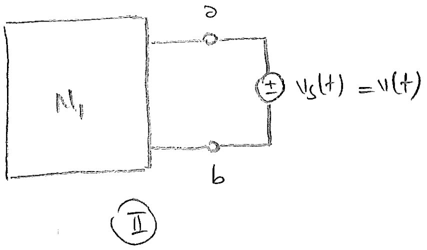
A circuit is resistive if it is NOT dynamic.

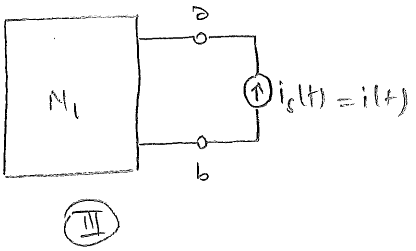
— o —

Substitution Thm. Consider the two circuits N_1, N_2 below:



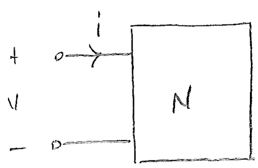
We can replace N_2 by either an independent voltage source with $v_s(t) = v(t)$ or by an independent current source with $i_s(t) = i(t)$ without affecting any branch voltage/current inside N_1.





In other words, an observer inside N_1 cannot tell the difference between the configurations I, II, III.

Definition A pair of associated terminals is called a port. One-port is a circuit or circuit element that has one pair of terminals accessible from outside:

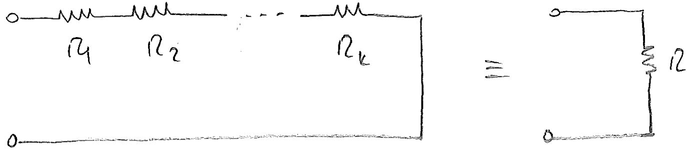


one-port circuit

Two one-ports are said to be <u>equivalent</u> if they have identical i-v characteristics at the port terminals. Let N_0, N_1, N_2 be one-ports and N_1 be equivalent to N_2 ($N_1 \equiv N_2$). Then for both of the following configurations the set of branch currents and voltages within N_0 will be the same.

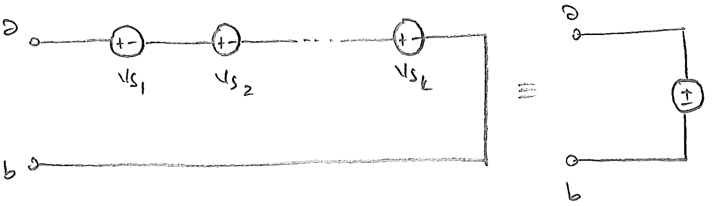
($N_1 \equiv N_2$)

Some equivalent one-ports

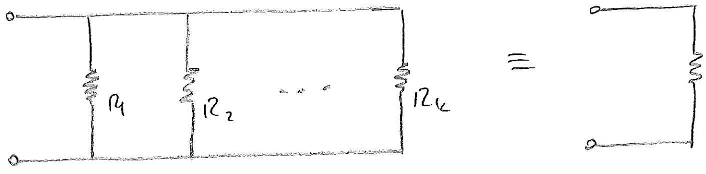


$R_{eq} = R_1 + R_2 + \cdots + R_k$

(series connection)

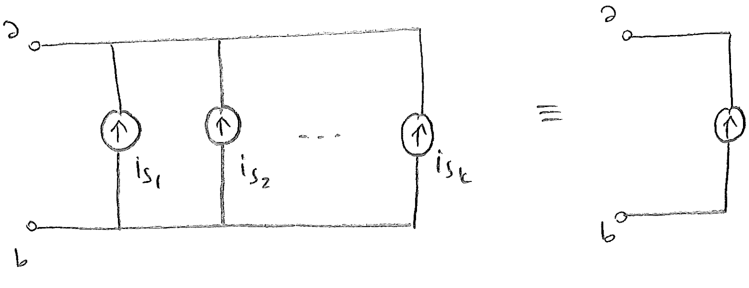


$v_{s_{eq}} = v_{s_1} + v_{s_2} + \cdots + v_{s_k}$



$R_{eq} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_k}\right)^{-1}$

(parallel connection)



$i_{s_{eq}} = i_{s_1} + i_{s_2} + \cdots + i_{s_k}$

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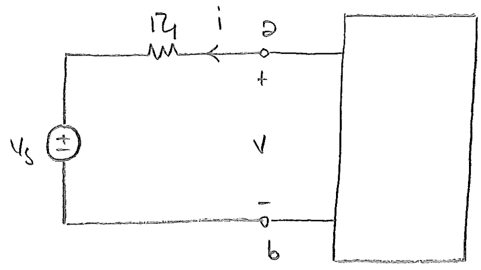
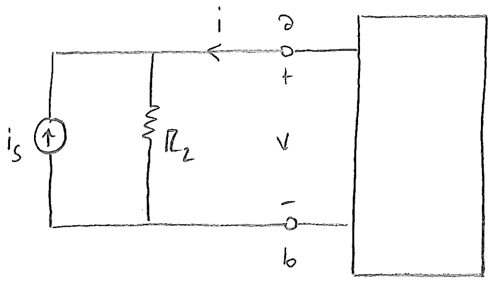


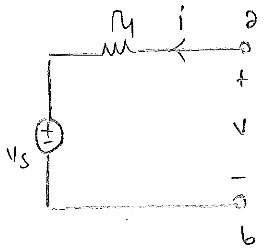


$v = R_1 i + v_s$ (1)

$v = R_2 (i + i_s)$

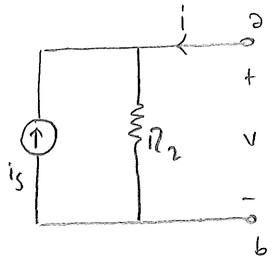
$\Rightarrow v = R_2 i + R_2 i_s$ (2)

→ The one-port



has the i-v char. given in Eq. (1)

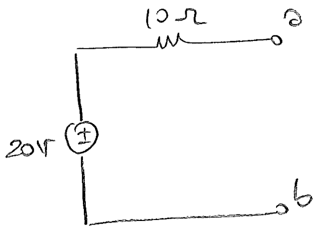
→ The one-port



has the i-v char. given in Eq. (2)

Now, take $R_1 = R_2$ and $v_s = R_1 i_s$. Then (1) and (2) become identical. Hence, one-ports become equivalent.

Ex



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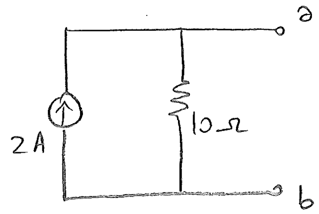

Solving LTI resistive circuits

There are two methods:

1) Node voltage analysis
2) Mesh current analysis

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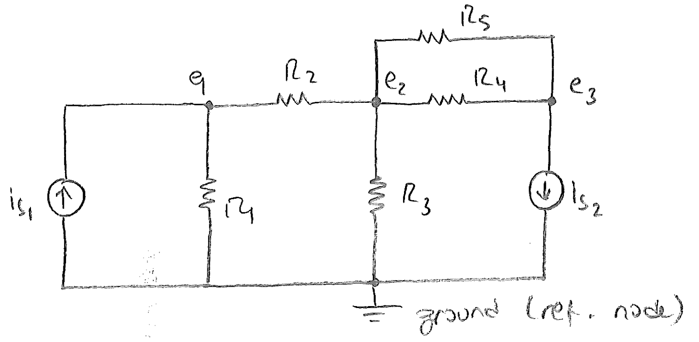
Node Voltage Analysis

The idea is to solve for node voltages by writing KCL at each node (except the reference node). Note that once the node voltages (e) are known, the branch voltages (v) can be computed easily, $v = A^T e$. And once the branch voltages are known, the branch currents can be computed via terminal equations.

Example



procedure:
- → choose reference node
- → label node voltages
- → write KCL

KCL at node ① : $-i_{s_1} + i_{R_1} + i_{R_2}$ $\Rightarrow$ $-i_{s_1} + \frac{e_1}{R_1} + \frac{e_1 - e_2}{R_2} = 0$ (1)

$\frac{e_1}{R_1} \Leftarrow \frac{v_{R_1}}{R_1}$ $\quad$ $\frac{v_{R_2}}{R_2} \Rightarrow \frac{e_1 - e_2}{R_2}$

KCL at node ② : $\frac{e_2 - e_1}{R_2} + \frac{e_2}{R_3} + \frac{e_2 - e_3}{R_4} + \frac{e_2 - e_3}{R_5} = 0$ (2)

KCL at node ③ : $\frac{e_3 - e_2}{R_4} + \frac{e_3 - e_2}{R_5} + i_{s_2} = 0$ (3)

In terms of conductances $G_k = 1/R_k$:

$$\left.\begin{aligned} &(1) \Rightarrow (G_1 + G_2) e_1 - G_2 e_2 = i_{s_1} \\ &(2) \Rightarrow -G_2 e_1 + (G_2 + G_3 + G_4 + G_5) e_2 - (G_4 + G_5) e_3 = 0 \\ &(3) \Rightarrow -(G_4 + G_5) e_2 + (G_4 + G_5) e_3 = -i_{s_2} \end{aligned}\right\} (*)$$

$$(*) \Rightarrow \underbrace{\begin{bmatrix} G_1 + G_2 & -G_2 & 0 \\ -G_2 & G_2 + G_3 + G_4 + G_5 & -(G_4 + G_5) \\ 0 & -(G_4 + G_5) & G_4 + G_5 \end{bmatrix}}_{Y_n:\text{ node admittance matrix}} \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}}_{e} = \underbrace{\begin{bmatrix} i_{s_1} \\ 0 \\ -i_{s_2} \end{bmatrix}}_{i_s}$$

⇐ node eqn.'s in matrix form

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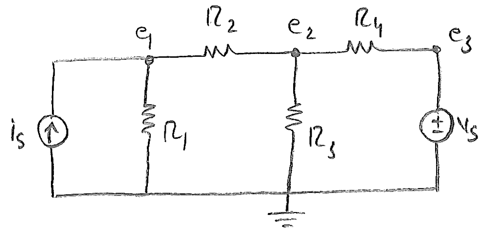
Node analysis with voltage sources

Example



formulation variables : e_1, e_2, e_3

# of eqn. needed = 3

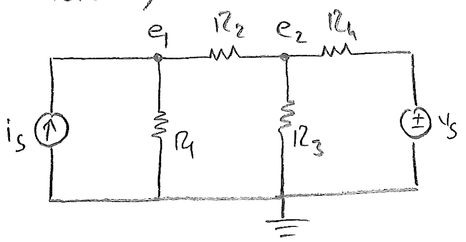
Node ①: $-i_s + \frac{e_1}{R_1} + \frac{e_1 - e_2}{R_2} = 0$

Node ②: $\frac{e_2 - e_1}{R_2} + \frac{e_2}{R_3} + \frac{e_2 - e_3}{R_4} = 0$

Node ③: KCL does not give us a useful equation (in terms of e_1, e_2, e_3) here! However, note that $e_3 = V_s$ (this is the third equation)

Remark When there is an ind. voltage source V_s between the ground and some node k, we do not consider the node voltage e_k as a formulation variable (since it is NOT an unknown, $e_k = V_s$)

Example (revisited)



formulation var. : e_1, e_2

# of eqn. needed = 2

Node ①: $-i_s + \frac{e_1}{R_1} + \frac{e_1 - e_2}{R_2} = 0$

Node ②: $\frac{e_2 - e_1}{R_2} + \frac{e_2}{R_3} + \frac{e_2 - V_s}{R_4} = 0$

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} i_s \\ \frac{V_s}{R_4} \end{bmatrix}$$

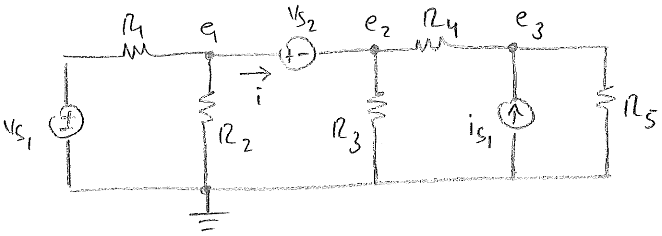
Modified node analysis

When there is a voltage source (dependent or independent) between two (non-ground) nodes a and b, we cannot express the current through it in terms of e_a and e_b. In such cases we do either of the following:

#6 13 Ekim 2017

Method 1 Let the (unknown) current i through the voltage source be one of your formulation variables.

Example



formulation var.: e_1, e_2, e_3, i

Node ①: $\frac{e_1 - v_{s_1}}{R_1} + \frac{e_1}{R_2} + i = 0$

Node ②: $-i + \frac{e_2}{R_3} + \frac{e_2 - e_3}{R_4} = 0$

Node ③: $\frac{e_3 - e_2}{R_4} - i_{s_1} + \frac{e_3}{R_5} = 0$

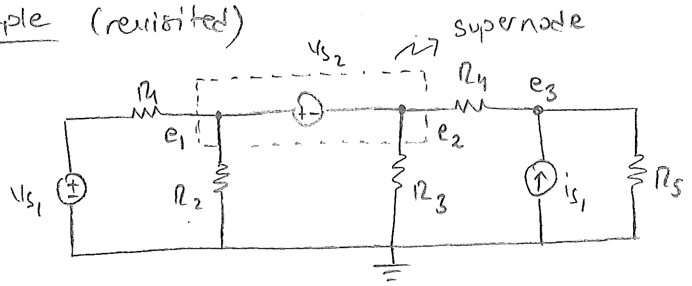
constraint eqn: $e_1 - e_2 = v_{s_2}$

$$\begin{bmatrix} G_1+G_2 & 0 & 0 & 1 \\ 0 & G_3+G_4 & -G_4 & -1 \\ 0 & -G_4 & G_4+G_5 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ i \end{bmatrix} = \begin{bmatrix} G_1 v_{s_1} \\ 0 \\ i_{s_1} \\ v_{s_2} \end{bmatrix}$$

$G_k = 1/R_k$

Method 2 Write KCL pretending that the two nodes (between which there is the voltage source) are a single node. Such node is called a <u>supernode</u>.

Example (revisited)



formulation var.: e_1, e_2, e_3

Supernode Ⓢ: $\frac{e_1 - v_{s_1}}{R_1} + \frac{e_1}{R_2} + \frac{e_2}{R_3} + \frac{e_2 - e_3}{R_4} = 0$

Node ③: $\frac{e_3 - e_2}{R_4} - i_{s_1} + \frac{e_3}{R_5} = 0$

Constraint: $e_1 - e_2 = v_{s_2}$

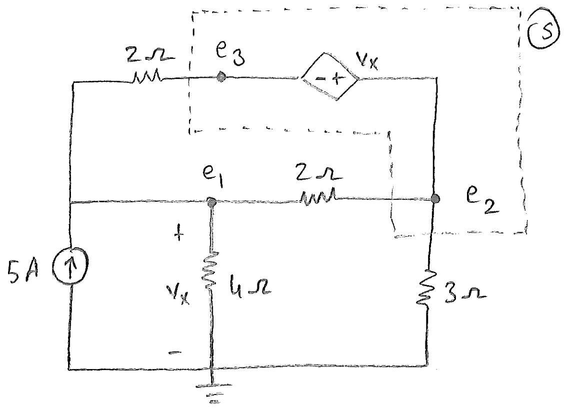
Node equations

$$\begin{bmatrix} G_1+G_2 & G_3+G_4 & -G_4 \\ 0 & -G_4 & G_4+G_5 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} G_1 v_{s_1} \\ i_{s_1} \\ v_{s_2} \end{bmatrix}$$

Node equations in matrix form

#7 13 Ekim 2017

Example



Find the power absorbed by the 3Ω resistor.

KCL at node ① : $-5 + \frac{e_1 - e_3}{2} + \frac{e_1}{4} + \frac{e_1 - e_2}{2} = 0$ (1)

KCL at node ⑤ : $\frac{e_3 - e_1}{2} + \frac{e_2 - e_1}{2} + \frac{e_2}{3} = 0$ (2)

constraint eqn. : $\left.\begin{array}{l} e_2 - e_3 = V_x \\ V_x = e_1 \end{array}\right\} \; e_2 - e_3 = e_1$ (3)

three eqn.'s three unknowns

$$(1) \Rightarrow \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{2}\right) e_1 - \frac{1}{2} e_2 - \frac{1}{2} e_3 = 5 \overset{(3)}{\Rightarrow} \frac{5}{4}(e_2 - e_3) - \frac{1}{2} e_2 - \frac{1}{2} e_3 = 5$$

$$\Rightarrow \frac{3}{4} e_2 - \frac{7}{4} e_3 = 5 \quad (4)$$

$$(2) \Rightarrow -\left(\frac{1}{2} + \frac{1}{2}\right) e_1 + \left(\frac{1}{2} + \frac{1}{3}\right) e_2 + \frac{1}{2} e_3 = 0 \overset{(3)}{\Rightarrow} -(e_2 - e_3) + \frac{5}{6} e_2 + \frac{1}{2} e_3 = 0$$

$$\Rightarrow -\frac{1}{6} e_2 + \frac{3}{2} e_3 = 0 \Rightarrow e_3 = \frac{1}{9} e_2 \quad (5)$$

$$(4) \& (5) \Rightarrow \frac{3}{4} e_2 - \frac{7}{4} \cdot \frac{1}{9} e_2 = 5 \Rightarrow \frac{\overset{4}{\cancel{20}}}{36} e_2 = \cancel{5} \Rightarrow e_2 = 9V \qquad (e_3 = 1V,\ e_1 = 8V)$$

$$P_{3\Omega} = \frac{e_2^2}{3} = \frac{9^2}{3} = \boxed{27W}$$

Remark When there is a dependent source between node k and ground, do NOT write KCL at node k (because you cannot). Figure out the constraint eqn. introduced by the dep. source.

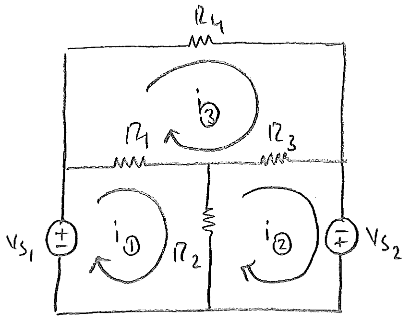
Exercise Redo the previous example by choosing node ③ as ground.

\# 8   13 Ekim 2017

Mesh Current Analysis [Dual of Node Voltage Analysis]

The idea is to solve for mesh currents by writing KVL at inner meshes.

Example



formulation variables $i_{\textcircled{1}}, i_{\textcircled{2}}, i_{\textcircled{3}}$
(mesh currents. by convention their directions are chosen cw.)

KVL at mesh ① : $-v_{s_1} + \underset{\substack{\Downarrow \\ R_1(i_{\textcircled{1}}-i_{\textcircled{3}}) \Leftarrow R_1 i_{R_1}}}{v_{R_1}} + \underset{\substack{\Downarrow \\ R_2 i_{R_2} \Rightarrow R_2(i_{\textcircled{1}}-i_{\textcircled{2}})}}{v_{R_2}} = 0$

$$\Rightarrow \quad -v_{s_1} + R_1(i_{\textcircled{1}} - i_{\textcircled{3}}) + R_2(i_{\textcircled{1}} - i_{\textcircled{2}}) = 0 \qquad (1)$$

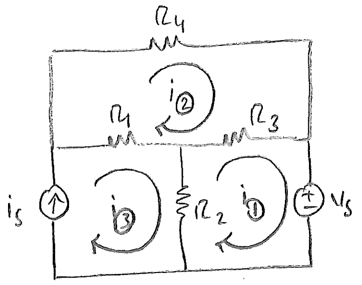
KVL at mesh ② : $-v_{s_2} + R_2(i_{\textcircled{2}} - i_{\textcircled{1}}) + R_3(i_{\textcircled{2}} - i_{\textcircled{3}}) = 0 \qquad (2)$

KVL at mesh ③ : $R_4 i_{\textcircled{3}} + R_3(i_{\textcircled{3}} - i_{\textcircled{2}}) + R_1(i_{\textcircled{3}} - i_{\textcircled{1}}) = 0 \qquad (3)$

$$(1),(2),(3) \Rightarrow \underbrace{\begin{bmatrix} R_1+R_2 & -R_2 & -R_1 \\ -R_2 & R_2+R_3 & -R_3 \\ -R_1 & -R_3 & R_1+R_3+R_4 \end{bmatrix}}_{Z_M:\ \text{mesh-impedance matrix}} \underbrace{\begin{bmatrix} i_{\textcircled{1}} \\ i_{\textcircled{2}} \\ i_{\textcircled{3}} \end{bmatrix}}_{i_M} = \underbrace{\begin{bmatrix} v_{s_1} \\ v_{s_2} \\ 0 \end{bmatrix}}_{v_s}$$

Mesh Analysis with current sources

Example



Note that $i_{\textcircled{3}} = i_s$, therefore known.

formulation var. : $i_{\textcircled{1}}, i_{\textcircled{2}}$

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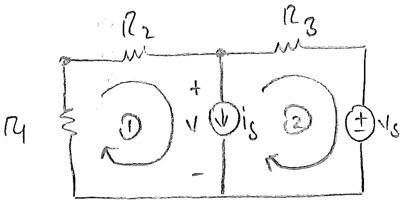
$$\left.\begin{array}{l}\text{KVL at } \textcircled{1}: \; R_2(i_{\textcircled{1}} - i_s) + R_3(i_{\textcircled{1}} - i_{\textcircled{2}}) + V_s = 0 \\ \text{KVL at } \textcircled{2}: \; R_1(i_{\textcircled{2}} - i_s) + R_4 i_{\textcircled{2}} + R_3(i_{\textcircled{2}} - i_{\textcircled{1}}) = 0\end{array}\right\} \Rightarrow \begin{bmatrix} R_2+R_3 & -R_3 \\ -R_3 & R_1+R_3+R_4 \end{bmatrix}\begin{bmatrix} i_{\textcircled{1}} \\ i_{\textcircled{2}} \end{bmatrix} = \begin{bmatrix} R_2 i_s - V_s \\ R_1 i_s \end{bmatrix}$$

Modified Mesh Analysis

When there is a current source between two inner meshes we cannot express its voltage in terms of mesh currents. In such a case we do either of the following:

Method 1 Let the (unknown) voltage v across the current source be one of our formulation variables.

Example

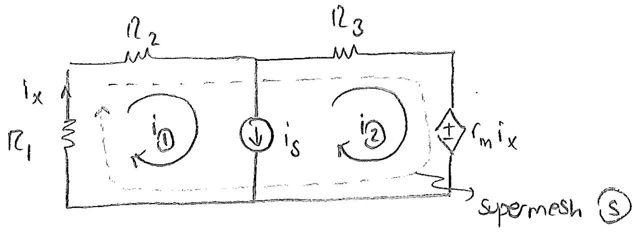


formulation var. : $i_{\textcircled{1}}, i_{\textcircled{2}}, v$

$$\left.\begin{array}{ll}\text{mesh } \textcircled{1}: & (R_1+R_2) i_{\textcircled{1}} + v = 0 \\ \text{mesh } \textcircled{2}: & R_3 i_{\textcircled{2}} + V_s - v = 0 \\ \text{constraint}: & i_{\textcircled{1}} - i_{\textcircled{2}} = i_s\end{array}\right\} \quad \begin{bmatrix} R_1+R_2 & 0 & 1 \\ 0 & R_3 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} i_{\textcircled{1}} \\ i_{\textcircled{2}} \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ -V_s \\ i_s \end{bmatrix}$$

Method 2 Write KVL pretending that the two meshes (between which there is the current source) are a single mesh. Such mesh is called a supermesh.

Example



formulation var. : $i_{\textcircled{1}}, i_{\textcircled{2}}$

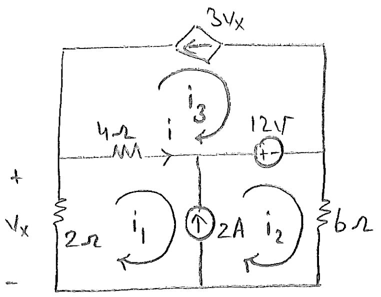
$$\left.\begin{array}{l}\text{KVL at supermesh Ⓢ}: (R_1+R_2) i_{\textcircled{1}} + R_3 i_{\textcircled{2}} + r_m i_x = 0 \\ \qquad i_x = i_{\textcircled{1}}\end{array}\right\} (R_1+R_2+r_m) i_{\textcircled{1}} + R_3 i_{\textcircled{2}} = 0 \quad (1)$$

constraint eqn. : $i_{\textcircled{1}} - i_{\textcircled{2}} = i_s \quad (2)$

$$(1) \; \& \; (2) \Rightarrow \begin{bmatrix} R_1+R_2+r_m & R_3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} i_{\textcircled{1}} \\ i_{\textcircled{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ i_s \end{bmatrix}$$

#10 13 Ekim 2017

Example



Find i.

Mesh Current Formulation

Supermesh (①&②) : $2i_1 + 4(i_1 - i_3) + 12 + 6i_2 = 0$

$\Rightarrow 6i_1 + 6i_2 - 4i_3 = -12$ (1)

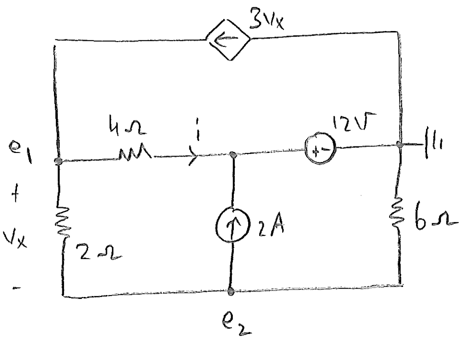
constraint 1 : $\left.\begin{array}{l} i_3 = -3V_x \\ V_x = -2i_1 \end{array}\right\}$ $i_3 = 6i_1$ (2)

constraint 2 : $i_2 - i_1 = 2 \Rightarrow i_2 = i_1 + 2$ (3)

(1),(2),(3) $\Rightarrow 6i_1 + 6(i_1+2) - 4(6i_1) = -12 \Rightarrow -12i_1 = -24 \Rightarrow i_1 = 2A$ (4)

(2) & (4) $\Rightarrow i = i_1 - i_3 = i_1 - 6i_1 = -5i_1 = \boxed{-10A}$

Node Voltage Formulation



Node ① : $\frac{e_1 - e_2}{2} + \frac{e_1 - 12}{4} - 3V_x = 0$ & $V_x = e_1 - e_2$

$\Rightarrow \left(\frac{1}{2} + \frac{1}{4} - 3\right)e_1 + \left(-\frac{1}{2} + 3\right)e_2 = 3$

$\Rightarrow -\frac{9}{4}e_1 + \frac{5}{2}e_2 = 3$ (1)

Node ② : $2 + \frac{e_2 - e_1}{2} + \frac{e_2}{6} = 0$

$\Rightarrow -\frac{1}{2}e_1 + \frac{2}{3}e_2 = -2$ (2)

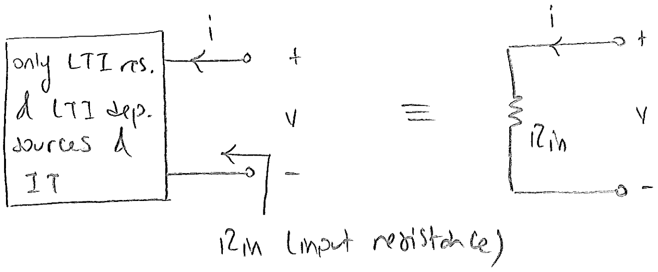
(1) & (2) $\Rightarrow \underbrace{\begin{bmatrix} -9/4 & 5/2 \\ -1/2 & 2/3 \end{bmatrix}}_{Y} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \underbrace{\frac{1}{-\frac{3}{2}+\frac{5}{4}} \begin{bmatrix} 2/3 & -5/2 \\ 1/2 & -9/4 \end{bmatrix}}_{Y^{-1}} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -28 \\ -24 \end{bmatrix}$

Hence, $i = \frac{e_1 - 12}{4} = \frac{-28 - 12}{4} = \boxed{-10A}$

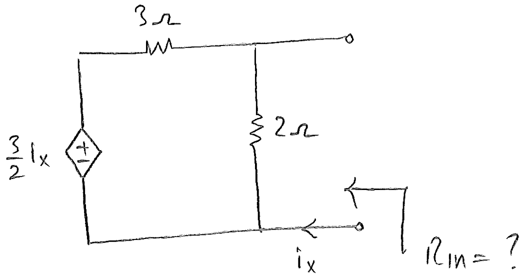
#11 13 Ekim 2017

Input Resistances of LTI Resistive One-Ports

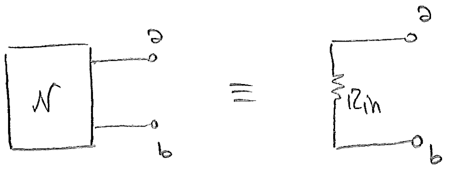
A one-port consisting only of ideal transformers, LTI resistors, and LTI dependent sources (whose control variables are within the one-port) is equivalent to a single LTI resistor.



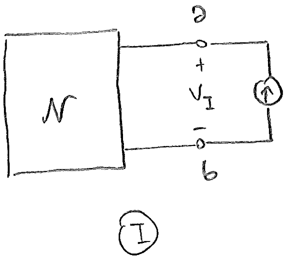
Example



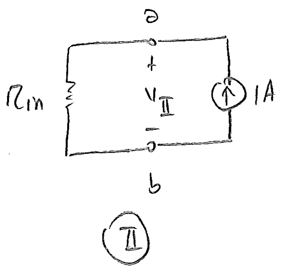
Idea If



then we must have $v_I = v_{II}$ for the following configurations



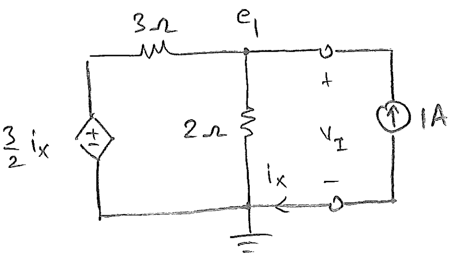
Ⓘ



(II)

$$\underset{\text{unknown}}{R_{in}} = \frac{v_{II}}{1} = v_I$$ ↳ can be computed by solving the circuit in config. I.

Sol'n



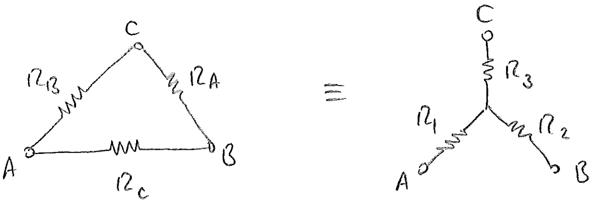
$$\frac{e_1 - \frac{3}{2}i_x}{3} + \frac{e_1}{2} - 1 = 0 \quad \& \quad i_x = -1A$$

$$\Rightarrow \frac{e_1 + 3/2}{3} + \frac{e_1}{2} - 1 = 0 \Rightarrow \frac{5}{6}e_1 = \frac{1}{2}$$

$$\Rightarrow e_1 = \frac{3}{5}V \Rightarrow v_I = \frac{3}{5}V \Rightarrow \boxed{R_{in} = \frac{3}{5}\Omega}$$

Δ-Y Transform

Suppose the below 3-T components are equivalent.



Find the transformation relations between the triples (R_A, R_B, R_C) & (R_1, R_2, R_3).

<u>Idea</u> Equate the equiv. resistances seen between the terminals A-B, B-C, & C-A.

$$R_{AB} = R_C \,//\, (R_A + R_B) = \frac{R_C(R_A+R_B)}{R_A+R_B+R_C} = R_1 + R_2 \quad (1)$$

$$R_{BC} = \frac{R_A(R_B+R_C)}{R_A+R_B+R_C} = R_2 + R_3 \quad (2) \quad \& \quad R_{CA} = \frac{R_B(R_C+R_A)}{R_A+R_B+R_C} = R_1 + R_3 \quad (3)$$

$$(1)+(2)+(3) \Rightarrow \cancel{2}\,\frac{R_AR_B + R_BR_C + R_CR_A}{R_A+R_B+R_C} = \cancel{2}(R_1+R_2+R_3) \quad (4)$$

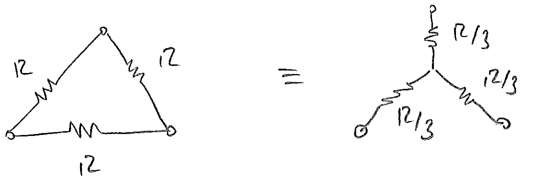
$$(4)-(2) \Rightarrow \boxed{R_1 = \frac{R_BR_C}{R_A+R_B+R_C}} \quad (4)-(3) \Rightarrow \boxed{R_2 = \frac{R_CR_A}{R_A+R_B+R_C}} \quad (4)-(1) \Rightarrow \boxed{R_3 = \frac{R_AR_B}{R_A+R_B+R_C}}$$

Now, write

$$R_1R_2 + R_2R_3 + R_3R_1 = \frac{R_AR_BR_C(R_A+R_B+R_C)}{(R_A+R_B+R_C)^2} = \frac{R_AR_BR_C}{R_A+R_B+R_C} = R_AR_1 = R_BR_2 = R_CR_3$$

$$\Rightarrow \boxed{R_A = \frac{R_1R_2+R_2R_3+R_3R_1}{R_1}}, \quad \boxed{R_B = \frac{R_1R_2+R_2R_3+R_3R_1}{R_2}}, \quad \& \quad \boxed{R_C = \frac{R_1R_2+R_2R_3+R_3R_1}{R_3}}$$

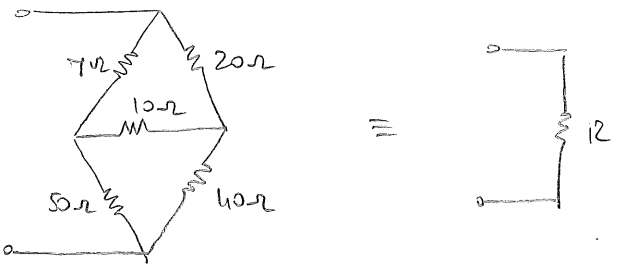
<u>Remark</u> :



<u>Exercise</u> Find the transformation relations between the triples (G_A, G_B, G_C) & (G_1, G_2, G_3)

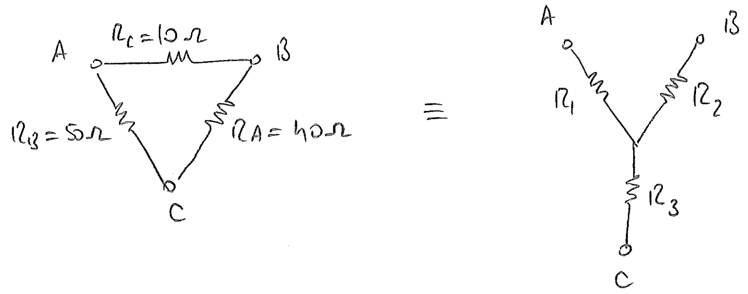
$[G_\square = 1/R_\square]$

Example



Find R.

Sol'n

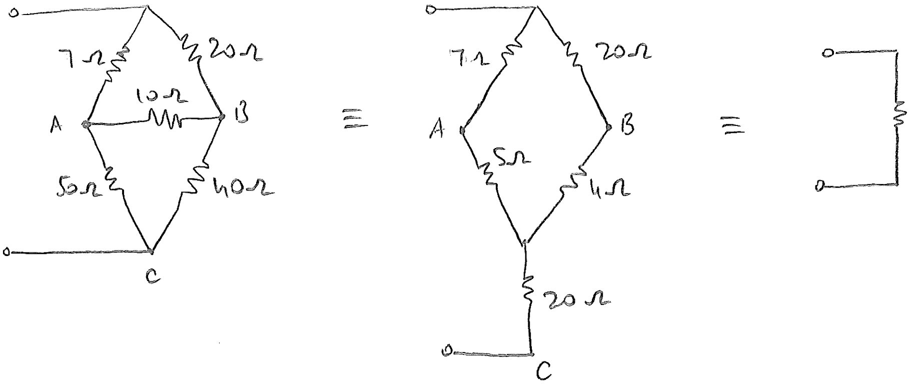


$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} = \frac{500}{100} = 5\Omega$$

$$R_2 = \frac{R_C R_A}{R_A + R_B + R_C} = \frac{400}{100} = 4\Omega$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} = \frac{2000}{100} = 20\Omega$$

Then



$R = (12 \,//\, 24) + 20$

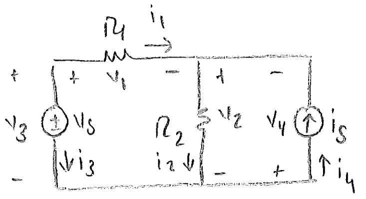
$= \boxed{28\Omega}$

#1 2 Kasım 2018

Linearity

Consider an LTI resistive circuit. Let output y be any branch voltage/current. Let the vector $u = [u_1 \; u_2 \cdots u_m]^T$ be the input vector whose entries are the voltages of the IVS's and the currents of the ICS's in the circuit.

Ex



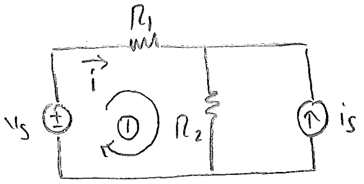
output y can be any one of these:

$i_1, v_1, i_2, v_2, i_3, v_3, i_4, v_4$

input vector: $u = \begin{bmatrix} v_s \\ i_s \end{bmatrix}$

Linearity: For each output y, there exists $k \in \mathbb{R}^{1\times m}$, $k = [k_1 \; k_2 \cdots k_m]$ such that $\boxed{y = ku}$ That is, $y(t) = k_1 u_1(t) + k_2 u_2(t) + \cdots + k_m u_m(t)$.

Example

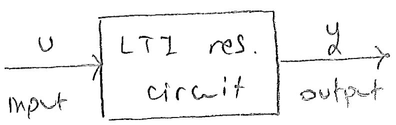


For the output $y = i$ find k_1 & k_2 such that $i = k_1 v_s + k_2 i_s$.

KVL at ①: $-v_s + R_1 i + R_2(i + i_s) = 0$

$$\Rightarrow \quad i = \frac{1}{R_1+R_2} v_s - \frac{R_2}{R_1+R_2} i_s \quad \Rightarrow \quad k_1 = \frac{1}{R_1+R_2}, \quad k_2 = -\frac{R_2}{R_1+R_2}$$

Block diagram representation



Note that y is a function of u. That is, $y = y(u)$.

A direct implication of linearity is

$$\boxed{y(\alpha u + \beta v) = \alpha y(u) + \beta y(v)}$$

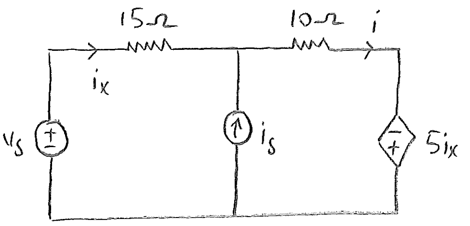
where u, v : input vectors ; α, β : real numbers.

#2 2 Kasım 2018

Example

$v_s = 20V$

$i_s = 4A$



Find i by superposition (i.e. consider one ind. source at a time)

input vector $u = \begin{bmatrix} v_s \\ i_s \end{bmatrix}_{2\times 1}$. Linearity implies: a $k = [k_1\ k_2]_{1\times 2}$ exists such that $i = ku$.

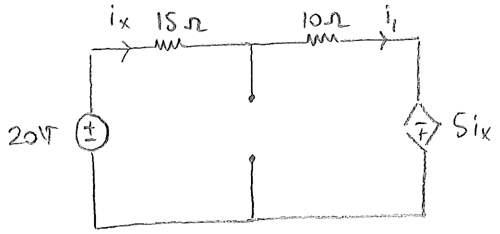
We can write $i = k\begin{bmatrix} v_s \\ i_s \end{bmatrix} = k\left(\begin{bmatrix} v_s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ i_s \end{bmatrix}\right)$

$= \underbrace{k\begin{bmatrix} v_s \\ 0 \end{bmatrix}}_{i_1} + \underbrace{k\begin{bmatrix} 0 \\ i_s \end{bmatrix}}_{i_2}$

i_1: the current passing through 10Ω resistor when $i_s = 0$ (when ind. current source is killed)

i_2: the current passing through 10Ω resistor when $v_s = 0$ (when ind. voltage source is killed)

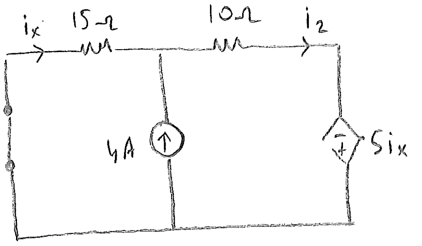
To compute i_1, kill the current source



KVL: $-20 + 15i_x + 10i_1 - 5i_x = 0$ $(i_x = i_1)$

$\Rightarrow 20i_1 = 20 \Rightarrow \boxed{i_1 = 1A}$

To compute i_2, kill the voltage source



KVL: $15i_x + 10i_2 - 5i_x = 0$ $(i_x = i_2 - 4)$

$\Rightarrow 20i_2 = 40 \Rightarrow \boxed{i_2 = 2A}$

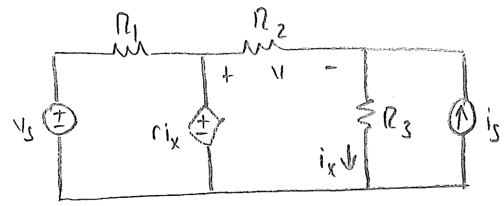
Finally, to find i we superpose i_1 & i_2: $i = i_1 + i_2 = \boxed{3A}$

Remark killing an IVS means replacing it with short circuit.
killing an ICS means replacing it with open circuit.

Warning: Do NOT kill dependent sources. (Because they are not inputs.)

#3 2 Kasım 2018

Example Three experiments are performed on the below circuit.



	v_s	i_s	v
Exp#1	12V	0	v_1
Exp#2	−6V	6A	v_2
Exp#3	4V	4A	v_3

Express v_3 in terms of v_1 & v_2.

Sol'n By linearity there exists $K \in \mathbb{R}^{1\times 2}$ such that $v = K\begin{bmatrix} v_s \\ i_s \end{bmatrix}$

$$\left.\begin{array}{l} \text{exp\#1} \Rightarrow \quad v_1 = K\begin{bmatrix}12\\0\end{bmatrix} \\ \text{exp\#2} \Rightarrow \quad v_2 = K\begin{bmatrix}-6\\6\end{bmatrix}\end{array}\right\} \quad [v_1 \ \ v_2] = K\begin{bmatrix}12 & -6\\0 & 6\end{bmatrix} \Rightarrow K = [v_1 \ \ v_2]\begin{bmatrix}12 & -6\\0 & 6\end{bmatrix}^{-1}$$

$$\text{exp\#3} \Rightarrow v_3 = K\begin{bmatrix}4\\4\end{bmatrix} = \overbrace{[v_1 \ \ v_2]\begin{bmatrix}\frac{1}{12} & \frac{1}{12}\\ 0 & \frac{1}{6}\end{bmatrix}}^{K}\begin{bmatrix}4\\4\end{bmatrix} = \boxed{\frac{2}{3}v_1 + \frac{2}{3}v_2}$$

$$\underbrace{\begin{bmatrix}\frac{1}{12} & \frac{1}{12}\\ 0 & \frac{1}{6}\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix}}_{\begin{bmatrix}2/3\\2/3\end{bmatrix}}$$

OR,
$$v_3 = v\left(\begin{bmatrix}4\\4\end{bmatrix}\right) = v\left(\alpha\begin{bmatrix}12\\0\end{bmatrix} + \beta\begin{bmatrix}-6\\6\end{bmatrix}\right) = \alpha \cdot v\left(\begin{bmatrix}12\\0\end{bmatrix}\right) + \beta \cdot v\left(\begin{bmatrix}-6\\6\end{bmatrix}\right) = \alpha v_1 + \beta v_2$$

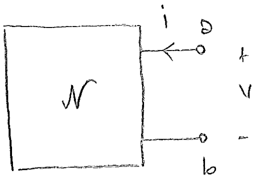
$$\Rightarrow \alpha\begin{bmatrix}12\\0\end{bmatrix} + \beta\begin{bmatrix}-6\\6\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix} \Rightarrow \left.\begin{array}{r}12\alpha - 6\beta = 4\\ 6\beta = 4\end{array}\right\} \alpha = \frac{2}{3}, \ \beta = \frac{2}{3}$$ as expected.

#4 2 Kasım 2018

Thevenin & Norton Equivalent Circuits

Recall: two one-ports are equivalent if they have identical i-v char.

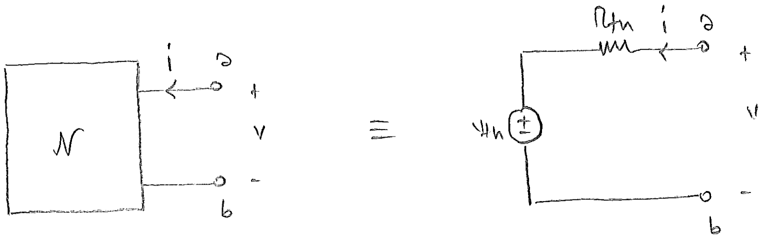
Thevenin's Theorem Consider the below one-port, where $\mathcal{N}$ is an LTI resistive circuit satisfying:



(A1) Any dep. source in $\mathcal{N}$ has its control var. also in $\mathcal{N}$.

(A2) One-port is both voltage- and current-controlled.

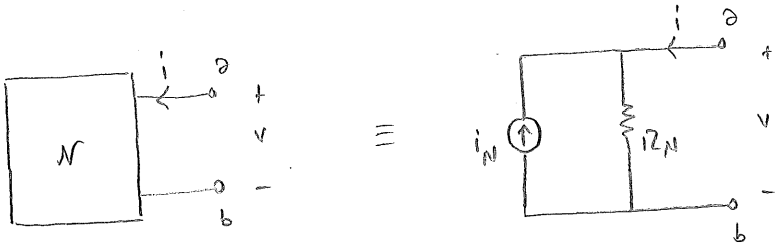
Then we can find v_{th}, R_{th} and establish the below equivalence



i-v characteristics:

$$\boxed{v = R_{th} i + v_{th}} \qquad (1)$$

Norton's Theorem Under the same conditions (A1) & (A2) we can find i_N, R_N and the below equivalence holds



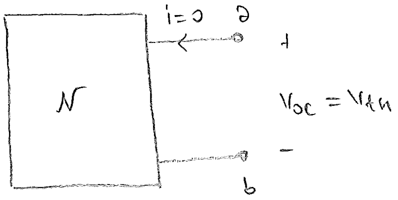
i-v char:

$$\boxed{v = R_N i + R_N i_N} \qquad (2)$$

Remark Since (1) ≡ (2) we have to have $R_{th} = R_N$ and $v_{th} = R_{th} i_N$.

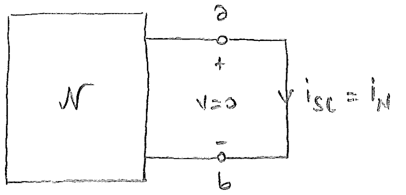
Computing v_{th}, R_{th}, i_N (for a given one-port $\mathcal{N}$)

1) open circuit voltage equals v_{th}:

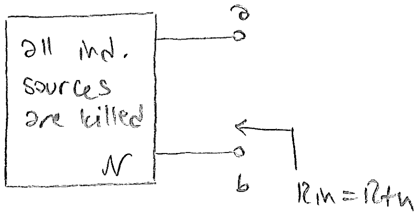


#5 2 Kasım 2018

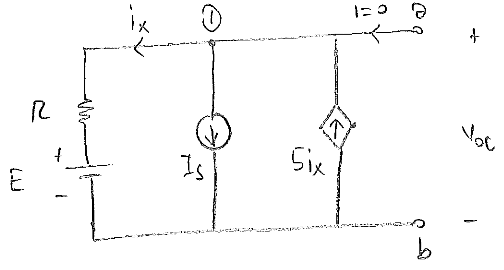
2) Short circuit current equals i_N :



3) Once v_{th} and i_N are known we have $R_{th} = v_{th}/i_N$. Another way to obtain R_{th} is to kill all independent sources within N and compute the input resistance:



Example Find the thevenin and norton equivalents of the below one-port.



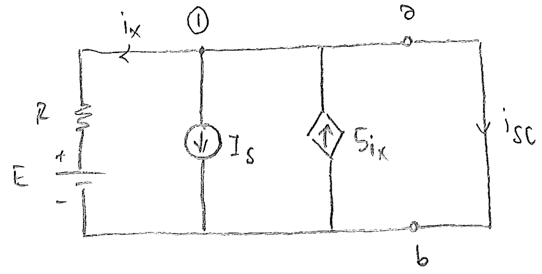
Step 1 Find the open-circuit voltage V_{oc}.

KCL at ①: $i_x + I_s - 5i_x = 0 \Rightarrow i_x = \frac{1}{4} I_s$

$V_{oc} = R i_x + E = \frac{R}{4} I_s + E$

$\Rightarrow \boxed{V_{th} = \frac{R}{4} I_s + E}$

Step 2 Find the short-circuit current i_{sc}.



$R i_x + E = 0 \Rightarrow i_x = -\frac{E}{R}$

KCL at ①: $i_x + I_s - 5i_x + i_{sc} = 0$

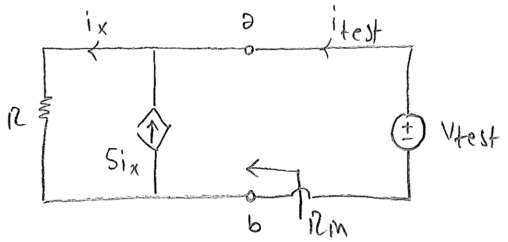
$\Rightarrow i_{sc} = -I_s + 4i_x = -I_s - \frac{4}{R} E$

$\Rightarrow \boxed{i_N = -I_s - \frac{4}{R} E}$

#6 2 Kasım 2018

Step3 Find R_{th}. $R_{th} = \frac{V_{th}}{i_N} = \frac{\frac{R}{4}I_s + E}{-(I_s + \frac{4}{R}E)} \Rightarrow \boxed{R_{th} = -\frac{R}{4}}$

Alternative way to compute R_{th} : kill all ind. sources.



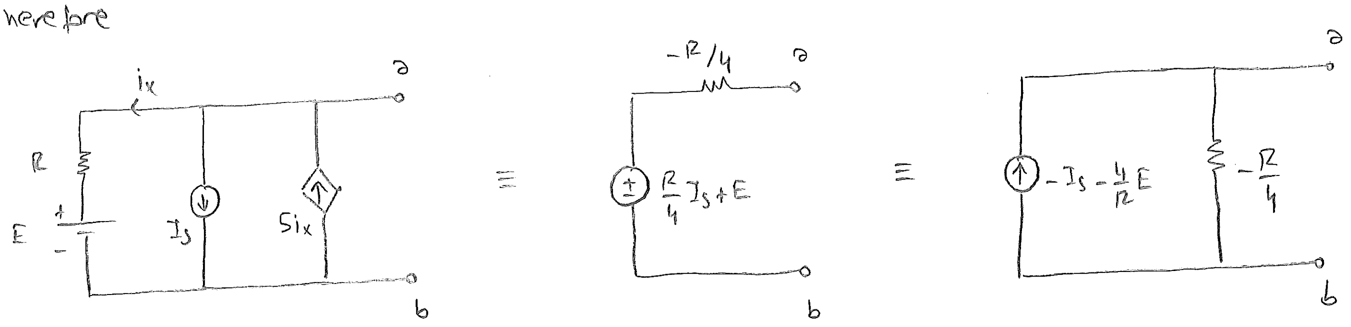
$R_{th} = \frac{V_{test}}{i_{test}}$

Choose $V_{test} = 1V$. Then $i_x = \frac{V_{test}}{R} = \frac{1}{R}$

KCL $\Rightarrow$ $i_x - 5i_x - i_{test} = 0 \Rightarrow i_{test} = -4i_x = -\frac{4}{R}$

$R_{th} = \frac{V_{test}}{i_{test}} = \frac{1}{-4/R} = -\frac{R}{4} \Rightarrow R_{th} = -\frac{R}{4}$ as expected.

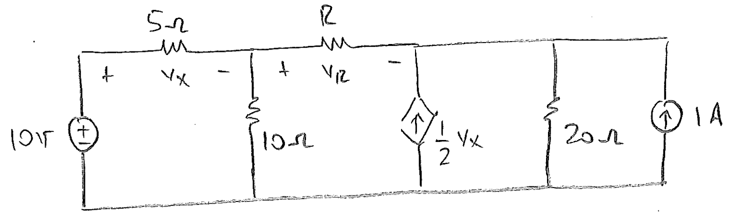
Therefore



thevenin equiv.   norton equiv.

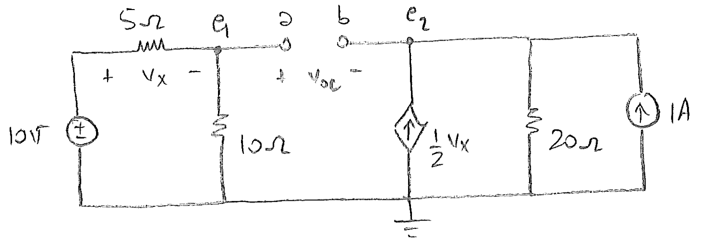
Example Find the thevenin equiv. circuit as viewed by the resistor R.

What should R be so that $V_R = -7V$.



#7 2 Kasım 2018

Sol'n First. Remove R and find V_{oc}.



Node ①: $\frac{e_1 - 10}{5} + \frac{e_1}{10} = 0 \Rightarrow e_1 = \frac{20}{3}$ V

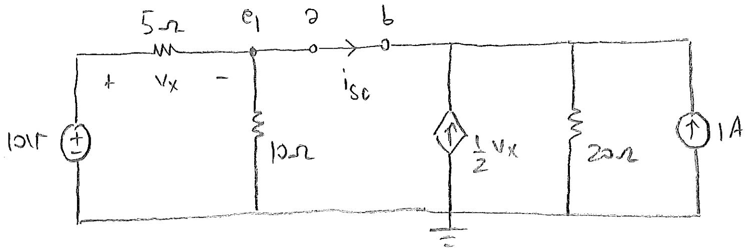
Node ②: $-\frac{1}{2}V_x + \frac{e_2}{20} - 1 = 0$

$V_x = 10 - e_1 = \frac{10}{3}$ V

$\Bigg\} \; -\frac{5}{3} + \frac{e_2}{20} - 1 = 0 \Rightarrow e_2 = \frac{160}{3}$ V

$V_{oc} = e_1 - e_2 = \frac{20}{3} - \frac{160}{3} = -\frac{140}{3}$ V $\Rightarrow \boxed{V_{th} = -\frac{140}{3} \text{ V}}$

Then. Find i_{sc}.



KCL at node ①: $\frac{e_1 - 10}{5} + \frac{e_1}{10} - \frac{1}{2}V_x + \frac{e_1}{20} - 1 = 0$

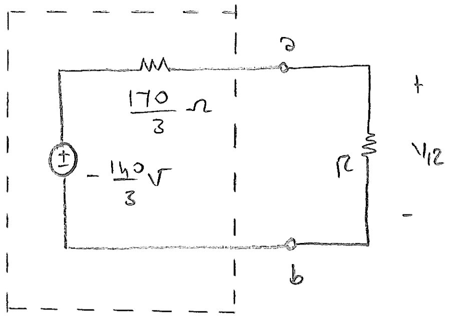
$V_x = 10 - e_1$

$\Bigg\} \; \underbrace{\left(\frac{1}{5} + \frac{1}{10} + \frac{1}{2} + \frac{1}{20}\right)}_{\frac{17}{20}} e_1 = 1 + 2 + 5 \Rightarrow e_1 = \frac{160}{17}$ V

once again KCL at node ①: $\frac{e_1 - 10}{5} + \frac{e_1}{10} + i_{sc} = 0 \Rightarrow i_{sc} = 2 - \frac{3}{10}e_1 = -\frac{14}{17}$ A $\Rightarrow \boxed{i_N = -\frac{14}{17} \text{ A}}$

Hence $R_{th} = \frac{V_{th}}{i_N} = \frac{-\frac{140}{3}}{-\frac{14}{17}} = \boxed{\frac{170}{3} \, \Omega}$

Finally.



↓

thevenin equiv. seen by R

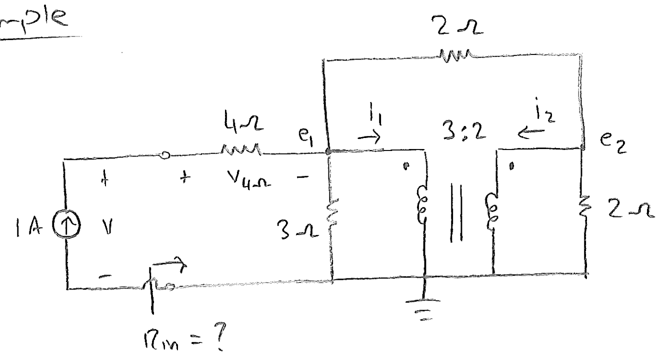
want: $v_R = -7V$

design parameter: R

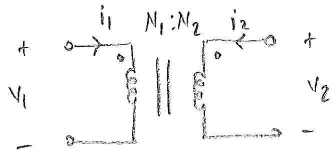
$$v_R = \frac{R}{R+\frac{170}{3}} \cdot \left(-\frac{140}{3}\right) = -7$$

$$\Rightarrow \frac{140R}{3R+170} = 7 \quad \Rightarrow \boxed{R = 10\Omega}$$

Example



Recall IT equations



$$\frac{v_1}{N_1} = \frac{v_2}{N_2} \quad \& \quad N_1 i_1 + N_2 i_2 = 0$$

Node ①: $-1 + \frac{e_1}{3} + i_1 + \frac{e_1 - e_2}{2} = 0$

Node ②: $i_2 + \frac{e_2}{2} + \frac{e_2 - e_1}{2} = 0$

IT eqn.'s : $\frac{e_1}{3} = \frac{e_2}{2}$, $3i_1 + 2i_2 = 0$

Four equations four unknowns

$$\Rightarrow e_1 = \frac{18}{11} V$$

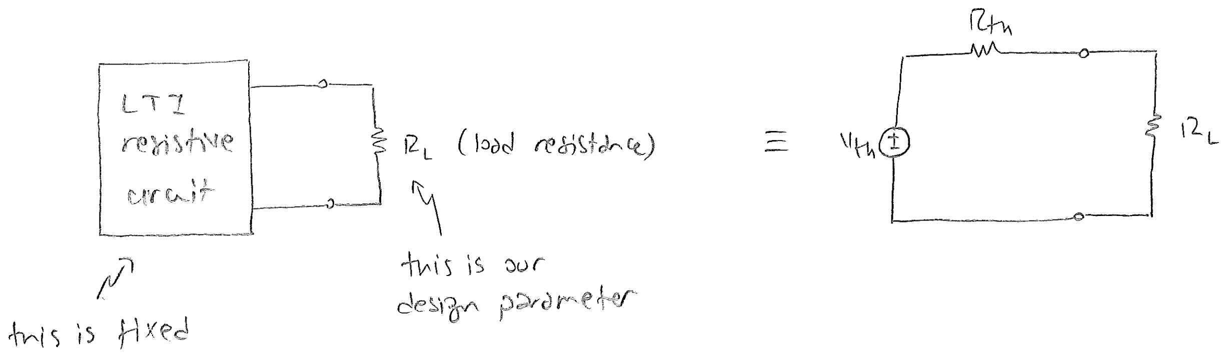
Then, $v = v_{4\Omega} + e_1 = 4 \cdot 1 + \frac{18}{11} = \frac{62}{11} V$

$$\Rightarrow \boxed{R_{in} = \frac{62}{11}\Omega}$$

#9 2 Kasım 2018

Maximum Power Transfer

There are many applications in circuit theory where it is desirable to extract the maximum possible power from a given circuit.



Now, the power delivered to the load is : $P_L = R_L \left[\frac{V_{th}}{R_{th}+R_L} \right]^2$

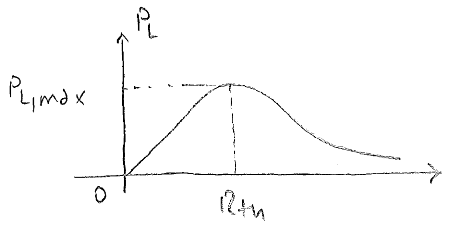
P_L : function of R_L (since V_{th} & R_{th} are fixed)

Question: $P_{L,max} = ?$

Answer: differentiate w.r.t. R_L .

$$\frac{dP_L}{dR_L} = \frac{V_{th}^2}{(R_L+R_{th})^2} - 2R_L \frac{V_{th}^2}{(R_L+R_{th})^3} = V_{th}^2 \left\{ \frac{(R_L+R_{th}) - 2R_L}{(R_L+R_{th})^3} \right\} = V_{th}^2 \frac{R_{th}-R_L}{(R_L+R_{th})^3}$$

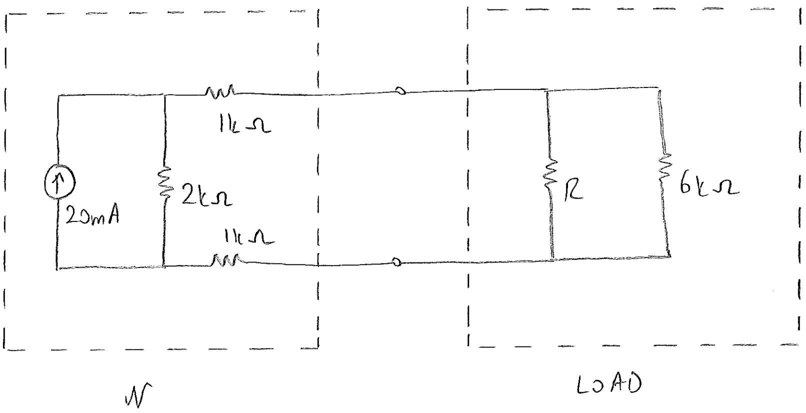
$$\frac{dP_L}{dR_L} = 0 \implies \boxed{R_L = R_{th}}$$



Conclusion For max power transfer to the load, R_L must equal R_{th} !

#10 2 Kasım 2018

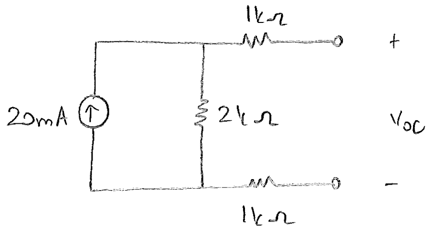
Example



a) Find R so that max power is delivered to the load.

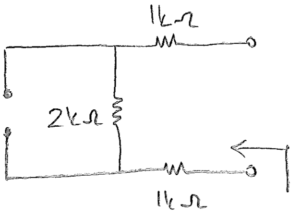
b) Find this max power.

Step1 Find the thevenin equiv. of the circuit N.



$V_{oc} = (20mA) \times (2k\Omega) = 40V$

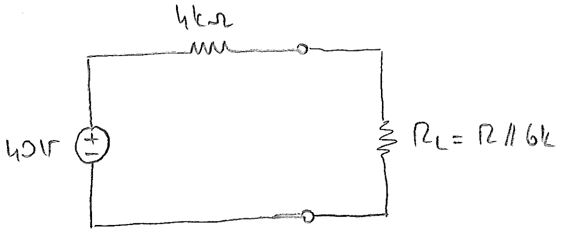
$\Rightarrow \boxed{V_{th} = 40V}$



$R_{th} = 1+2+1 = 4k\Omega$

$\Rightarrow \boxed{R_{th} = 4k\Omega}$

Step2 Study the equiv. circuit



For max power transfer we need $R_L = 4k\Omega$

$\Rightarrow \frac{1}{R} + \frac{1}{6} = \frac{1}{4} \Rightarrow \boxed{R = 12k\Omega}$

$$P_{L,max} = R_L \left(\frac{40}{4k + R_L} \right)^2 \Bigg|_{R_L = 4k} = 4k \left(\frac{40}{8k} \right)^2 = (4k\Omega) \times (5mA)^2$$

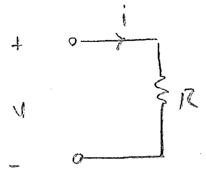
$$= 4000 \times (5 \times 10^{-3})^2 = 0.1W$$

$\Rightarrow \boxed{P_{L,max} = 0.1W}$

#1 30 Ekim 2018

Two Ports

Consider a LTI resistor

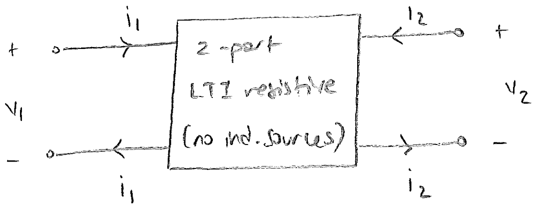


i-v char : $\underbrace{v}_{\text{scalar}} = \overbrace{R}^{\text{scalar}} \underbrace{i}_{\text{scalar}}$ (1)

Eq. (1) has a natural generalization:

$$\underbrace{v}_{\text{vector in } \mathbb{R}^n} = \underbrace{R}_{n\times n \text{ matrix}} \overbrace{i}^{\text{vector in } \mathbb{R}^n}$$

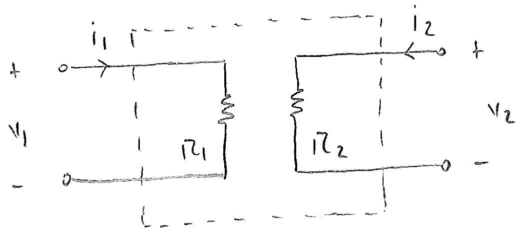
We will be interested in the case $n=2$ (two ports).



$$\left.\begin{aligned} v_1 &= r_{11} i_1 + r_{12} i_2 \\ v_2 &= r_{21} i_1 + r_{22} i_2 \end{aligned}\right\} \quad \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}}_{i}$$

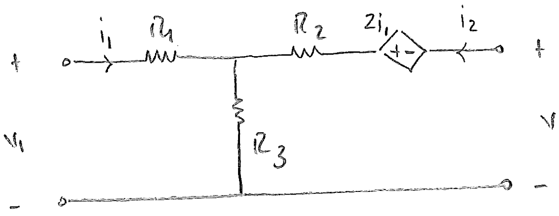
R : resistance matrix

Example (trivial case)



$$\left.\begin{aligned} v_1 &= R_1 i_1 + 0 \cdot i_2 \\ v_2 &= 0 \cdot i_2 + R_2 i_2 \end{aligned}\right\} \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}}_{R} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Example Find the resistance matrix R



Sol'n by KVL:

$$\left.\begin{aligned} v_1 &= R_1 i_1 + R_3(i_1+i_2) \\ v_2 &= -2i_1 + R_2 i_2 + R_3(i_1+i_2) \end{aligned}\right\} \quad (*)$$

$$(*) \Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1+R_3 & R_3 \\ R_3-2 & R_2+R_3 \end{bmatrix}}_{R} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

#2 30 Ekim 2018

Another approach

$$v_1 = r_{11} i_1 + r_{12} i_2 \Rightarrow r_{11} = \left.\frac{v_1}{i_1}\right|_{i_2=0}$$

that is, r_{11} equals the ratio v_1/i_1 when the 2nd port is open-circuited ($i_2=0$)

$$\& \quad r_{12} = \left.\frac{v_1}{i_2}\right|_{i_1=0}$$

likewise, $v_2 = r_{21} i_1 + r_{22} i_2 \Rightarrow r_{21} = \left.\frac{v_2}{i_1}\right|_{i_2=0}$ & $r_{22} = \left.\frac{v_2}{i_2}\right|_{i_1=0}$.

Let's apply this approach to the previous example:

when $i_2 = 0$

$$v_1 = R_1 i_1 + R_3 i_1 = (R_1+R_3) i_1 \Rightarrow r_{11} = R_1 + R_3$$

$$v_2 = -2i_1 + R_3 i_1 = (R_3 - 2) i_1 \Rightarrow r_{21} = R_3 - 2$$

when $i_1 = 0$

$$v_1 = R_3 i_2 \Rightarrow r_{12} = R_3$$

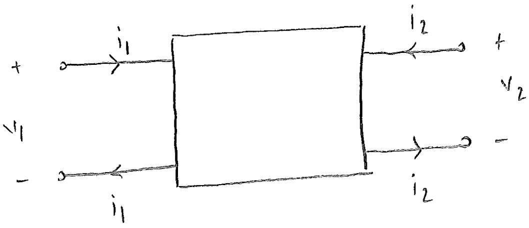
$$v_2 = R_2 i_2 + R_3 i_2 \Rightarrow r_{22} = R_2 + R_3$$

$$R = \begin{bmatrix} R_1+R_3 & R_3 \\ R_3-2 & R_2+R_3 \end{bmatrix}$$

as expected

Other representations are also possible & used:

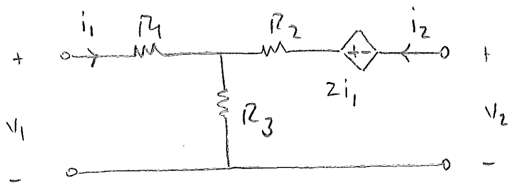
Conductance matrix



$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}}_{\text{conductance matrix, } G} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

#8 30 Ekim 2018

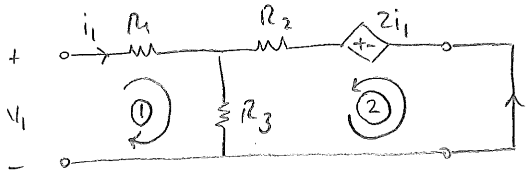
Example Find G.



Note that $g_{11} = \left.\frac{i_1}{v_1}\right|_{v_2=0}$ (ratio $\frac{i_1}{v_1}$ when 2nd port is short-circuited ($v_2=0$))

& $g_{21} = \left.\frac{i_2}{v_1}\right|_{v_2=0}$, $g_{12} = \left.\frac{i_1}{v_2}\right|_{v_1=0}$, $g_{22} = \left.\frac{i_2}{v_2}\right|_{v_1=0}$

When $v_2=0$



①: $-v_1 + R_1 i_1 + R_3(i_1+i_2) = 0$ (1)

②: $-2i_1 + R_2 i_2 + R_3(i_1+i_2) = 0$ (2)

(1)&(2) $\Rightarrow$ $$\begin{bmatrix} R_1+R_3 & R_3 \\ R_3-2 & R_2+R_3 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_1 \qquad (3)$$

(3) $\Rightarrow$ $$\begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} = \left.\begin{bmatrix} i_1/v_1 \\ i_2/v_1 \end{bmatrix}\right|_{v_2=0} = \begin{bmatrix} R_1+R_3 & R_3 \\ R_3-2 & R_2+R_3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} R_2+R_3 & -R_3 \\ 2-R_3 & R_1+R_3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (4)$$

$$\Delta := (R_1+R_3)(R_2+R_3) - R_3(R_3-2) = R_1R_2 + R_2R_3 + R_1R_3 + 2R_3 .$$

(4) $\Rightarrow$ $g_{11} = \frac{R_2+R_3}{\Delta}$ & $g_{21} = \frac{2-R_3}{\Delta}$.

Exercise: compute g_{12} & g_{22}. ($g_{12} = -\frac{R_3}{\Delta}$ & $g_{22} = \frac{R_1+R_3}{\Delta}$)

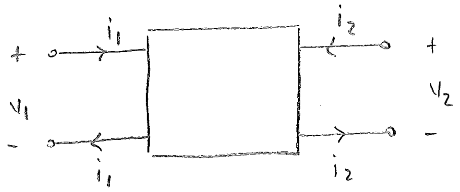
Hence, $$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{R_2+R_3}{\Delta} & -\frac{R_3}{\Delta} \\ \frac{2-R_3}{\Delta} & \frac{R_1+R_3}{\Delta} \end{bmatrix}}_{G}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

(observe that $G = R^{-1}$, the inverse of the resistance matrix)

Remark In general the resistance (conductance) matrix R (G) need not be invertible. When the inverse exists we have $G = R^{-1}$ ($R = G^{-1}$).

#4 30 Ekim 2018

Hybrid matrices



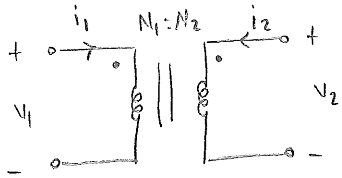
$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}}_{H} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \quad \& \quad \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \underbrace{\begin{bmatrix} h'_{11} & h'_{12} \\ h'_{21} & h'_{22} \end{bmatrix}}_{H'} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

$H' = H^{-1}$ (If the inverse exists)

Transmission matrix (chained parameters)

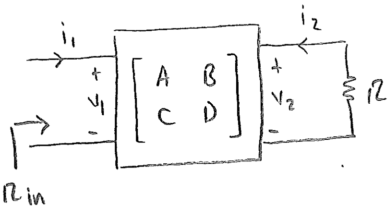
$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{T} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$$

Example Find the matrix T for the ideal transformer.



$$\left.\begin{array}{l} \dfrac{v_1}{N_1} = \dfrac{v_2}{N_2} \Rightarrow v_1 = \dfrac{N_1}{N_2} \cdot v_2 \\[2ex] N_1 i_1 + N_2 i_2 = 0 \Rightarrow i_1 = -\dfrac{N_2}{N_1} i_2 \end{array}\right\} \begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{N_1}{N_2} & 0 \\ 0 & \frac{N_2}{N_1} \end{bmatrix}}_{T} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$$

Example Find the input resistance



where $\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$

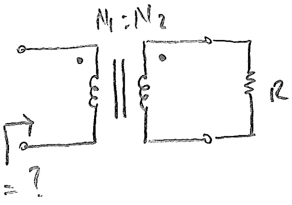
$$R_{in} = \frac{v_1}{i_1}$$

$$v_1 = Av_2 - Bi_2$$
$$i_1 = Cv_2 - Di_2$$
$$v_2 = -Ri_2$$

$$v_1 = -ARi_2 - Bi_2 = -(AR+B)i_2$$
$$i_1 = -CRi_2 - Di_2 = -(CR+D)i_2$$

$$\boxed{\frac{v_1}{i_1} = \frac{AR+B}{CR+D}}$$

Example

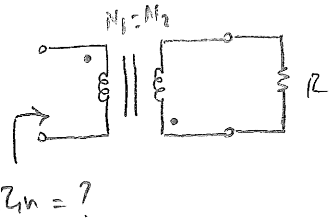


Combining the previous two examples:

$$R_{in} = \frac{AR+B}{CR+D} = \frac{\frac{N_1}{N_2} \cdot R + 0}{0 + \frac{N_2}{N_1}}$$

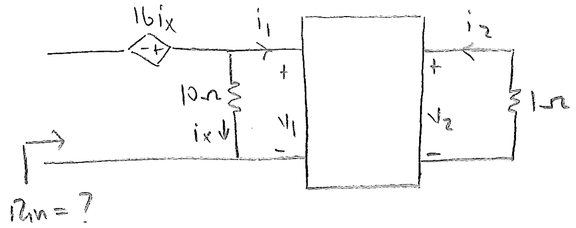
$$\boxed{= \left(\frac{N_1}{N_2}\right)^2 R}$$

Exercise :



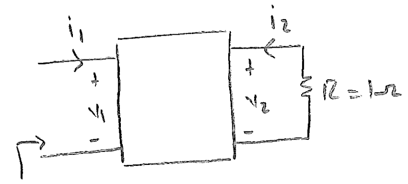
#5 30 Ekim 2018

Example



where $\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$ (1)

Step1 Find R_1



$v_2 = -R i_2$ (2)

$(1)\&(2) = \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} i_1 \\ -R i_2 \end{bmatrix}$

$R_1 = v_1/i_1$

$\Rightarrow v_1 = 4i_1 + 2R i_2$

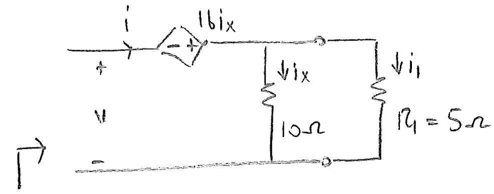
$\& \quad i_2 = 2i_1 - 3R i_2 \Rightarrow i_2 = \frac{2}{3R+1} i_1$

Then,

$v_1 = 4i_1 + 2R\left(\frac{2}{3R+1}\right) i_1 = \left[4 + \frac{4R}{3R+1}\right] i_1$

$\Rightarrow R_1 = \left.\frac{v_1}{i_1}\right|_{R=1\Omega} = \left[4 + \frac{4R}{3R+1}\right]_{R=1} = \boxed{5\Omega}$

Step2 Compute R_{in} using



$R_{in} = \frac{v}{i}$

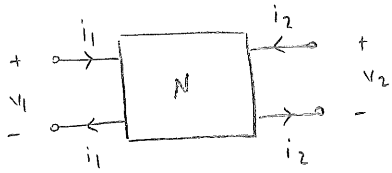
$10 i_x = 5 i_1 \Rightarrow i_1 = 2 i_x$

$i = i_x + i_1 = 3 i_x$

$v = -16 i_x + 10 i_x = -6 i_x$

$\Big\} \; R_{in} = \frac{v}{i} = \frac{-6 i_x}{3 i_x} = \boxed{-2\Omega}$

Power of two-port



Let P_N be the sum of powers of all individual components inside N.

Theorem $P_N = i_1 v_1 + i_2 v_2$

Proof Exercise. (You may assume N contains only 2-terminal components)

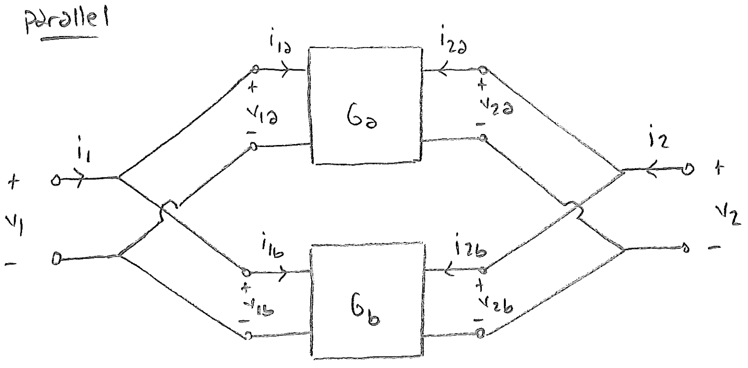
Hint: use Tellegen's Thm.

Definition A two-port is passive if $P_N \geq 0$ for all possible sets of port variables (i_1, v_1, i_2, v_2). Active if not passive.

#6 30 Ekim 2018

Interconnections of two-ports

parallel



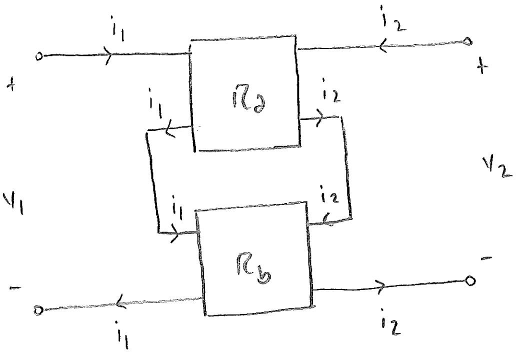
$v_1 = v_{1a} = v_{1b}$ & $v_2 = v_{2a} = v_{2b}$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} i_{1a} \\ i_{2a} \end{bmatrix} + \begin{bmatrix} i_{1b} \\ i_{2b} \end{bmatrix}$$

$$= G_a \begin{bmatrix} v_{1a} \\ v_{2a} \end{bmatrix} + G_b \begin{bmatrix} v_{1b} \\ v_{2b} \end{bmatrix}$$

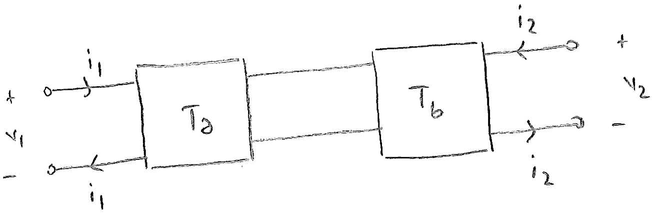
$$= [G_a + G_b] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Series



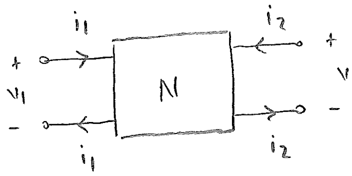
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = [R_a + R_b] \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Cascade



$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = T_a T_b \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$$

Example



Given the following measurements on N, find R.

	v_1	v_2	i_1	i_2
Exp#1	1	2	3	4
Exp#2	5	6	7	8

Sol'n

Exp#1 $\Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} = R \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Exp#2 $\Rightarrow \begin{bmatrix} 5 \\ 6 \end{bmatrix} = R \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix} = R \begin{bmatrix} 3 & 7 \\ 4 & 8 \end{bmatrix} \Rightarrow \boxed{R = \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 4 & 8 \end{bmatrix}^{-1}}$$

Exercise Given $H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$, find T.

Hint:

$H \Rightarrow$

	v_1	v_2	i_1	i_2
Exp#1	h_{11}	0	1	h_{21}
Exp#2	h_{12}	1	0	h_{22}

$$\text{Table} \Rightarrow \begin{bmatrix} h_{11} \\ 1 \end{bmatrix} = T \begin{bmatrix} 0 \\ -h_{21} \end{bmatrix} \;\&\; \begin{bmatrix} h_{12} \\ 0 \end{bmatrix} = T \begin{bmatrix} 1 \\ -h_{22} \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} h_{11} & h_{12} \\ 1 & 0 \end{bmatrix}}_{X} = T \underbrace{\begin{bmatrix} 0 & 1 \\ -h_{21} & -h_{22} \end{bmatrix}}_{Y} \Rightarrow \boxed{T = X Y^{-1}}$$

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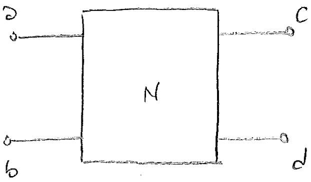
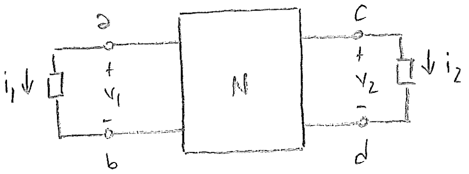
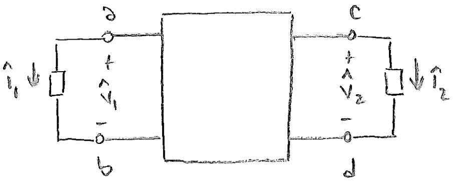
Reciprocity



Let the two-port N contain only LTI resistors and ideal transformers. Such N is called **reciprocal** and satisfies :



case C



case Ĉ

: arbitrary one-ports (including open circuit & short circ.)

$$\boxed{\hat{i}_1 v_1 + \hat{i}_2 v_2 = i_1 \hat{v}_1 + i_2 \hat{v}_2} \quad (1)$$

Also, for reciprocal N we have

$$\boxed{\begin{array}{ll} r_{12} = r_{21} & h_{12} = -h_{21} \\ g_{12} = g_{21} & h'_{12} = -h'_{21} \end{array}}$$

Proof of eq. (1) [For simplicity assume N does not contain any IT's.]

Since both cases have the same graph, by Tellegen's Thm we can write

$$0 = \sum_{k=1}^{b} \hat{i}_k v_k = \sum_{k=1}^{b} i_k \hat{v}_k \quad (2)$$

i_k ($\hat{v}_k$) : kth branch current (voltage) of C (Ĉ)

Note that for k ≥ 3 we have $v_k = R_k i_k$ ($\hat{v}_k = R_k \hat{i}_k$) since N is reciprocal. Rewrite (2) as $\hat{i}_1 v_1 + \hat{i}_2 v_2 + \sum_{k=3}^{b} \hat{i}_k v_k = i_1 \hat{v}_1 + i_2 \hat{v}_2 + \sum_{k=3}^{b} i_k \hat{v}_k$. Then

$$\begin{aligned} \hat{i}_1 v_1 + \hat{i}_2 v_2 &= i_1 \hat{v}_1 + i_2 \hat{v}_2 + \sum_{k=3}^{b} i_k \hat{v}_k - \sum_{k=3}^{b} \hat{i}_k v_k \\ &= i_1 \hat{v}_1 + i_2 \hat{v}_2 + \sum_{k=3}^{b} \{ i_k \hat{v}_k - \hat{i}_k v_k \} \\ &= i_1 \hat{v}_1 + i_2 \hat{v}_2 + \sum_{k=3}^{b} \underbrace{\{ i_k R_k \hat{i}_k - \hat{i}_k R_k i_k \}}_{0} \\ &= i_1 \hat{v}_1 + i_2 \hat{v}_2 . \end{aligned}$$

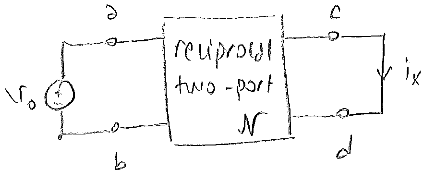
▨

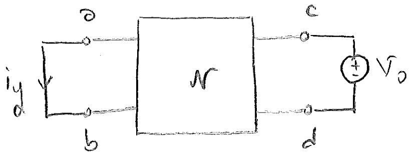
Exercise Prove the general case, where we also have IT's.

#8 30 Ekim 2018

Implications of Reciprocity

①





$\boxed{i_x = i_y}$

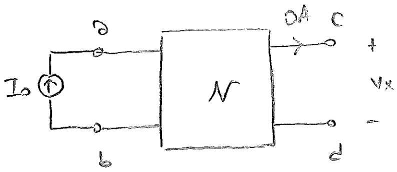
why?

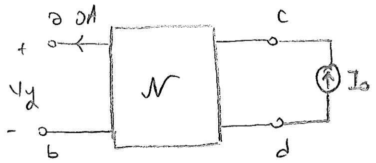
$i_1 = ?$ $\quad i_2 = i_x$ $\quad \hat{i}_1 = i_y$ $\quad \hat{i}_2 = ?$

$v_1 = V_o$ $\quad v_2 = 0$ $\quad \hat{v}_1 = 0$ $\quad \hat{v}_2 = V_o$

$$i_1\hat{v}_1 + i_2\hat{v}_2 = \hat{i}_1 v_1 + \hat{i}_2 v_2 \Rightarrow i_1 \cdot 0 + i_x \cdot V_o = i_y V_o + \hat{i}_2 \cdot 0 \Rightarrow i_x = i_y$$

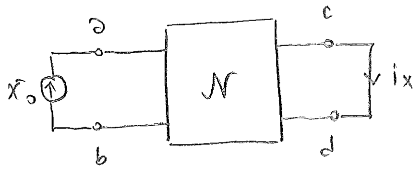
②

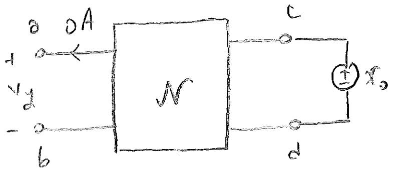




$\boxed{v_x = v_y}$ (why?)

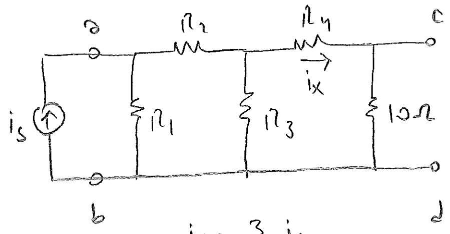
③



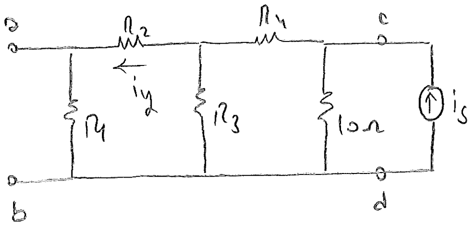


$\boxed{i_x = v_y}$ (why?)

Example Following measurements are made. Determine R_1



$i_x = \frac{3}{10} i_s$



$i_y = \frac{2}{10} i_s$

$i_1 = -i_s$ $\quad i_2 = 0$ $\quad \hat{i}_1 = 0$ $\quad \hat{i}_2 = -i_s$

$v_1 = ?$ $\quad v_2 = 10 i_x$ $\quad \hat{v}_1 = R_1 i_y$ $\quad \hat{v}_2 = ?$

By reciprocity we can write

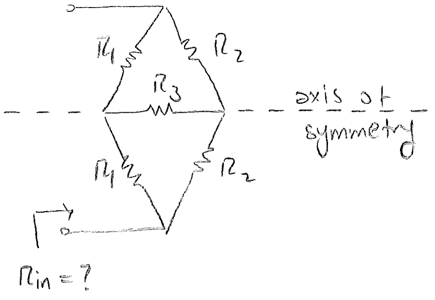
$$i_1\hat{v}_1 + i_2\hat{v}_2 = \hat{i}_1 v_1 + \hat{i}_2 v_2 \Rightarrow -i_s R_1 i_y + 0 \cdot \hat{v}_2 = 0 \cdot v_1 - i_s \cdot 10 i_x$$

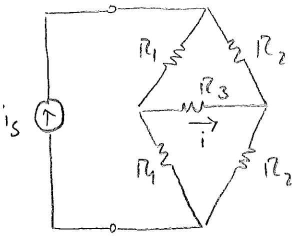
$$\Rightarrow R_1 i_y = 10 i_x \Rightarrow R_1\left(\frac{2}{10} i_s\right) = 10\left(\frac{3}{10} i_s\right) \Rightarrow \boxed{R_1 = 15\Omega}$$

#9 30 Ekim 2018

Symmetric Circuits

Ex

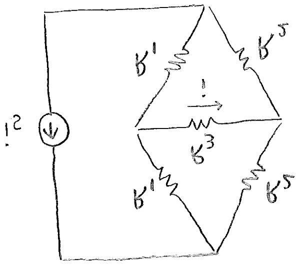




Question : $i = ?$

Answer : $i = 0$. (why?)

Because By linearity $i = \alpha \cdot i_s$ (for some α). Rotate the circuit about the axis of sym.



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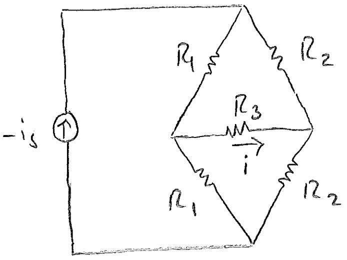
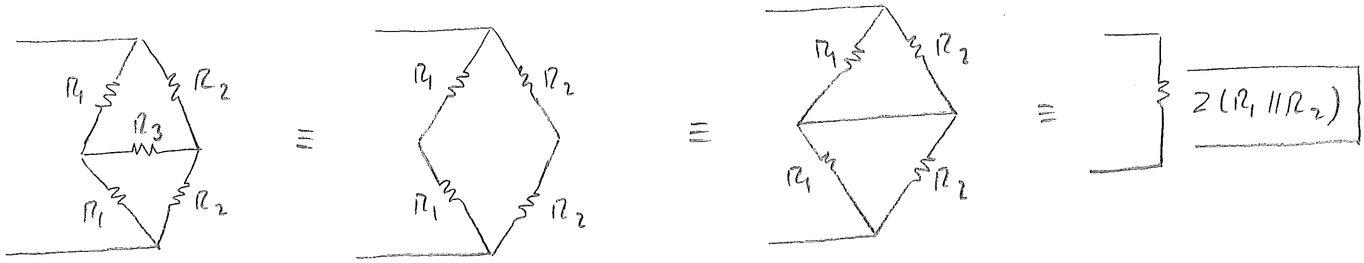


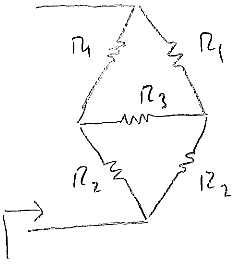
$\Rightarrow i = \alpha \cdot (-i_s)$

$\alpha i_s = -\alpha i_s \Rightarrow \alpha = 0 \Rightarrow i = 0.$

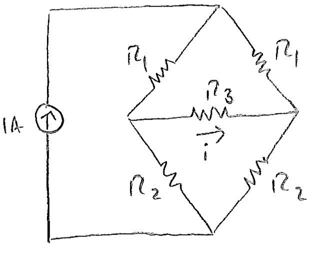
Remark Since $i = 0$, we can replace R_3 with open circuit. Also, the voltage across R_3 equals $v = R_3 \cdot i = 0$. Hence we can replace R_3 with short circuit, too. [Substitution thm.] Therefore



Ex

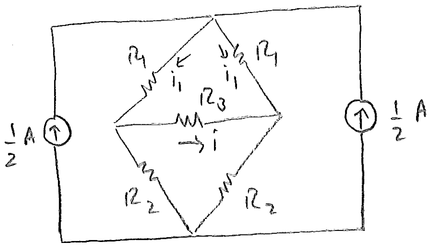


$R_{in} = ?$



$i = ?$

≡

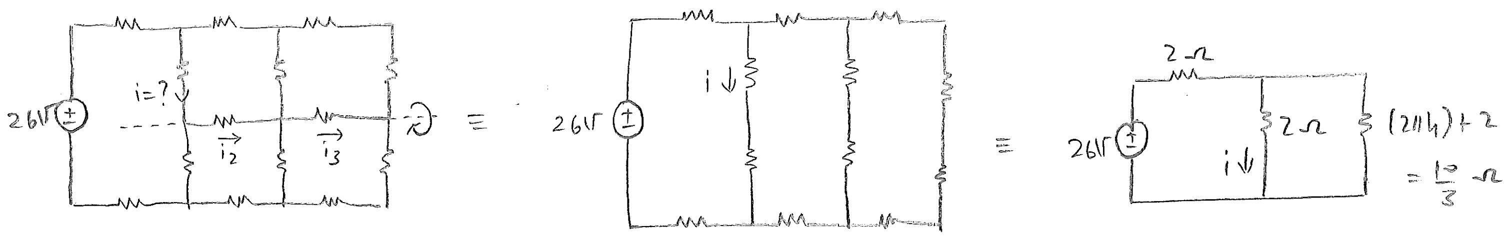


KVL: $\cancel{R_1 i_1} + R_3 i - \cancel{R_1 i_1} = 0 \Rightarrow i = 0$

$$\Rightarrow \boxed{R_{in} = \frac{R_1 + R_2}{2}}$$

#10 30 Ekim 2018

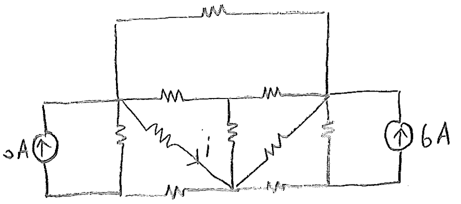
Example (All resistors are 1Ω)



symmetry $\Rightarrow$ $i_2 = i_3 = 0$

$\Rightarrow \boxed{i = 5A}$

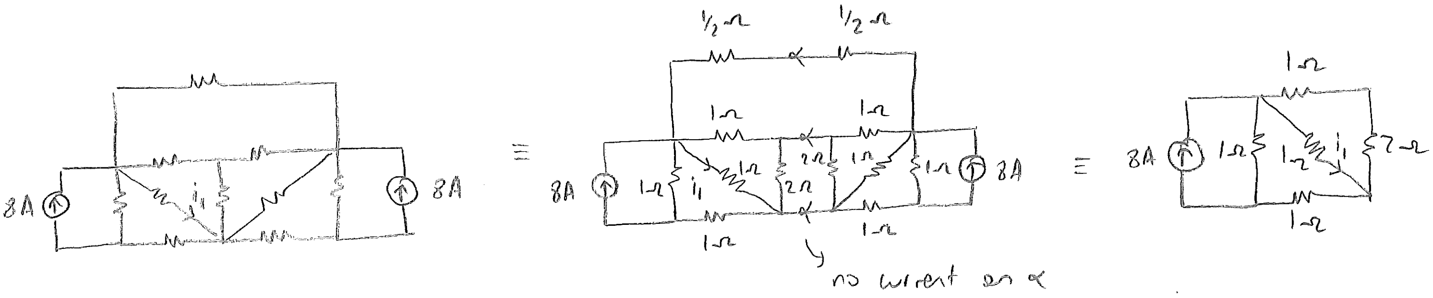
Exercise (All resistors are 1Ω) Find i.



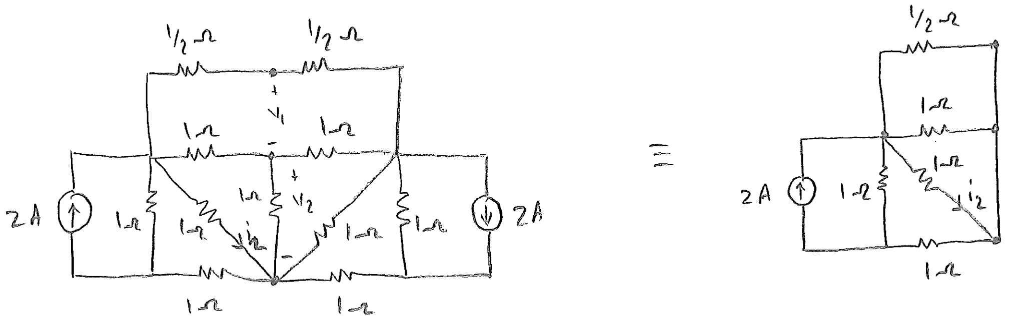
Hint The input vector $\begin{bmatrix}10\\6\end{bmatrix}$ can be written as

$$\begin{bmatrix}10\\6\end{bmatrix} = \begin{bmatrix}8\\8\end{bmatrix} + \begin{bmatrix}2\\-2\end{bmatrix}.$$

Step 1 Find i_1 in the following circuit



Step 2 Find i_2 in the following circuit



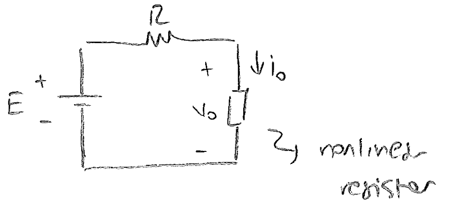
$(v_1 = v_2 = 0)$

Step 3 $i = i_1 + i_2$ (linearity)

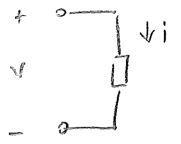
#1 11 Kasım 2018

Ch. III

Circuits with a single nonlinear resistor



i-v char. of the nonlinear element :



$i = g(v)$ (or $v = f(i)$)

KVL ⇒ $E = Ri_0 + v_0$

$= Rg(v_0) + v_0$ (or $E = Ri_0 + f(i_0)$)

Problem Find (i_0, v_0)

Solution 1 (Algebraic Solution) Solve $E = Rg(v_0) + v_0$ for v_0

Then $(i_0, v_0) = (g(v_0), v_0)$ is the solution

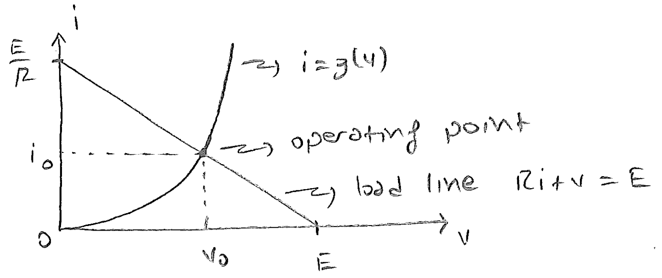
Remark Sometimes the i-v char. of the nonlinear element is available only as a measured curve. Then :

Solution 2 (Graphical Solution) We have two constraints on the set of possible (i_0, v_0) pairs :

→ Constraint 1 : $Ri + v - E = 0$ (due to circuit)

→ Constraint 2 : $i - g(v) = 0$ (due to nonlinear resistor)

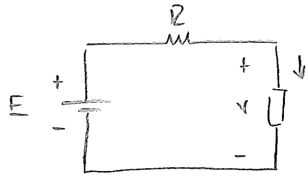
Any solution (i_0, v_0) must simultaneously satisfy both constraints



Remark Sometimes solution does not exist. Sometimes multiple operating points can exist.

#2 11 Kasım 2018

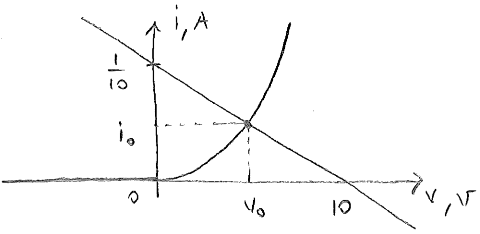
Example



$$i = \begin{cases} \frac{3}{100}v^2 & \text{for } v \geq 0 \\ 0 & \text{for } v < 0 \end{cases}, \quad E = 10V, \quad R = 100\Omega$$

Find the operating point.

Sol'n



By KVL we have

$$0 = Ri + v - E$$
$$= 100i + v - 10 \quad (1)$$

Case $v < 0$ ($i = 0$)

(1) ⇒ $v = 10\,V$ but $v < 0$. Hence no solution exists with $v < 0$.

Case $v \geq 0$ $\left(i = \frac{3}{100}v^2\right)$

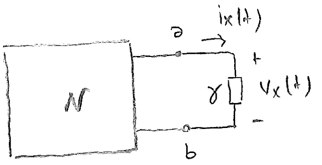
(1) ⇒ $0 = 100\left(\frac{3}{100}v^2\right) + v - 10 = 3v^2 + v - 10 = (3v-5)(v+2)$

⇒ $v = \frac{5}{3}$ or $\cancel{-2}$, because $v \geq 0$

Hence $V_0 = \frac{5}{3}V$ ⇒ $i_0 = \frac{3}{100}V_0^2 = \frac{1}{12}A$

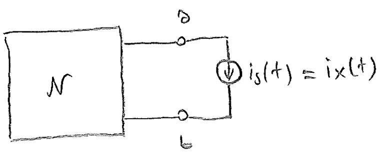
⇒ operating point $\boxed{(i_0, V_0) = \left(\frac{1}{12}, \frac{5}{3}\right)}$

Remark Let us have a circuit as

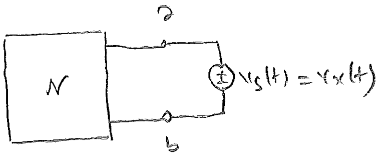


[γ: arbitrary component]

Then (by Substitution Thm) we can make the following substitutions



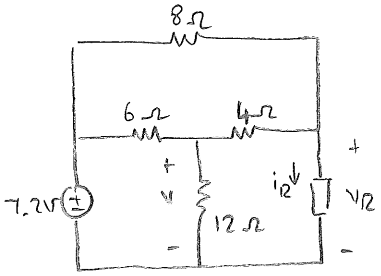
OR



without affecting any branch current or voltage within N.

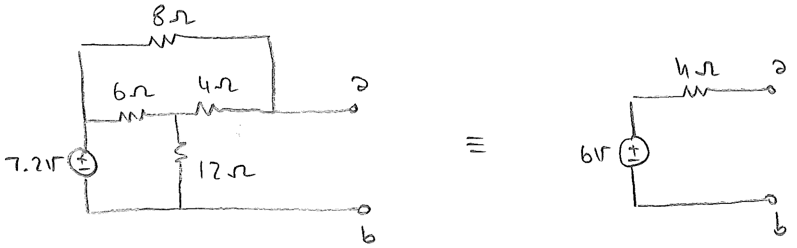
3   11 Kasım 2018

Example   Find v

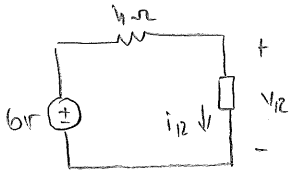


$$i_R = \begin{cases} \frac{1}{4}v_R^2 & \text{for } v_R \geq 0 \\ 0 & \text{for } v_R < 0 \end{cases}$$

Step1   obtain the thevenin equiv. seen by the nonlinear resistor.



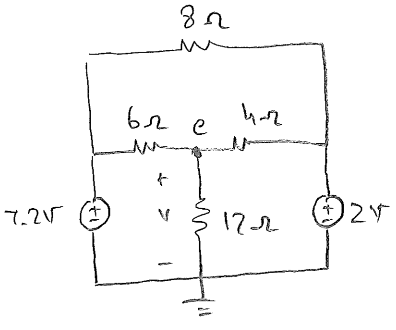
Step2   obtain the operating point using the Thevenin equiv. circuit



$$0 = 4i_R + v_R - 6 = 4\frac{v_R^2}{4} + v_R - 6 = v_R^2 + v_R - 6 = (v_R+3)(v_R-2)$$

$\Rightarrow \boxed{v_R = 2V}$ (because $v_R \geq 0$) & $\boxed{i_R = 1A}$

Step 3   Substitute the nonlinear resistor with a voltage source $v_s = 2V$ or a current source $i_s = 1A$ and solve for v.



$$\frac{e-7.2}{6} + \frac{e}{12} + \frac{e-2}{4} = 0$$

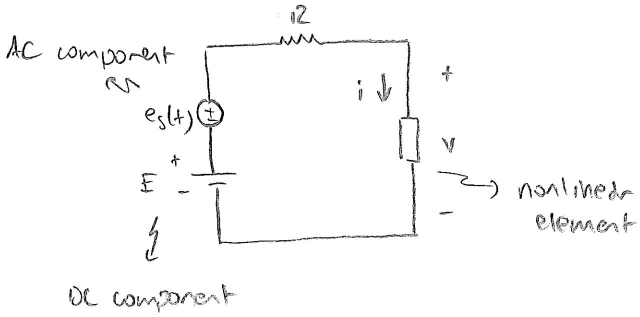
$\Rightarrow e = 3.4V$

$\Rightarrow \boxed{v = 3.4V}$

#4 11 Kasım 2018

Small Signal Analysis

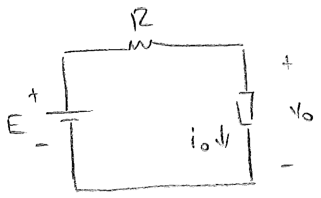
A circuit with a nonlinear resistive element and a time-varying input can be analyzed using small signal analysis method if the magnitude of the AC component of the input is sufficiently smaller than that of the DC component.



known : $i = g(v)$ & $|e_s(t)| << E$

asked : $v(t), i(t) = ?$ (approximately)

Step1 Ignore $e_s(t)$ and obtain the "DC operating point"



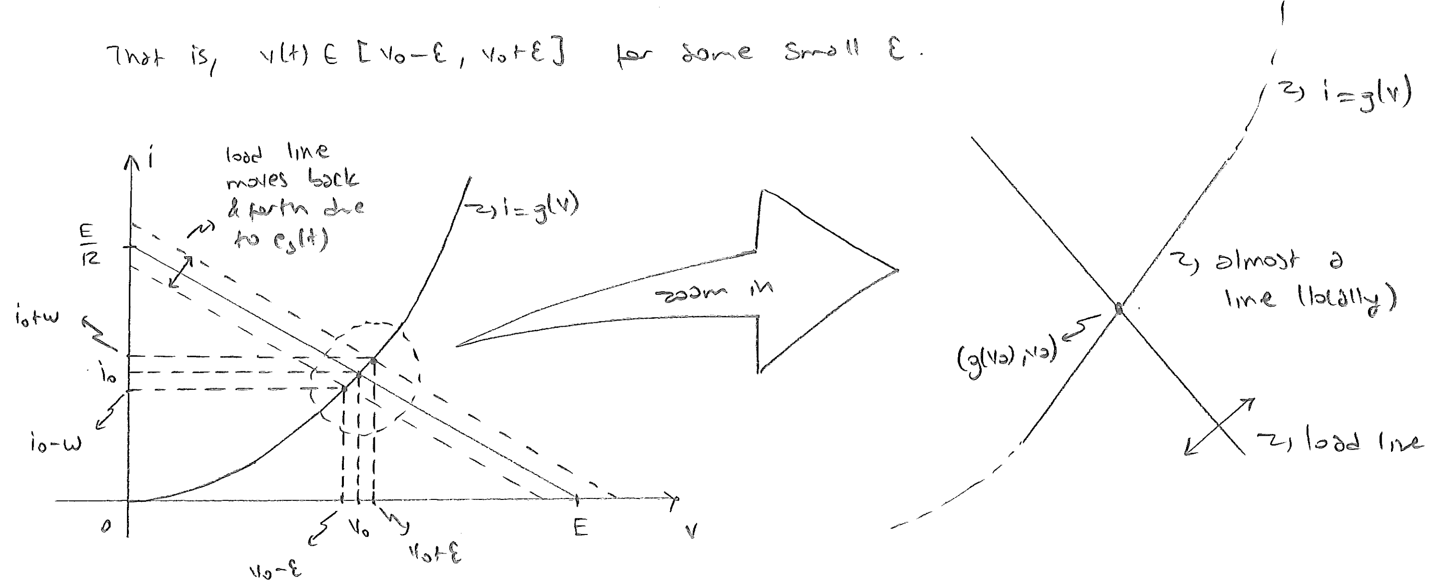
DC circuit

Solve $E = Rg(V_0) + V_0$ for V_0

Then the DC oper. point is $(i_0, V_0) = (g(V_0), V_0)$

Step2 Since $|e_s(t)|$ is small $v(t) \cong V_0(t)$ for all t.

That is, $v(t) \in [V_0 - \varepsilon, V_0 + \varepsilon]$ for some small ε.



#5   11 Kasım 2018

First order approximation of $g(v)$ around $v=v_0$ is :

$$g(v) \cong g(v_0) + \left.\frac{dg}{dv}\right|_{v=v_0} \times (v-v_0) \qquad \text{let us define } g_m := \left.\frac{dg}{dv}\right|_{v=v_0}$$

(g_m = slope of the line tangent to $g(v)$ at $v=v_0$)

$\Rightarrow g(v) \cong g(v_0) + g_m \cdot (v-v_0)$ for $v \in [v_0-\varepsilon, v_0+\varepsilon]$, ε small

Now, we can write

$E + e_s(t) = R g(v) + v$ ⤹ replace $g(v)$ with its approximation

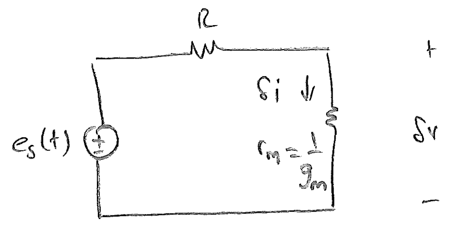
$\Rightarrow E + e_s(t) \cong R\left[g(v_0) + g_m(v-v_0)\right] + v$ ⤹ $E = Rg(v_0) + v_0$

$\Rightarrow \cancel{Rg(v_0)} + v_0 + e_s(t) \cong \cancel{Rg(v_0)} + Rg_m(v-v_0) + v$

$\Rightarrow e_s(t) \cong Rg_m(v-v_0) + (v-v_0)$ let us define $\delta v := v - v_0$ (i.e. the deviation of $v(t)$ around v_0)

$$\Rightarrow \boxed{e_s(t) \cong Rg_m \delta v + \delta v} \qquad (1)$$

Eq. (1) suggests the following <u>linear</u> circuit (R resistance, g_m conductance)



($\delta i = g_m \delta v$)

because :

$$\delta v = \frac{1/g_m}{R + 1/g_m} \cdot e_s(t) \qquad \text{(voltage division)}$$

$\Rightarrow e_s(t) = Rg_m \delta v + \delta v$ (same as (1))

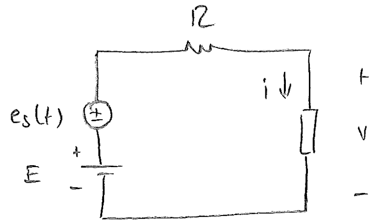
AC circuit (small signal circuit)

<u>Step 3</u> $v(t) = v_0 + \delta v(t)$ is the approximate solution for $v(t)$

$i(t) = i_0 + \delta i(t)$ " " " " " $i(t)$.

#6 11 Kasım 2018

Example



$R = 100\Omega$

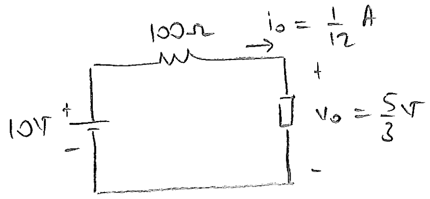
$E = 10V$

$e_s(t) = \sin t \ V$

$$i = \begin{cases} 0.03v^2 & \text{for } v \geq 0 \\ 0 & \text{for } v < 0 \end{cases}$$

Find $i(t), v(t)$.

Step1 Solve the DC circuit.



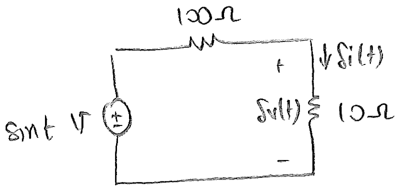
How about AC circuit?

First find g_m.

$g(v) = 0.03v^2$ & $v_o = \frac{5}{3} V$

$$\Rightarrow g_m = \left.\frac{dg}{dv}\right|_{v=v_o} = 0.06\left(\frac{5}{3}\right) = 0.1 \mho$$

Step2 Solve the AC circuit.



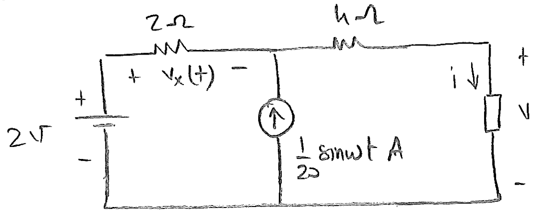
$\delta v(t) = \frac{1}{11} \sin t \ V$

$\delta i(t) = \frac{1}{110} \sin t \ A$

Step3 Write the approximate solution

$$\boxed{\begin{aligned} v(t) &\cong v_o + \delta v(t) = \frac{5}{3} + \frac{1}{11}\sin t \ V \\ i(t) &\cong i_o + \delta i(t) = \frac{1}{12} + \frac{1}{110}\sin t \ A \end{aligned}}$$

Example

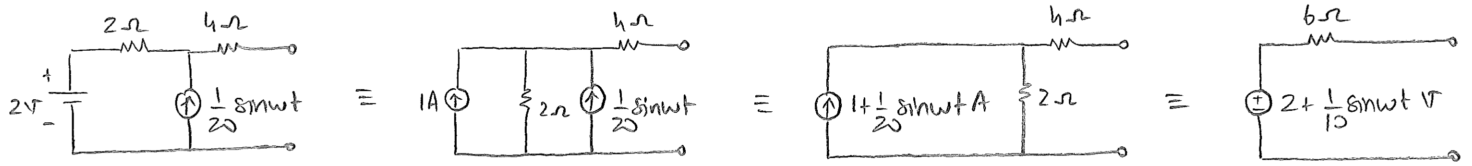


$$i = \begin{cases} v^2 & \text{for } v \geq 0 \\ 0 & \text{for } v < 0 \end{cases}$$

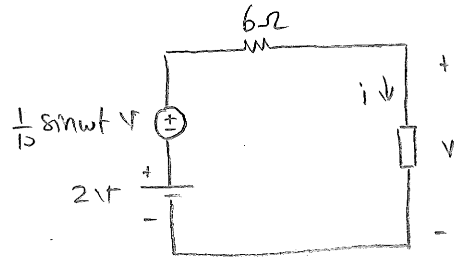
Find $v_x(t)$.

#7 11 Kasım 2018

Step 0 Find the thevenin equiv. seen by the nonlinear resistor.



Step 1 Find v(t) & i(t) by small signal analysis



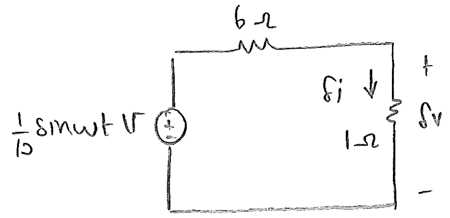
DC oper. point? $2 = 6i + v = 6v^2 + v$

$\Rightarrow v^2 + \frac{1}{6}v - \frac{1}{3} = 0 \Rightarrow (v - \frac{1}{2})(v + \frac{2}{3}) = 0$

$\Rightarrow V_0 = \frac{1}{2} V \Rightarrow i_0 = \frac{1}{4} A$

Compute g_m : $g_m = \frac{d}{dv} v^2 \Big|_{v=V_0} = 2v \Big|_{v=\frac{1}{2}} = 1 \mho$

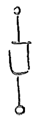
Small signal circuit :

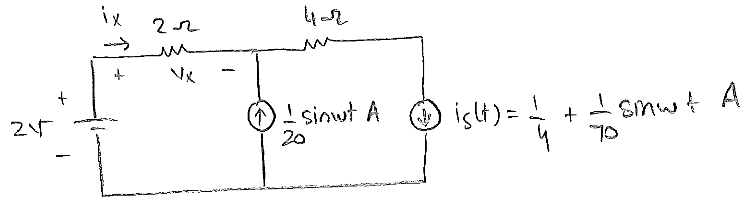


$\delta v(t) = \frac{1}{70}\sin\omega t\ V$ & $\delta i(t) = \frac{1}{70}\sin\omega t\ A$

$\Rightarrow v(t) = \frac{1}{2} + \frac{1}{70}\sin\omega t\ V$ & $i(t) = \frac{1}{4} + \frac{1}{70}\sin\omega t\ A$

Step 2 Go back to the original circuit & apply substitution thm. to





$\Rightarrow i_x + \frac{1}{20}\sin\omega t = \frac{1}{4} + \frac{1}{70}\sin\omega t \Rightarrow i_x(t) = \frac{1}{4} - \frac{1}{28}\sin\omega t\ A$ & $v_x(t) = 2 i_x(t)$

$$\boxed{\Rightarrow v_x(t) \cong \frac{1}{2} - \frac{1}{14}\sin\omega t\ V}$$

Remark Exact sol'n is

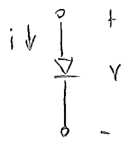
$$v_x(t) = \frac{25}{36} - \frac{1}{15}\sin\omega t - \frac{1}{3}\sqrt{\frac{49}{144} + \frac{1}{60}\sin\omega t}\ V \Rightarrow \text{error} \lesssim 60\mu V$$

$(\lesssim 0.01\%)$

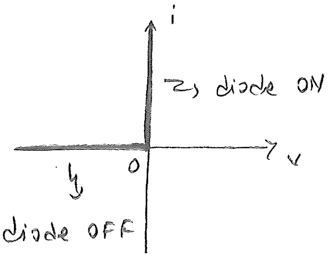
#8   11 Kasım 2018

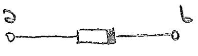
Piecewise linear Resistive Circuits

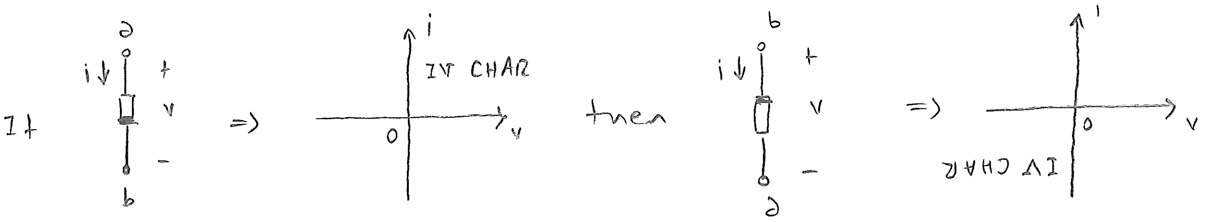
Ideal Diode



$$\text{i-v Char}: \begin{cases} v=0 \text{ (short circuit) when } i \geq 0 \\ i=0 \text{ (open circuit) when } v \leq 0 \end{cases}$$

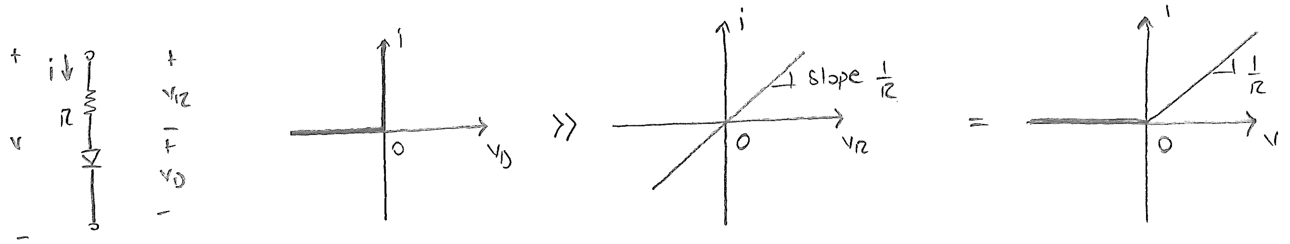


Remark Let  be an arbitrary resistive component.



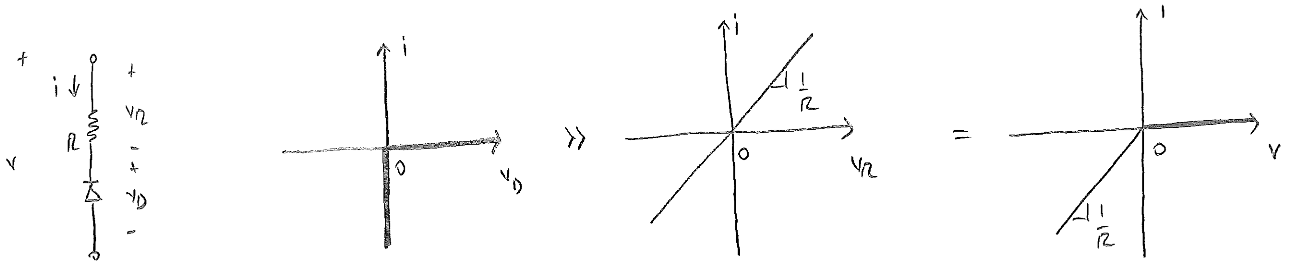
Reflections of each other w.r.t. the origin

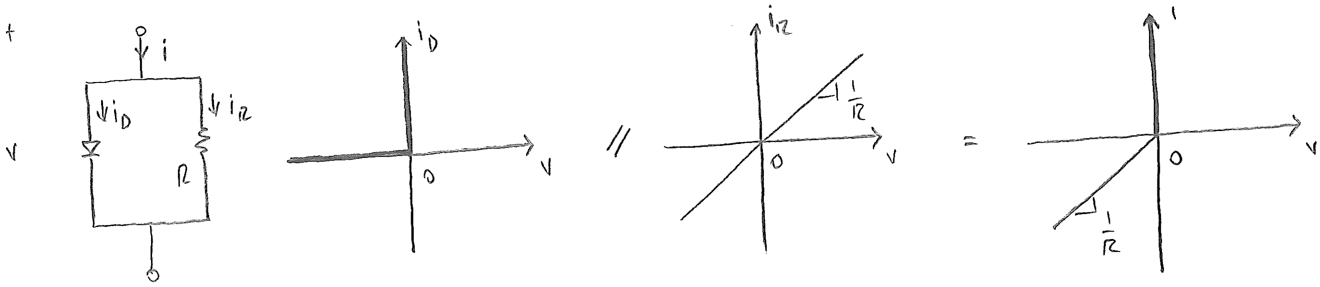
Series & parallel connections of diodes, LTI resistors, & constants sources



$v = v_R + v_D$

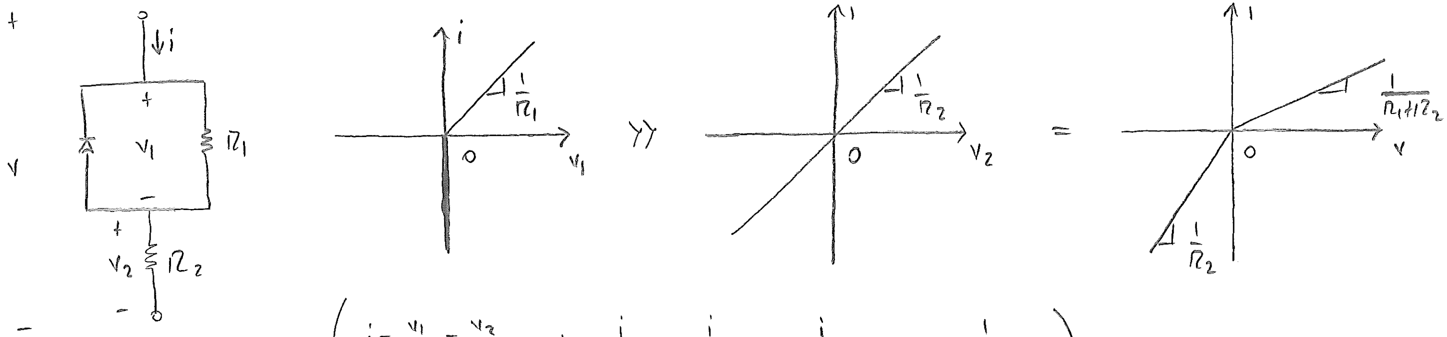
add voltages



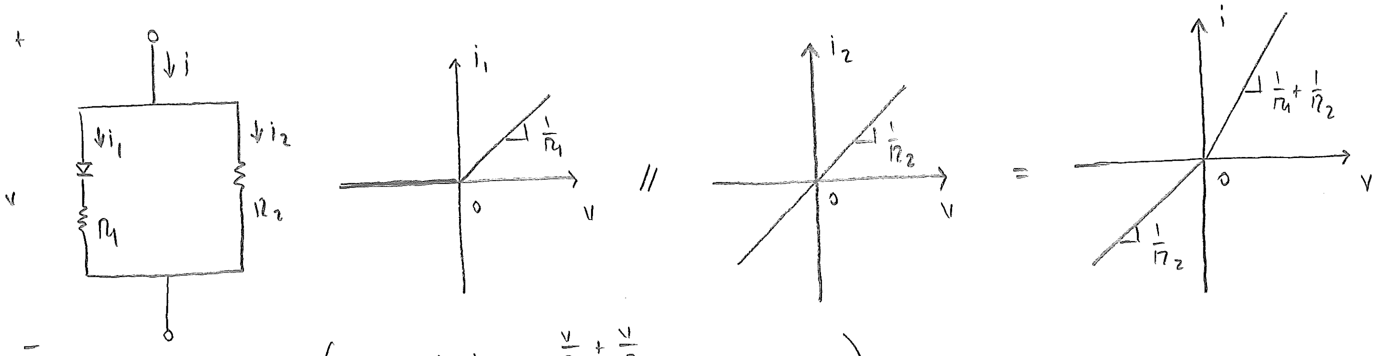


$i = i_D + i_R$

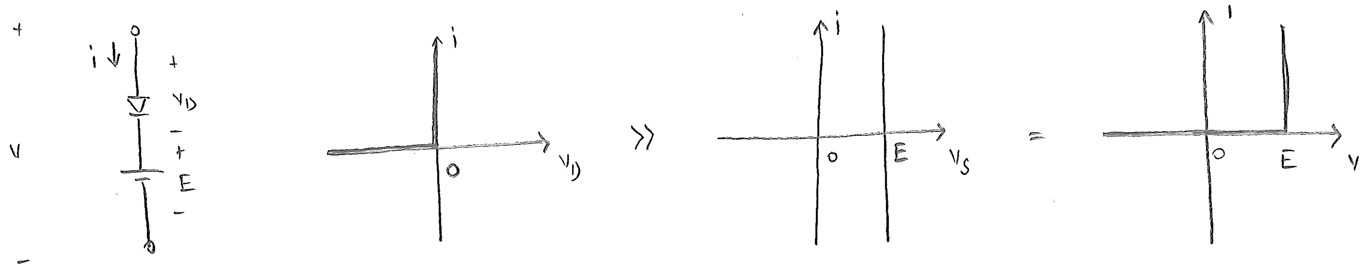
add currents

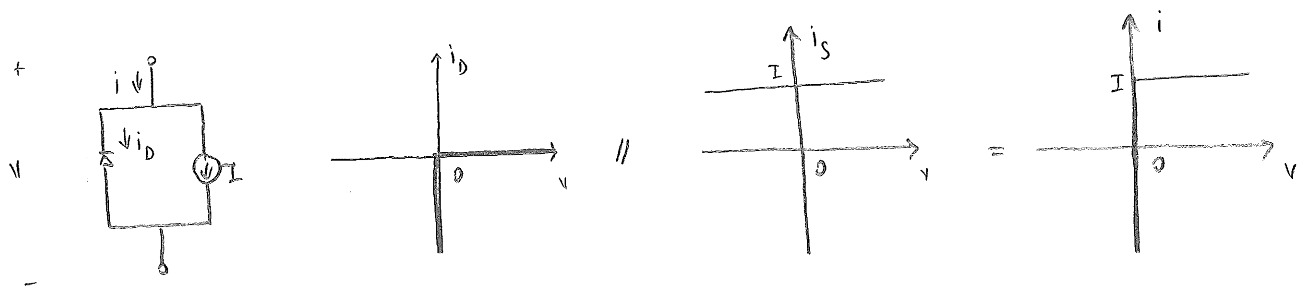


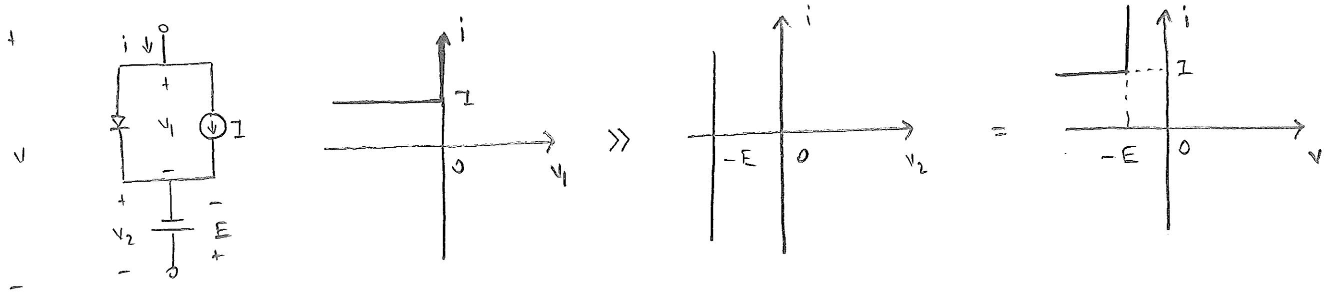
$$\left(i = \frac{v_1}{R_1} = \frac{v_2}{R_2} \Rightarrow \frac{i}{v} = \frac{i}{v_1+v_2} = \frac{i}{R_1 i + R_2 i} = \frac{1}{R_1+R_2} \right)$$



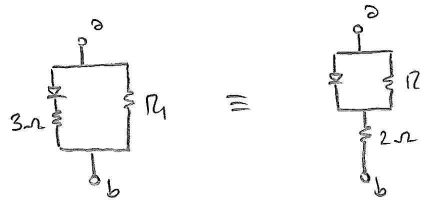
$$\left(\frac{i}{v} = \frac{i_1+i_2}{v} = \frac{\frac{v}{R_1}+\frac{v}{R_2}}{v} = \frac{1}{R_1}+\frac{1}{R_2} \right)$$







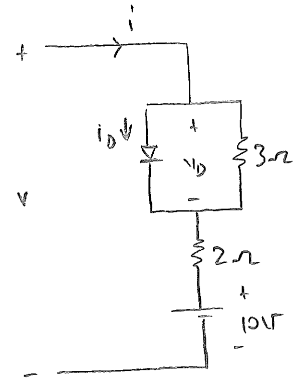
Exercise



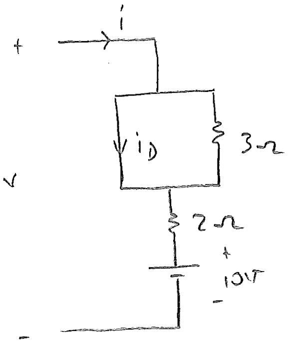
Find R_1, R_2.

\# 10   11 Kasım 2018

Example [Non-graphical method]   Plot i-v char. of



Assume diode ON   (condition : $i_D \geq 0$)



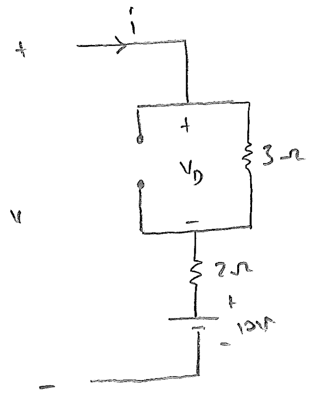
$\boxed{v = 2i + 10}$ when diode ON

check condition :

$i_D = i \Rightarrow v = 2i_D + 10 \Rightarrow i_D = \frac{v-10}{2}$

$i_D \geq 0 \Rightarrow \boxed{v \geq 10}$ for diode to be ON

Assume diode OFF   (condition : $v_D < 0$)



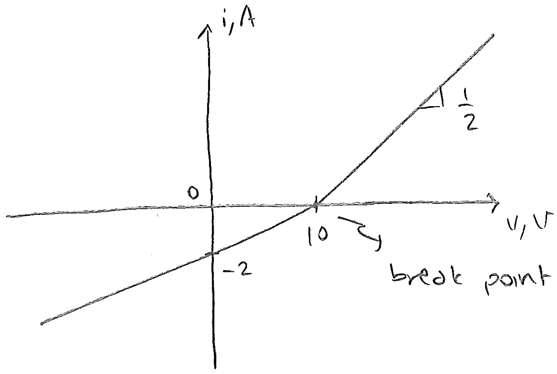
$\boxed{v = 5i + 10}$ when diode OFF

check condition :

$v_D = 3i = 3\left(\frac{v-10}{5}\right)$

$v_D < 0 \Rightarrow \boxed{v < 10}$ for diode to be OFF

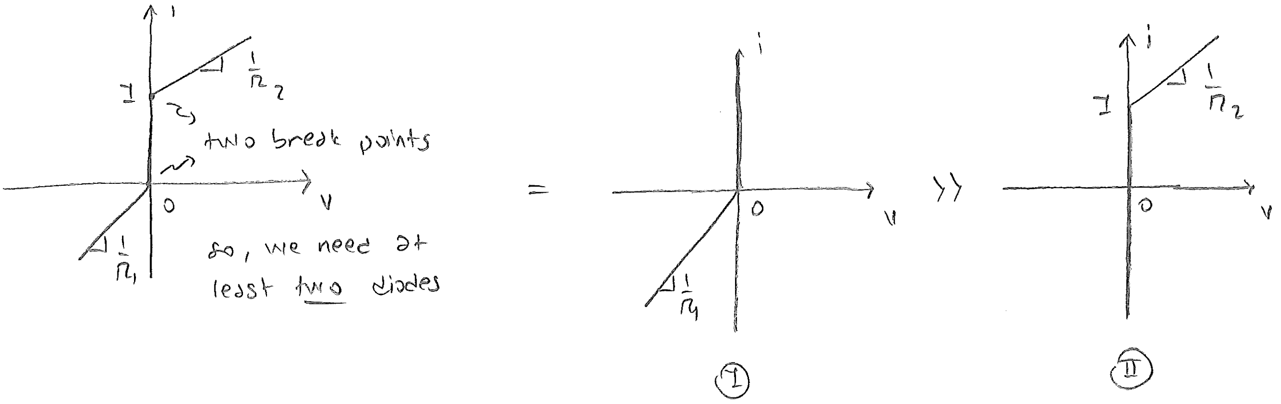
Hence, $i = \begin{cases} \frac{v-10}{2} & \text{for } v \geq 10 \\ \frac{v-10}{5} & \text{for } v < 10 \end{cases}$



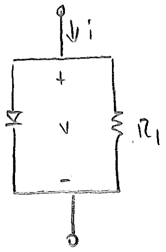
break point when both $i_D = 0$ & $v_D = 0$.

#11 11 Kasım 2018

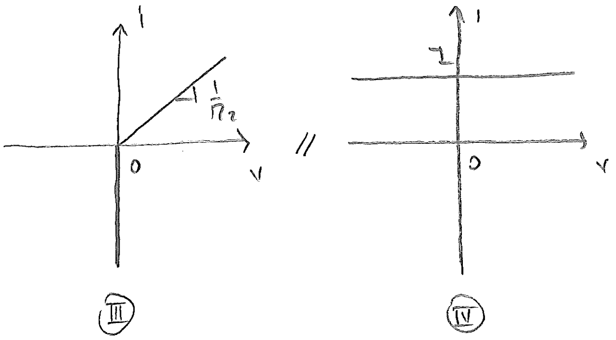
Example [synthesis, i.e., obtaining the circuit from i-v char.]



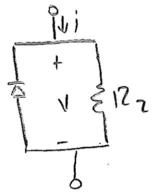
(I) ⇒



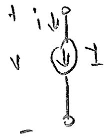
(II) =



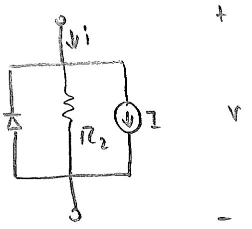
(III) ⇒



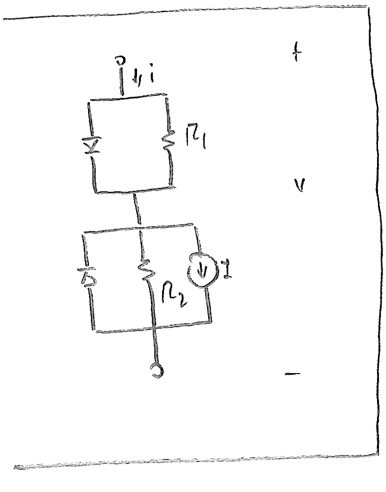
(IV) ⇒



Therefore (II) ⇒

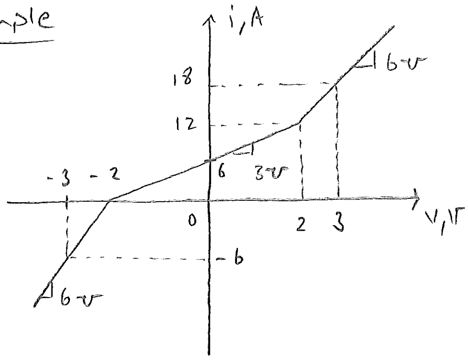


Finally, (I) >> (II) ⇒

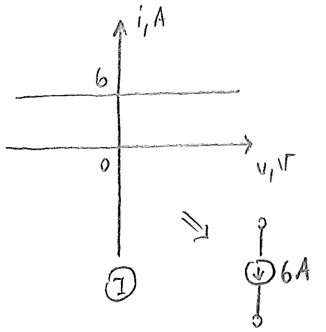


Remark Two different circuits may have the same i-v char. (nonuniqueness)

Example



=



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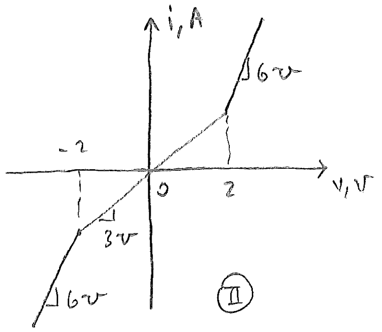
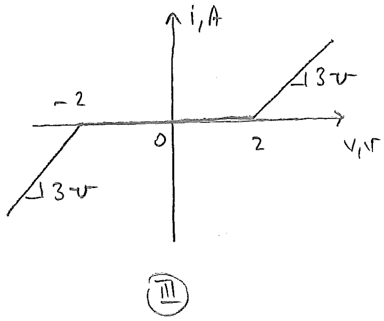


II =



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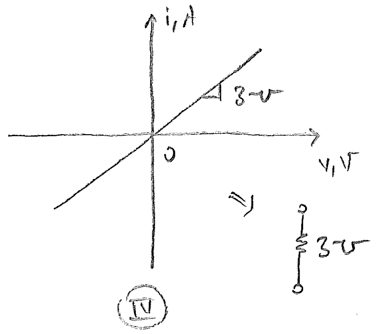
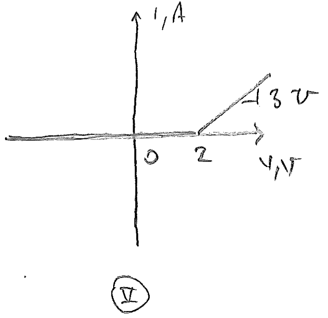


III =



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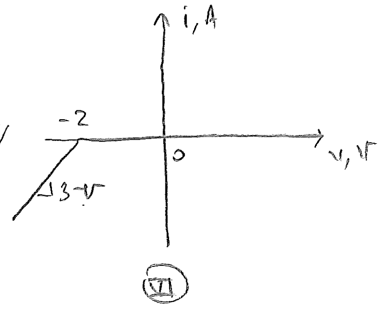
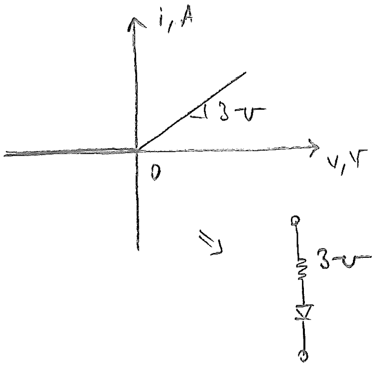
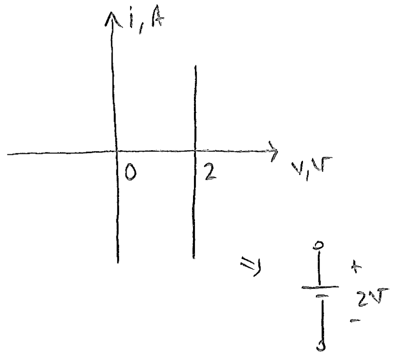


V =

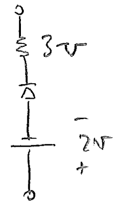


⇒

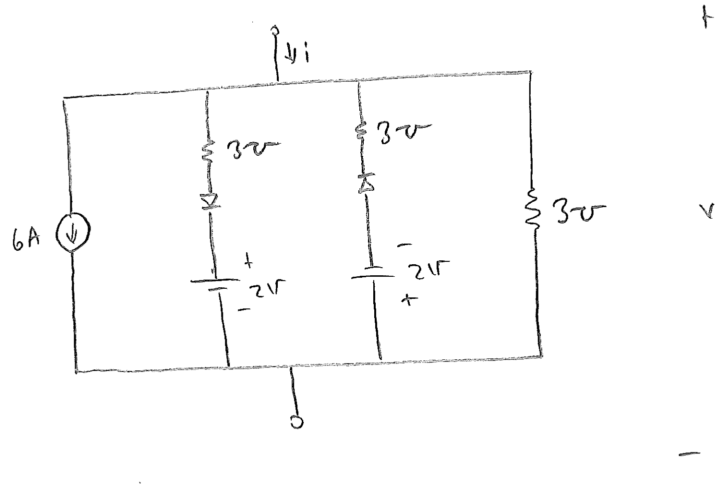


VI is the reflection of V w.r.t. the origin. Hence

VI ⇒

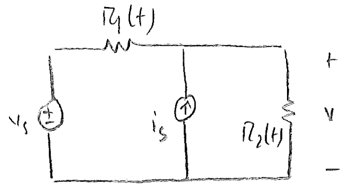


Now, we glue the pieces together :



#13   11 Kasım 2018

Example [Linear Time-Varying Resistive Circuit]



KCL : $\frac{v - v_s}{R_1} - i_s + \frac{v}{R_2} = 0$

$$\Rightarrow \quad v = \underbrace{\frac{R_2(t)}{R_1(t)+R_2(t)}}_{k_1(t)} v_s + \underbrace{\frac{R_1(t)R_2(t)}{R_1(t)+R_2(t)}}_{k_2(t)} i_s$$

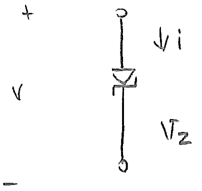
$$\Rightarrow \quad v = \underbrace{[\,k_1(t) \quad k_2(t)\,]}_{k(t)} \underbrace{\begin{bmatrix} v_s \\ i_s \end{bmatrix}}_{u} \qquad (1)$$

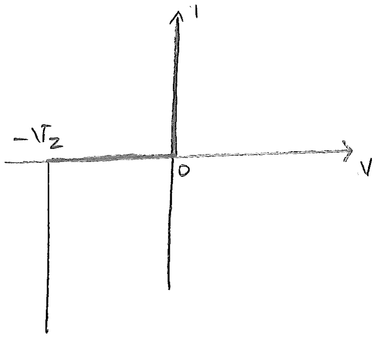
Remark Note that (1) implies

$$v(\alpha u_a + \beta u_b) = \alpha v(u_a) + \beta v(u_b)$$ α, β: scalars ; u_a, u_b: input vectors

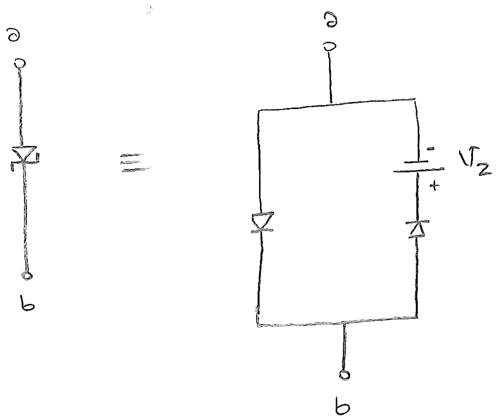
Hence, superposition is still applicable for LTV resistive circuits.

ideal Zener diode

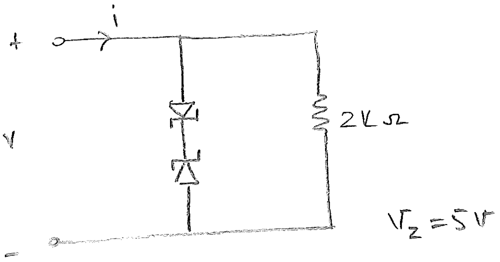




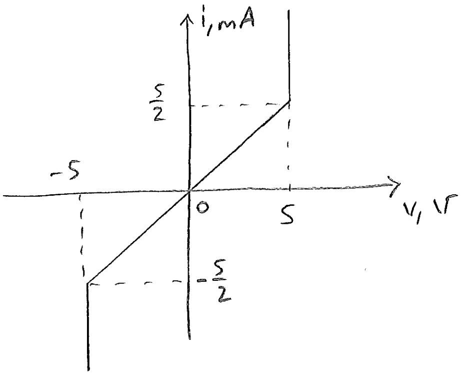
Note that we can represent the Zener diode with two diodes and a battery



Exercise [2PS III_9(c)] Sketch the port (i-v) characteristics for :



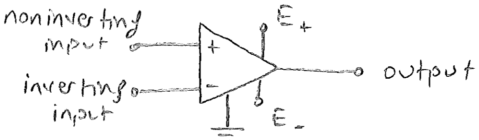
Answer :



#1 18 Kasım 2018

Ch. IV Operational Amplifiers

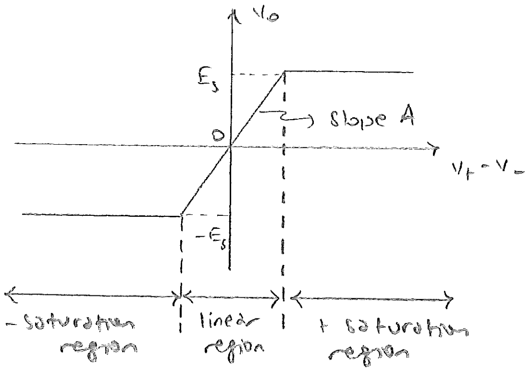
Six (or more) terminal device



E_+ & E_- terminals are connected to DC voltage supplies. The values $E_+ > E_-$ determine the upper & lower limits of the output voltage v_o.

That is, $E_- \leq v_o \leq E_+$

Transfer Characteristics (let $E_+ = E_s$ & $E_- = -E_s$)



A : open-loop voltage gain

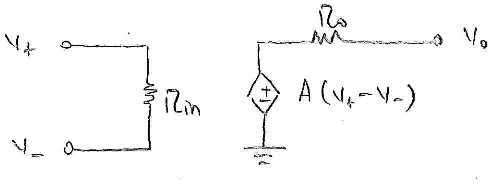
Typically $A \cong 200{,}000$

v_o : voltage of the output terminal
v_+ : voltage of the ⊕ terminal
v_- : voltage of the ⊖ terminal
} wrt some common ground

OP-AMP has three operating modes:

1) linear mode, $v_o = A(v_+ - v_-)$ and $-E_s < v_o < E_s$
2) + sat mode, $v_o = E_s$ and $A(v_+ - v_-) > E_s$
3) - sat mode, $v_o = -E_s$ and $A(v_+ - v_-) < -E_s$

Dependent source model of an OPAMP operating in linear region:



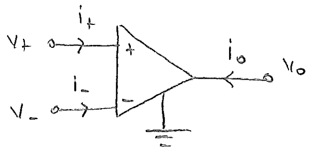
R_{in} : input resistance, $\sim 10^6 - 10^{12}\ \Omega$
R_o : output resistance, $\sim 10 - 100\ \Omega$
A : voltage gain, $\sim 10^5 - 10^8$ V/V

<u>Remark</u> For many applications it is reasonable to work with the ideal OP-AMP model. There are two idealizations:

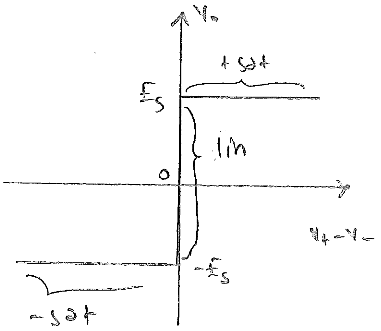
1) Finite-gain ideal OP-AMP model : $R_{in} = \infty$, $R_o = 0$, $A < \infty$
2) Infinite-gain ideal OP-AMP model : $R_{in} = \infty$, $R_o = 0$, $A = \infty$

#2 18 Kasım 2018

Infinite-gain ideal OP-AMP



linear region	+sat region	−sat region
$v_+ = v_-$	$v_+ > v_-$	$v_+ < v_-$
$-E_s \le v_o \le E_s$	$v_o = E_s$	$v_o = -E_s$

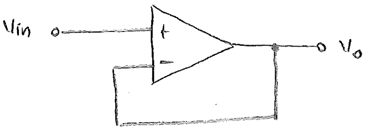


$i_+ = i_- = 0$ (all regions)

Remark : $i_o \neq 0$!

Some useful OPAMP circuits:

Voltage follower (buffer)



⤳ negative feedback

$v_+ = v_{in}$, $v_- = v_o$

linear $v_+ = v_-$ & $-E_s \le v_o \le E_s$

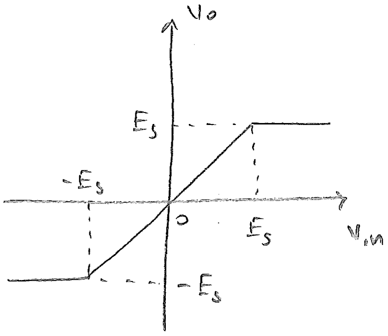
⇒ $v_o = v_{in}$ & $-E_s \le v_{in} \le E_s$

+sat $v_+ > v_-$ & $v_o = E_s$

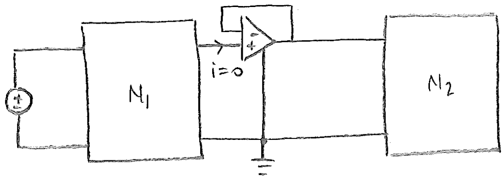
⇒ $v_{in} > E_s$

−sat $v_+ < v_-$ & $v_o = -E_s$

⇒ $v_{in} < -E_s$



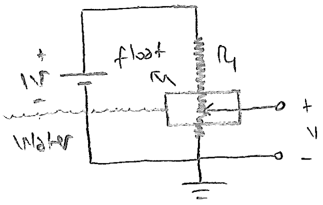
Remark Buffers are widely used to isolate 2 two-ports



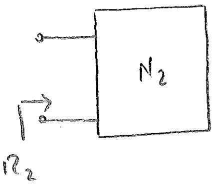
Here, the buffer prevents N_2 from "loading down" N_1.

Ex :

N_1 :



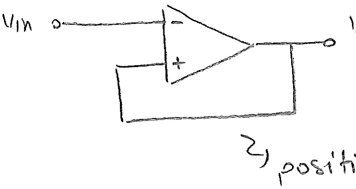
⤳ voltage proportional to water level



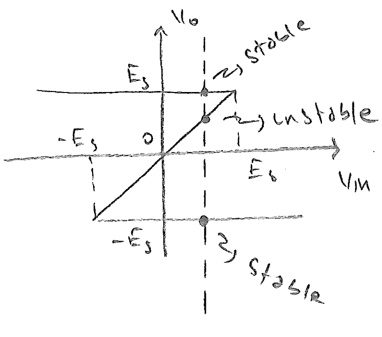
⤳ circuitry that controls the water pump to regulate the water level

where $R_2 \ll R_1$

How about ?



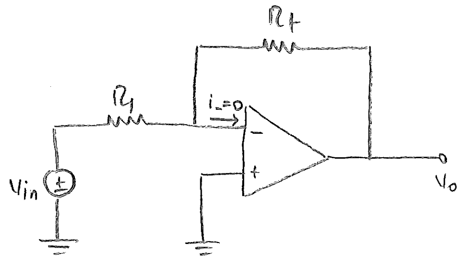
⤳ positive feedback



(i.e. in practice the OP-AMP will quickly jump to either to +sat or to -sat region under any disturbance.)

#3 18 Kasım 2018

Inverting amplifier



KCL : $\frac{v_- - v_{in}}{R_1} + \frac{v_- - v_0}{R_f} = 0$

$$\Rightarrow \left(\frac{1}{R_1}+\frac{1}{R_f}\right) v_- = \frac{1}{R_1} v_{in} + \frac{1}{R_f} v_0$$

$$\Rightarrow v_- = \left(\frac{1}{R_1}+\frac{1}{R_f}\right)^{-1} \left\{ \frac{1}{R_1} v_{in} + \frac{1}{R_f} v_0 \right\} \quad (1)$$

$v_+ = 0$ (2). Note that (1) & (2) are valid at all regions.

<u>linear</u> $v_- = v_+$ & $-E_s \le v_0 \le E_s$

$$\left(\frac{1}{R_1}+\frac{1}{R_f}\right)^{-1} \left\{ \frac{1}{R_1} v_{in} + \frac{1}{R_f} v_0 \right\} = 0 \Rightarrow \boxed{v_0 = -\frac{R_f}{R_1} v_{in}}$$

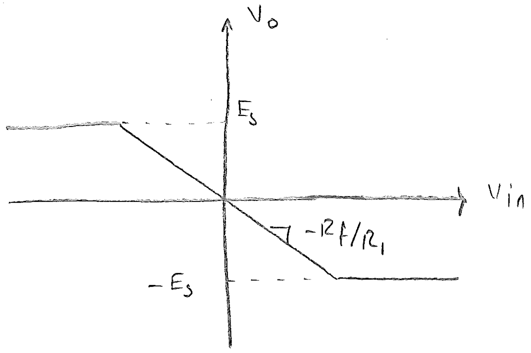
$$\& \quad -E_s \le -\frac{R_f}{R_1} v_{in} \le E_s \Rightarrow \boxed{-\frac{R_1}{R_f} E_s \le v_{in} \le \frac{R_1}{R_f} E_s}$$

<u>+sat</u> $v_- < v_+$ & $\boxed{v_0 = E_s}$

$$\left(\frac{1}{R_1}+\frac{1}{R_f}\right)^{-1} \left\{ \frac{1}{R_1} v_{in} + \frac{1}{R_f} E_s \right\} < 0 \Rightarrow \boxed{v_{in} < -\frac{R_1}{R_f} E_s}$$

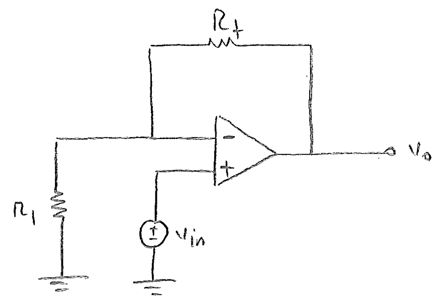
<u>-sat</u> $v_- > v_+$ & $\boxed{v_0 = -E_s}$ $\Rightarrow$ $\boxed{v_{in} > \frac{R_1}{R_f} E_s}$

Hence,



#4 18 Kasım 2018

Noninverting amplifier



KCL : $\frac{V_-}{R_1} + \frac{V_- - V_o}{R_f} = 0 \Rightarrow V_- = \left(1+\frac{R_f}{R_1}\right)^{-1} V_o$ (1)

& $V_+ = V_{in}$ (2)

linear $V_- = V_+$ & $-E_s \leq V_o \leq E_s$

$\Rightarrow \left(1+\frac{R_f}{R_1}\right)^{-1} V_o = V_{in} \Rightarrow \boxed{V_o = \left(1+\frac{R_f}{R_1}\right) V_{in}}$

& $-E_s \leq \left(1+\frac{R_f}{R_1}\right) V_{in} \leq E_s \Rightarrow \boxed{-\left(1+\frac{R_f}{R_1}\right)^{-1} E_s \leq V_{in} \leq \left(1+\frac{R_f}{R_1}\right)^{-1} E_s}$

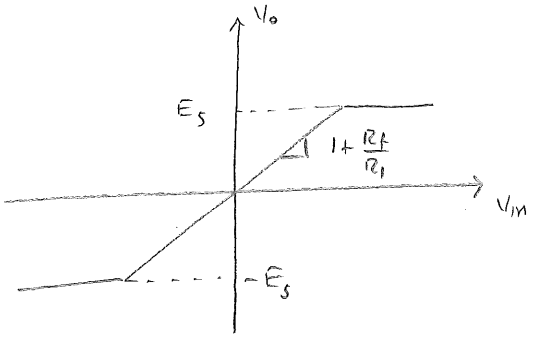
+sat $V_- < V_+$ & $\boxed{V_o = E_s}$

$\boxed{\left(1+\frac{R_f}{R_1}\right)^{-1} E_s < V_{in}}$

-sat $V_- > V_+$ & $\boxed{V_o = -E_s}$

$\boxed{-\left(1+\frac{R_f}{R_1}\right)^{-1} E_s > V_{in}}$

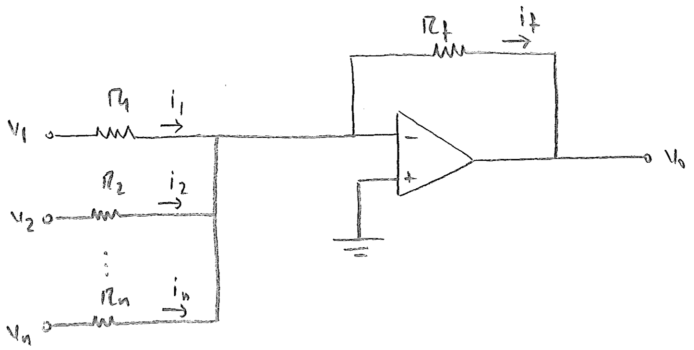
Hence,



Remark Note that the voltage follower is a special case of the noninverting amplifier with $R_f = 0$ & $R_1 = \infty$.

#5 18 Kasım 2018

Summing amplifier



In linear region $v_- = v_+ = 0$

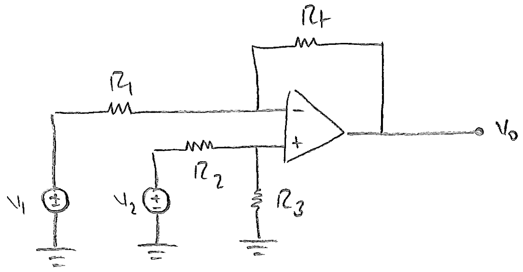
$$\Rightarrow \quad i_k = \frac{v_k - v_-}{R_k} = \frac{v_k}{R_k} \quad \& \quad i_f = \frac{v_- - v_o}{R_f} = -\frac{v_o}{R_f}$$

$$\text{KCL} \Rightarrow \quad i_1 + i_2 + \cdots + i_n = i_f$$

$$\Rightarrow \quad \frac{v_1}{R_1} + \frac{v_2}{R_2} + \cdots + \frac{v_n}{R_n} = -\frac{v_o}{R_f}$$

$$\Rightarrow \quad \boxed{v_o = -\left(\frac{R_f}{R_1} v_1 + \frac{R_f}{R_2} v_2 + \cdots + \frac{R_f}{R_n} v_n\right)} \quad \text{(in linear region)}$$

Difference amplifier



$$v_+ = \frac{R_3}{R_2 + R_3} v_2 = \left[1 + \frac{R_2}{R_3}\right]^{-1} v_2 \quad (1)$$

$$\frac{v_- - v_1}{R_1} + \frac{v_- - v_o}{R_f} = 0 \Rightarrow \left(\frac{1}{R_1} + \frac{1}{R_f}\right) v_- = \frac{v_1}{R_1} + \frac{v_o}{R_f}$$

$$\Rightarrow v_o = \frac{R_f}{R_1} \left\{ \left[1 + \frac{R_1}{R_f}\right] v_- - v_1 \right\} \quad (2)$$

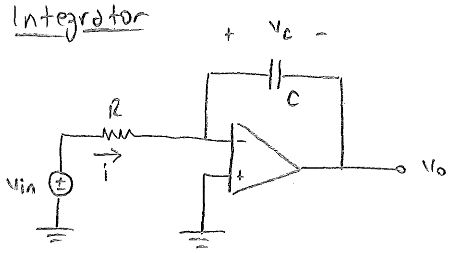
in linear region $v_+ = v_-$

$$(1) \& (2) \Rightarrow \quad v_o = \frac{R_f}{R_1} \left\{ \left[1 + \frac{R_1}{R_f}\right] \left[1 + \frac{R_2}{R_3}\right]^{-1} v_2 - v_1 \right\}$$

Then choosing $\frac{R_1}{R_f} = \frac{R_2}{R_3}$ yields $\boxed{v_o = \frac{R_f}{R_1}(v_2 - v_1)}$ (in lin. region)

#6 18 Kasım 2018

Integrator



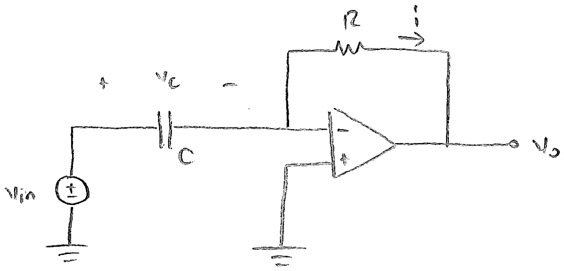
In linear region $V_- = V_+ = 0$

$$\Rightarrow i = \frac{V_{in}}{R} \quad (1) \quad \& \quad V_c = -V_o \quad (2)$$

Also, $V_c(t) = V_c(0) + \frac{1}{C}\int_0^t i(\tau)d\tau \quad (3)$

(1),(2),(3) $\Rightarrow$ $\boxed{V_o(t) = V_o(0) - \frac{1}{RC}\int_0^t V_{in}(\tau)d\tau}$ (in lin. region)

Differentiator



In linear region $V_- = V_+ = 0$

$V_{in} = V_c$

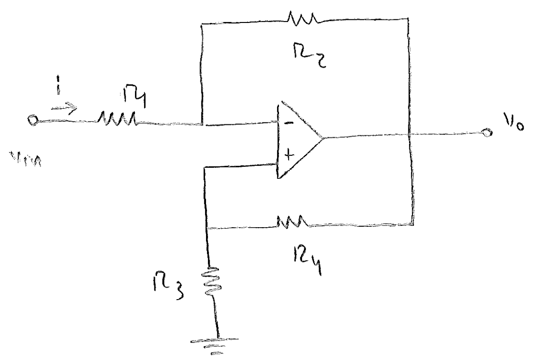
$i = C\frac{dV_c}{dt} = C\frac{dV_{in}}{dt}$

$V_o = -Ri$

$\boxed{V_o(t) = -RC\frac{dV_{in}}{dt}}$ (in lin. region)

#7 18 Kasım 2018

Example obtain the input ($v_{in}-i$) & transfer ($v_{in}-v_o$) char.



$$v_+ = \frac{R_3}{R_3+R_4} v_o \qquad (1)$$

$$\frac{v_- - v_{in}}{R_1} + \frac{v_- - v_o}{R_2} = 0$$

$$\Rightarrow v_- = \frac{R_2}{R_1+R_2} v_{in} + \frac{R_1}{R_1+R_2} v_o \qquad (2)$$

and $$i = \frac{1}{R_1+R_2}(v_{in} - v_o) \qquad (3)$$

Define $\beta := \frac{R_3}{R_3+R_4}$, $\gamma := \frac{R_1}{R_1+R_2}$ (Note that $0<\beta<1$ & $0<\gamma<1$.)

Linear region $v_+ = v_-$ & $-E_S \le v_o \le E_S$

(1) & (2) $\Rightarrow$ $\beta v_o = (1-\gamma) v_{in} + \gamma v_o$ $\Rightarrow$ $\boxed{v_o = \frac{1-\gamma}{\beta-\gamma} v_{in}}$

Define $\bar{V} := \left| \frac{\beta-\gamma}{1-\gamma} \right| E_S$ $\Rightarrow$ In linear region $\boxed{-\bar{V} \le v_{in} \le \bar{V}}$

$$(3) \Rightarrow i = \frac{1}{R_1+R_2}\left(1 - \frac{v_o}{v_{in}}\right) v_{in}$$

$$= \frac{1}{R_1+R_2}\left(1 - \frac{1-\gamma}{\beta-\gamma}\right) v_{in}$$

$$\Rightarrow \boxed{i = \frac{1}{R_1+R_2} \cdot \frac{\beta-1}{\beta-\gamma} v_{in}}$$

Define $G := \left| \frac{1}{R_1+R_2} \cdot \frac{\beta-1}{\beta-\gamma} \right|$

#8 18 Kasım 2018

+ sat region $v_+ > v_-$ & $v_o = E_s$

(1) & (2) $\Rightarrow \beta v_o > (1-\gamma) v_{in} + \gamma v_o$

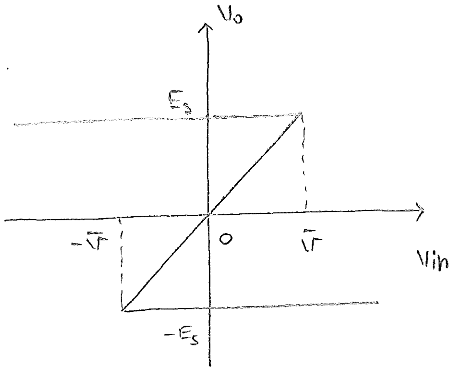
$\Rightarrow \beta E_s > (1-\gamma) v_{in} + \gamma E_s \Rightarrow \boxed{v_{in} < \frac{\beta-\gamma}{1-\gamma} E_s}$

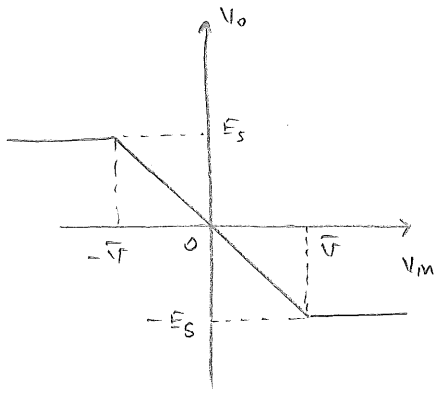
(3) $\Rightarrow \boxed{i = \frac{1}{R_1+R_2}(v_{in}-E_s)}$

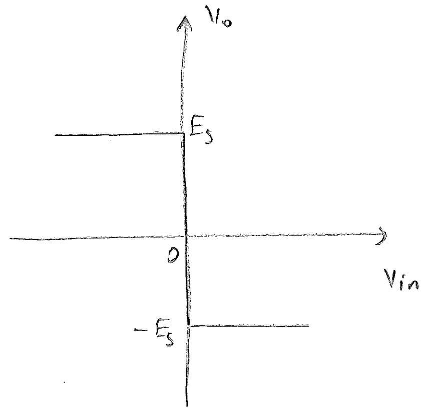
- sat region $v_+ < v_-$ & $v_o = -E_s$

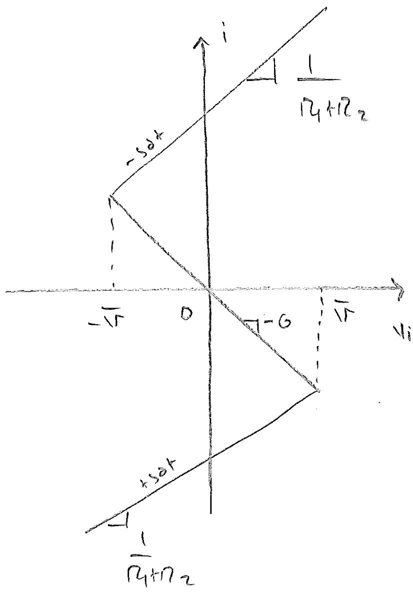
$\boxed{v_{in} > -\frac{\beta-\gamma}{1-\gamma} E_s}$ & $\boxed{i = \frac{1}{R_1+R_2}(v_{in}+E_s)}$

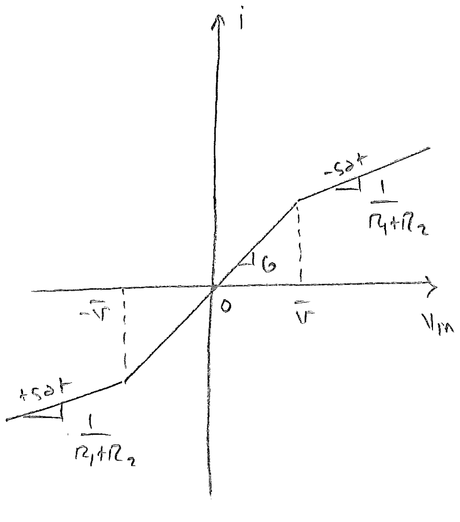
Cases: $\beta > \gamma$ $\beta < \gamma$ $\beta = \gamma$

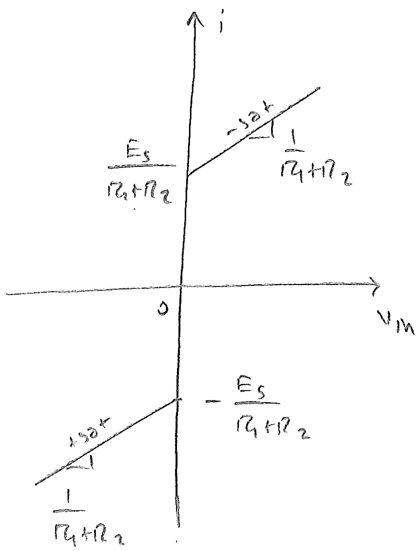






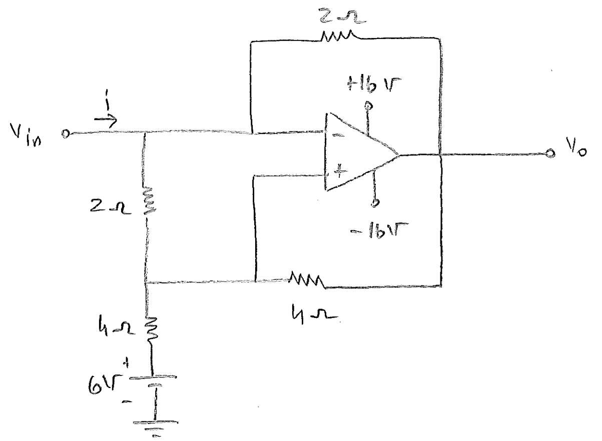






#9 18 Kasım 2018

Example obtain the input ($v_{in}-i$) & transfer ($v_{in}-v_o$) char.



$v_- = v_{in}$ (1)

$v_+ = ?$

$$\frac{v_+ - v_{in}}{2} + \frac{v_+ - 6}{4} + \frac{v_+ - v_o}{4} = 0$$

$$\Rightarrow \left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right\} v_+ = \frac{1}{2}v_{in} + \frac{1}{4}v_o + \frac{3}{2}$$

$$\Rightarrow v_+ = \frac{1}{2}v_{in} + \frac{1}{4}v_o + \frac{3}{2} \quad (2)$$

$i = ?$ $$i = \frac{v_{in}-v_+}{2} + \frac{v_{in}-v_o}{2}$$

$$= v_{in} - \frac{1}{2}v_o - \frac{1}{2}\left\{\frac{1}{2}v_{in} + \frac{1}{4}v_o + \frac{3}{2}\right\}$$

$$\Rightarrow i = \frac{3}{4}v_{in} - \frac{5}{8}v_o - \frac{3}{4} \quad (3)$$

Eq. (1), (2), (3) are valid in all regions!

Linear region $v_- = v_+$ & $-16 \le v_o \le 16$

$$v_- = v_+ \Rightarrow v_{in} = \frac{1}{2}v_{in} + \frac{1}{4}v_o + \frac{3}{2} \Rightarrow \boxed{v_o = 2v_{in} - 6}$$

$$-16 \le v_o \le 16 \Rightarrow -16 \le 2v_{in} - 6 \le 16 \Rightarrow \boxed{-5 \le v_{in} \le 11}$$

$$(3) \Rightarrow i = \frac{3}{4}v_{in} - \frac{5}{8}(2v_{in} - 6) - \frac{3}{4} \Rightarrow \boxed{i = -\frac{1}{2}v_{in} + 3}$$

#10   18 Kasım 2018

+sat $v_- < v_+$ & $\boxed{v_o = 16V}$

$v_- < v_+ \Rightarrow v_{in} < \frac{1}{2} v_{in} + \frac{1}{4}(16) + \frac{3}{2} \Rightarrow \boxed{v_{in} < 11}$

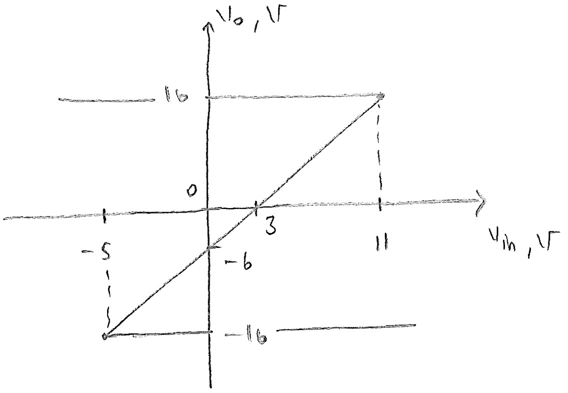
(3) $\Rightarrow i = \frac{3}{4} v_{in} - \frac{5}{8}(16) - \frac{3}{4} \Rightarrow \boxed{i = \frac{3}{4} v_{in} - \frac{43}{4}}$

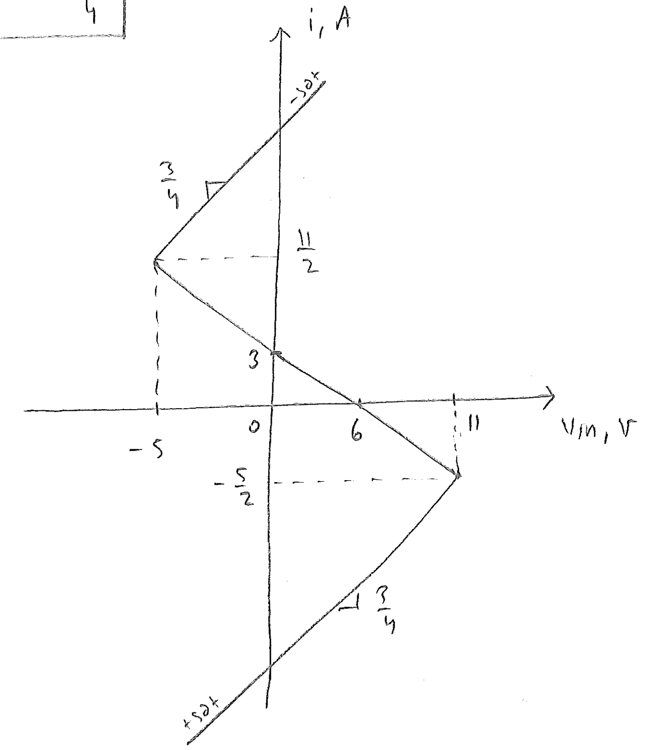
-sat $v_- > v_+$ & $\boxed{v_o = -16V}$

$v_- > v_+ \Rightarrow v_{in} > \frac{1}{2} v_{in} + \frac{1}{4}(-16) + \frac{3}{2} \Rightarrow \boxed{v_{in} > -5}$

(3) $\Rightarrow i = \frac{3}{4} v_{in} - \frac{5}{8}(-16) - \frac{3}{4} \Rightarrow \boxed{i = \frac{3}{4} v_{in} + \frac{37}{4}}$

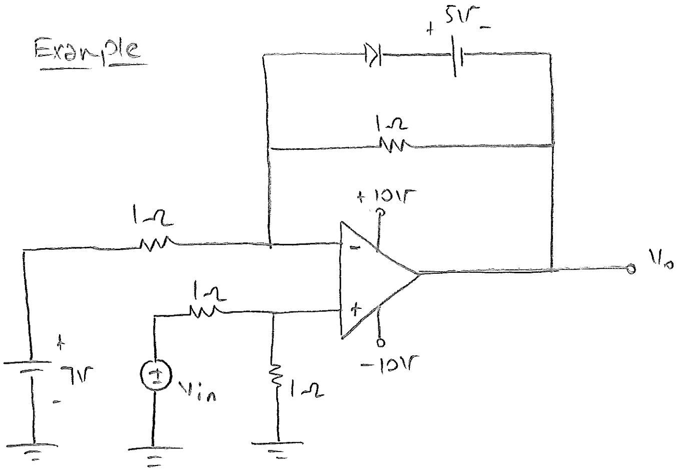
Hence,





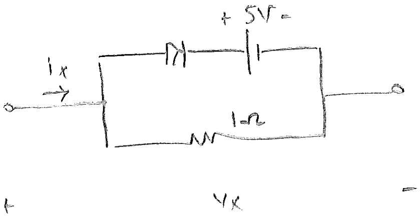
#11 18 Kasım 2018

Example

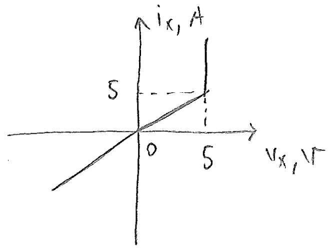


obtain the transfer char. (v_{in} - v_o curve)

Sol'n

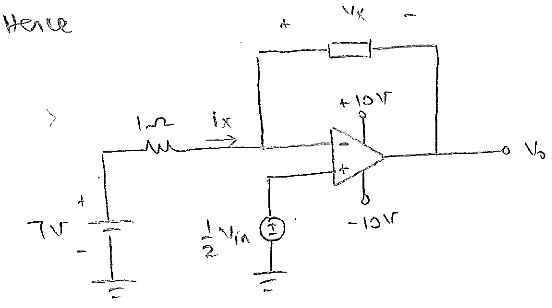


=>



$$\Rightarrow \quad v_x = \begin{cases} i_x & \text{for } \quad i_x \leq 5 \quad (\text{diode OFF}) \\ 5 & \text{for } \quad i_x > 5 \quad (\text{diode ON}) \end{cases} \qquad (1)$$

Hence



$v_+ = \frac{1}{2} v_{in}$ (2) , $v_- = ?$

$v_- = v_o + v_x$ (3)

$i_x = \dfrac{7 - v_-}{1}$ (4)

Using (1),(3),(4):

$$v_- = \begin{cases} v_o + (7 - v_-) & \text{for } \quad 7 - v_- \leq 5 \\ v_o + 5 & \text{for } \quad 7 - v_- > 5 \end{cases}$$

$\Rightarrow \quad v_- = v_o + 7 - v_- \quad \Rightarrow \quad v_- = \frac{1}{2}(v_o + 7)$ under $7 - v_- \leq 5 \Rightarrow 7 - \frac{1}{2}(v_o + 7) \leq 5 \Rightarrow v_o \geq -3$

and $v_- = v_o + 5$ under $7 - v_- > 5 \Rightarrow 7 - (v_o + 5) > 5 \Rightarrow v_o < -3$

$$\text{Therefore} \quad v_- = \begin{cases} \frac{1}{2}(v_o + 7) & \text{for } \quad v_o \geq -3 \quad (\text{OFF}) \\ v_o + 5 & \text{for } \quad v_o < -3 \quad (\text{ON}) \end{cases} \qquad (5)$$

Note that (2) & (5) are valid in all regions!

#12 18 Kasım 2018

linear $v_+ = v_-$, $-10 \le v_o \le 10$

$(2) \& (5) \Rightarrow \frac{1}{2} v_{in} = \begin{cases} \frac{1}{2}(v_o+7) & \text{for } -3 \le v_o \le 10 \\ v_o+5 & \text{for } -10 \le v_o < -3 \end{cases}$

$\Rightarrow \frac{1}{2} v_{in} = \frac{1}{2}(v_o+7) \Rightarrow \boxed{v_o = v_{in}-7}$ when $-3 \le v_{in}-7 \le 10 \Rightarrow \boxed{4 \le v_{in} \le 17}$

& $\frac{1}{2} v_{in} = v_o+5 \Rightarrow \boxed{v_o = \frac{1}{2} v_{in} - 5}$ when $-10 \le \frac{1}{2} v_{in} - 5 < -3 \Rightarrow \boxed{-10 \le v_{in} < 4}$

+sat $v_+ > v_-$, $\boxed{v_o = 10V}$

Since $v_o = 10V$, $(5) \Rightarrow v_- = \frac{1}{2}(10+7) = \frac{17}{2} V$

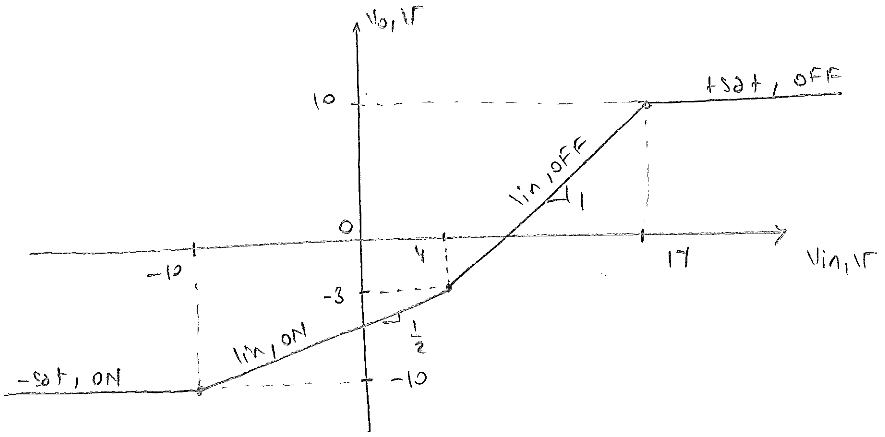
Then $(2) \Rightarrow \frac{1}{2} v_{in} > \frac{17}{2} \Rightarrow \boxed{v_{in} > 17}$

-sat $v_+ < v_-$, $\boxed{v_o = -10V}$

$(5) \Rightarrow v_- = -10+5 = -5V$

$(2) \Rightarrow \frac{1}{2} v_{in} < -5 \Rightarrow \boxed{v_{in} < -10}$

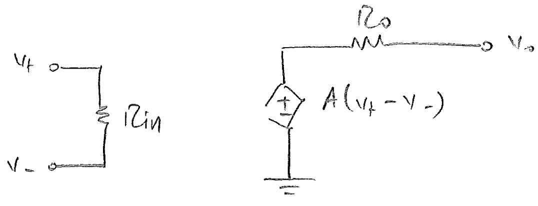
Hence,



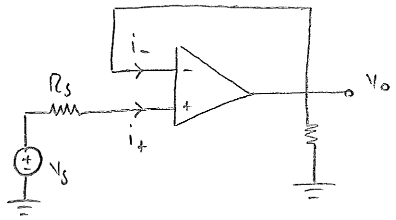
#13 18 Kasım 2018

More realistic model of OPAMP

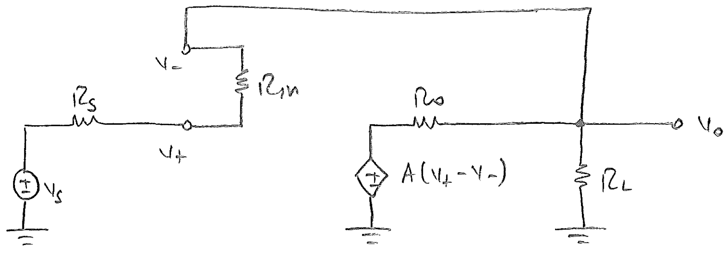
Finite gain model in linear region



Example Analyze the voltage follower circuit using the finite-gain model



$\Rightarrow$



Let $A = 10^5$, $R_s = 1k\Omega$, $R_{in} = 100k\Omega$

$R_o = 100\Omega$, $R_L = 10k\Omega$

Remark Recall that using the ideal model we've obtained $\frac{v_o}{v_s} = 1$ under the assumption that $i_+ = i_- = 0$.

Sol'n Write KCL at the output node

$$\left.\begin{aligned} &\frac{v_o - A(v_+ - v_-)}{R_o} + \frac{v_o}{R_L} + \frac{v_o - v_s}{R_s + R_{in}} = 0 \\ &\& \quad v_+ - v_- = \frac{R_{in}}{R_{in} + R_s}(v_s - v_o) \end{aligned}\right\} \frac{1}{R_o}\left\{ v_o - \frac{A R_{in}(v_s - v_o)}{R_{in} + R_s} \right\} + \frac{v_o}{R_L} + \frac{v_o - v_s}{R_s + R_{in}} = 0$$

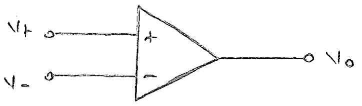
$$\Rightarrow \frac{v_o}{v_s} = \frac{\left[\frac{A R_{in}}{R_o(R_{in} + R_s)} + \frac{1}{R_s + R_{in}} \right]^{\sim 990.1}}{\left[\frac{A R_{in}}{R_o(R_{in} + R_s)} + \frac{1}{R_s + R_{in}} \right] + \left[\frac{1}{R_o} + \frac{1}{R_L} \right]_{\sim 0.0101}} = 0.999898 \cong 1$$

$$\& \quad i_+ = \frac{v_s - v_o}{R_s + R_{in}} = \frac{\left[1 - \frac{v_o}{v_s}\right]}{R_s + R_{in}} \cdot v_s$$, for $v_s = 10V$ we have $i_+ = 1.01 \times 10^{-9} A \cong 0$

Conclusion The (infinite-gain) ideal OPAMP model works well!

#14 18 Kasım 2018

Common Mode Rejection Ratio (CMRR)



The output voltage for actual OPAMPs satisfy

$$V_o = A_+ V_+ - A_- V_- \quad (1)$$

[up to now we've taken $A_+ = A_- = A$ but $A_+ \neq A_-$ in reality.]

$$(1) \Rightarrow \quad V_o = \underbrace{A_d (V_+ - V_-)}_{\text{differential voltage}} + \underbrace{A_c \left(\frac{V_+ + V_-}{2} \right)}_{\text{common mode voltage}}$$

where $A_d := \frac{A_+ + A_-}{2}$ differential gain

$A_c := A_+ - A_-$ common mode gain

[up to now we've taken $A_c = 0$]

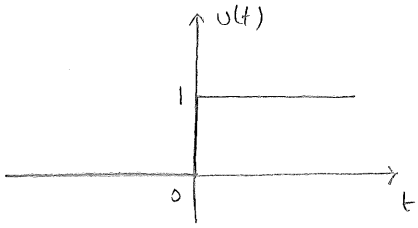
Definition $CMRR := \left| \frac{A_d}{A_c} \right|$ or $CMRR_{dB} := 20\log \left| \frac{A_d}{A_c} \right|$

Remark The higher the CMRR the better it is. Generally CMRR > 70 dB works fine for most applications. Widely used OP-AMP chip 741 has a CMRR of 90 dB.

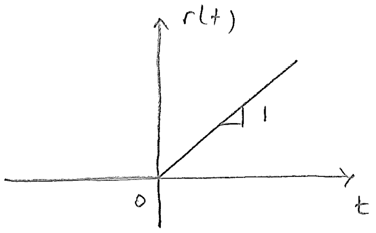
#1 25 Kasım 2018

$\boxed{\text{Ch. V}}$ Dynamic Elements

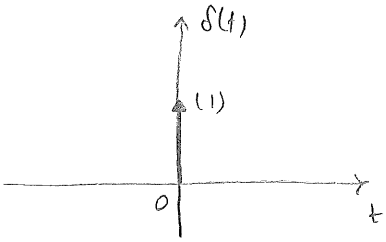
Elementary functions :



unit step function

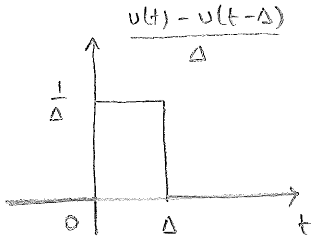


unit ramp function



unit impulse function (delta function)

$\delta(t) = ?$



delta function $\delta(t) := \lim_{\Delta \to 0} \frac{u(t)-u(t-\Delta)}{\Delta}$

$$\Rightarrow \delta(t) := \begin{cases} 0 & \text{for } t \neq 0 \\ \underbrace{\text{singular}}_{?} & \text{for } t = 0 \end{cases}$$

→ singularity is such that $\int_{-\alpha}^{\beta} \delta(t)dt = 1$ for all $\alpha, \beta > 0$.

observe : $\int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$ and $\int_{-\infty}^{t} u(\tau)d\tau = r(t)$.

By convention: $\delta(t) = \frac{d}{dt} u(t)$ and $u(t) = \frac{d}{dt} r(t)$.

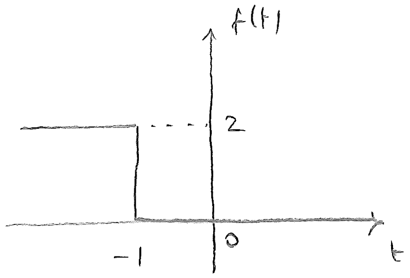
Example Sketch $f(t) = 2u(-t-1)$

let $s = -t-1$ $\Rightarrow$ $2u(s) = \begin{cases} 2 & \text{for } s > 0 \\ 0 & \text{for } s < 0 \end{cases}$

$s > 0 \Rightarrow -t-1 > 0 \Rightarrow t < -1$

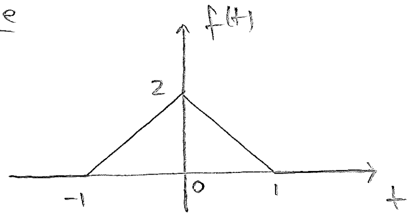
$s < 0 \Rightarrow -t-1 < 0 \Rightarrow t > -1$

Hence $f(t) = \begin{cases} 2 & \text{for } t < -1 \\ 0 & \text{for } t > -1 \end{cases}$



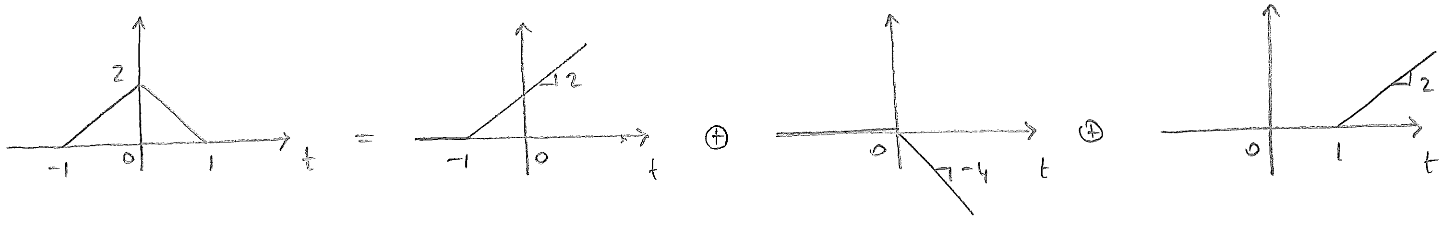
\# 2   25 Kasım 2018

Example



Express f(t) in terms of elementary functions δ(t), u(t), r(t).

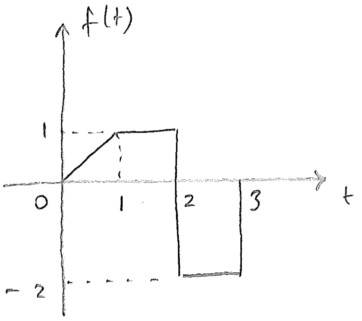
Sol'n



$2r(t+1)$ $-4r(t)$ $2r(t-1)$

$\Rightarrow \quad f(t) = 2r(t+1) - 4r(t) + 2r(t-1)$.

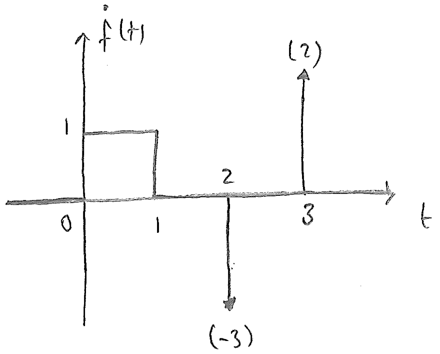
Example



$f(t) = r(t) - r(t-1) - 3u(t-2) + 2u(t-3)$

$\dot{f}(t) := \frac{d}{dt} f(t) = ?$

$\dot{f}(t) = u(t) - u(t-1) - 3\delta(t-2) + 2\delta(t-3)$



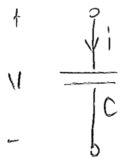
Exercise [sifting property of the impulse function]

Show that $\int_{-T}^{T} f(t)\delta(t)\,dt = f(0)$ for continuous f $(T>0)$

Hence, in general $\boxed{f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)}$ (for continuous f.)

#3 25 Kasım 2018

LTI capacitor



$q(t) = Cv(t)$

& $i(t) = \frac{d}{dt}q(t)$

$$\boxed{\begin{aligned} i(t) &= C\frac{dv(t)}{dt} \\ v(t) &= v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau \end{aligned}}$$

C : capacitance measured in Farads (F)

instantaneous power: $p(t) = v(t)i(t)$

energy accumulated during interval $[t_0, t]$:

$$w(t_0,t) = \int_{t_0}^{t} v(\tau)i(\tau)d\tau$$

$$= \int_{t_0}^{t} v(\tau)\left\{C\frac{dv(\tau)}{d\tau}\right\}d\tau$$

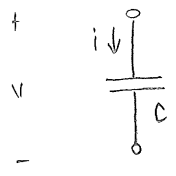
$$= \int_{v(t_0)}^{v(t)} Cv\,dv = \frac{1}{2}Cv^2\Big|_{v(t_0)}^{v(t)} = \frac{1}{2}Cv(t)^2 - \frac{1}{2}Cv(t_0)^2$$

energy stored at time t :

$$w(t) := w(-\infty,t) = \frac{1}{2}Cv(t)^2 - \underbrace{\frac{1}{2}Cv(-\infty)^2}_{=0 \text{ by assumption}} = \boxed{\frac{1}{2}Cv(t)^2}$$

Initial condition models

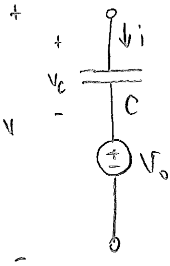
model 1



$v(t_0^-) = V_0$

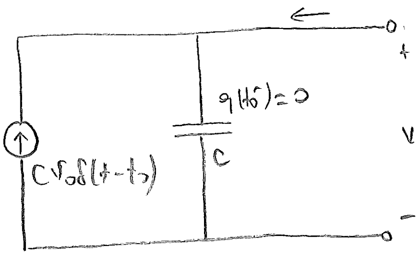
(for $t > t_0$) $\equiv$

model 2



$\equiv$

model 3



$\Rightarrow q(t_0^-) = CV_0$

↳ initial charge of the capacitor

#4 25 Kasım 2018

By equivalence "≡" we mean the following : all three models will have the same v(t), i(t) readings for $t > t_0$. [Note that in models 2 & 3 the capacitors are initially uncharged.]

proof model 1: $v(t) = v(t_0^-) + \frac{1}{C}\int_{t_0^-}^{t} i(\tau)d\tau = V_0 + \frac{1}{C}\int_{t_0^-}^{t} i(\tau)d\tau$

model 2: $v(t) = \underbrace{v_c(t_0^-)}_{=0} + \frac{1}{C}\int_{t_0^-}^{t} i(\tau)d\tau + \underbrace{V_0\, u(t-t_0)}_{=V_0 \text{ for } t>t_0}$

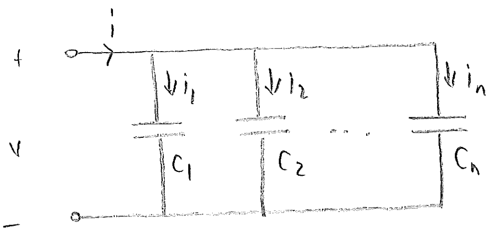
$$= V_0 + \frac{1}{C}\int_{t_0^-}^{t} i(\tau)d\tau \quad (\text{for } t>t_0)$$

model 3: $v(t) = \underbrace{v(t_0^-)}_{=0} + \frac{1}{C}\int_{t_0^-}^{t} [i(\tau) + C V_0 \delta(\tau - t_0)]d\tau$

$$= V_0 \underbrace{\int_{t_0^-}^{t} \delta(\tau - t_0)d\tau}_{=1 \text{ for } t>t_0} + \frac{1}{C}\int_{t_0^-}^{t} i(\tau)d\tau$$

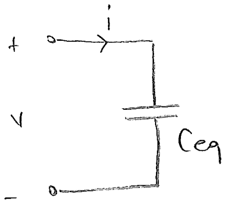
$$= V_0 + \frac{1}{C}\int_{t_0^-}^{t} i(\tau)\, d\tau \quad (\text{for } t>t_0)$$

Capacitors in parallel connection



$v(t_0) = V_0$

≡



$v(t_0) = V_0$

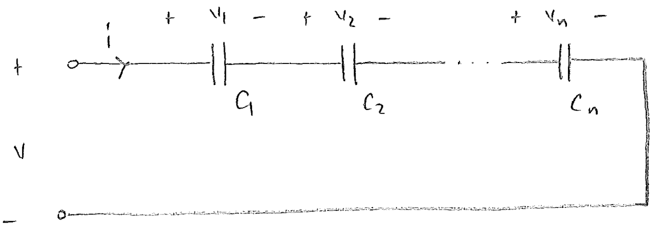
$C_{eq} = ?$

$$i(t) = i_1(t) + i_2(t) + \cdots + i_n(t) = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + \cdots + C_n\frac{dv}{dt}$$

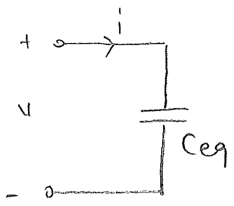
$$= \underbrace{(C_1 + C_2 + \cdots + C_n)}_{C_{eq}}\frac{dv}{dt} \Rightarrow \boxed{C_{eq} = C_1 + C_2 + \cdots + C_n}$$

#5 25 Kasım 2018

Capacitors in series connection



$\equiv$



$C_{eq} = ?$

$v_1(t_0) = V_{10}\ ,\ v_2(t_0) = V_{20}\ , \cdots ,\ v_n(t_0) = V_{n0}$ $\qquad v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_n(t_0)$

$$v(t) = v_1(t) + v_2(t) + \cdots + v_n(t)$$

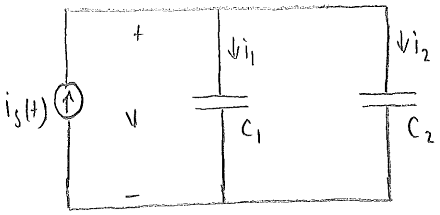
$$= \left\{ v_1(t_0) + \frac{1}{C_1}\int_{t_0}^{t} i(\tau)d\tau \right\} + \cdots + \left\{ v_n(t_0) + \frac{1}{C_n}\int_{t_0}^{t} i(\tau)d\tau \right\}$$

$$= \underbrace{\left\{ V_{10} + V_{20} + \cdots + V_{n0} \right\}}_{v(t_0)} + \underbrace{\left\{ \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n} \right\}}_{\frac{1}{C_{eq}}} \int_{t_0}^{t} i(\tau)d\tau$$

$$\Rightarrow \boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}}$$

Current division

$i_1(t),\ i_2(t) = ?$



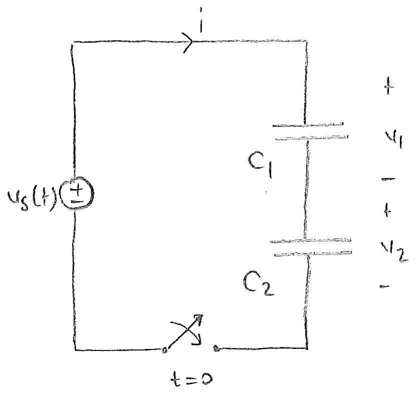
$$\left.\begin{array}{l} i_1(t) = C_1 \frac{dv}{dt} \Rightarrow \frac{dv}{dt} = \frac{1}{C_1} i_1(t) \\ i_2(t) = C_2 \frac{dv}{dt} \Rightarrow \frac{dv}{dt} = \frac{1}{C_2} i_2(t) \end{array}\right\} \quad \frac{i_1(t)}{C_1} = \frac{i_2(t)}{C_2} \qquad (1)$$

$$\text{KCL} : \quad i_1(t) + i_2(t) = i_s(t) \qquad (2)$$

$$(1)\ \&\ (2) \Rightarrow \boxed{i_1(t) = \frac{C_1}{C_1+C_2} i_s(t) \quad \& \quad i_2(t) = \frac{C_2}{C_1+C_2} i_s(t)}$$

#6 25 Kasım 2018

Voltage division



$v_1(0^-) = V_{10}$, $v_2(0^-) = V_{20}$

$v_1(t), v_2(t) = ?$ for $t>0$

Since the same current visits both capacitors

$$v_1(t) = v_1(0^-) + \frac{1}{C_1}\int_{0^-}^{t} i(\tau)d\tau \Rightarrow \int_{0^-}^{t} i(\tau)d\tau = C_1(v_1(t)-V_{10})$$

$$v_2(t) = v_2(0^-) + \frac{1}{C_2}\int_{0^-}^{t} i(\tau)d\tau \Rightarrow \int_{0^-}^{t} i(\tau)d\tau = C_2(v_2(t)-V_{20})$$

$$\Rightarrow \quad C_1(v_1(t)-V_{10}) = C_2(v_2(t)-V_{20}) \qquad (1)$$

$$\text{KVL:} \quad v_1(t)+v_2(t) = v_s(t) \qquad (2)$$

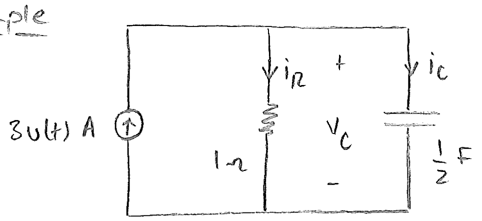
(1) & (2) $\Rightarrow$

$$\boxed{\begin{aligned} v_1(t) &= \frac{C_2}{C_1+C_2}v_s(t) + \frac{C_1V_{10}-C_2V_{20}}{C_1+C_2} \\ \& \quad v_2(t) &= \frac{C_1}{C_1+C_2}v_s(t) + \frac{C_2V_{20}-C_1V_{10}}{C_1+C_2} \end{aligned}}$$

for $t>0$

#7 25 Kasım 2018

Example



Find $v_c(0^+)$, $i_R(0^+)$, $i_c(0^+)$ for

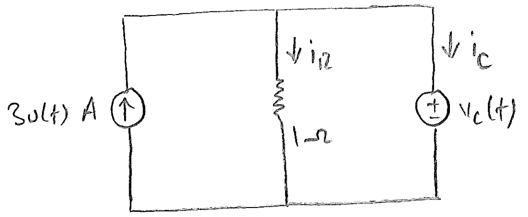
a) $v_c(0^-) = 0$

b) $v_c(0^-) = 4V$

Assumption Henceforth we will assume that the energy stored at a capacitor (or an inductor) is always finite.

Remark Finite energy assumption is not really necessary, we make it because it simplifies analysis.

Sol'n $v_c(0^-) = 0$ For $0^- < t < 0^+$ let's replace the capacitor with IVS. (substitution thm.)



What do we know about $v_c(t)$?

Recall stored energy $= \frac{1}{2} C v_c(t)^2$

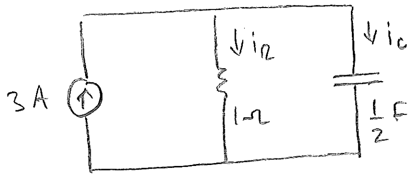
Hence finite energy $\Rightarrow$ bounded (finite) $v_c(t)$

Now, bounded $v_c(t)$ $\Rightarrow$ bounded $i_R(t)$

Then KCL $\Rightarrow$ $\underbrace{3u(t)}_{\text{bounded}} = \underbrace{i_R(t)}_{\text{bounded}} + i_c(t)$ $\Rightarrow$ bounded $i_c(t)$.

$$\Rightarrow v_c(0^+) = v_c(0^-) + \frac{1}{1/2} \underbrace{\int_{0^-}^{0^+} i_c(\tau)\, d\tau}_{=0 \text{ since } i_c \text{ bounded}} \quad \Rightarrow v_c(0^+) = v_c(0^-) = 0$$

At $t = 0^+$



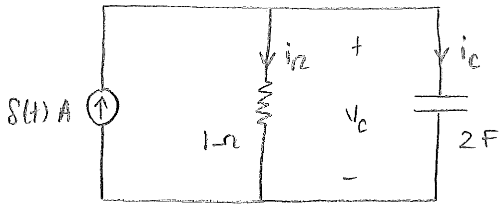
$v_c(0^+) = 0 \Rightarrow i_R(0^+) = 0$

$\Rightarrow i_c(0^+) = 3$ A.

b) $v_c(0^-) = 4V$. Exercise. Answer: $v_c(0^+) = 4V$, $i_R(0^+) = 4A$, $i_c(0^+) = -1A$.

#8 25 Kasım 2018

Example

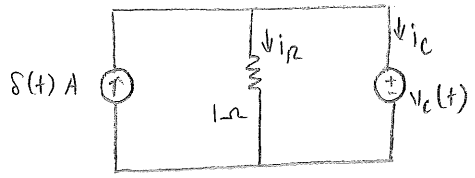


$v_C(0^-) = V_0$

Find $v_C(0^+)$, $i_R(0^+)$, $i_C(0^+)$

$0^- < t < 0^+$

Finite stored energy ⇒ bounded $v_C(t)$ ⇒ bounded $i_R(t)$



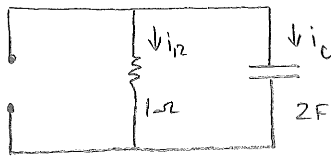
$$v_C(0^+) = v_C(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_C(\tau)d\tau$$

$$= V_0 + \frac{1}{2}\int_{0^-}^{0^+} \{\delta(\tau) - i_R(\tau)\}d\tau$$

$$= V_0 + \underbrace{\frac{1}{2}\int_{0^-}^{0^+} \delta(\tau)d\tau}_{=\frac{1}{2}} - \underbrace{\frac{1}{2}\int_{0^-}^{0^+} i_R(\tau)d\tau}_{=0 \text{ since } i_R \text{ bounded}}$$

Hence $v_C(0^+) = V_0 + \frac{1}{2}$ Volts.

$t = 0^+$



$i_R(0^+) = \frac{v_C(0^+)}{1} = V_0 + \frac{1}{2}$ Amps

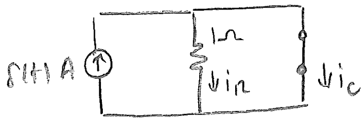
$i_C(0^+) = -i_R(0^+) = -V_0 - \frac{1}{2}$ Amps

Remark Note that for $0^- < t < 0^+$ we obtained $i_C(t) = \delta(t) + \{-i_R(t)\}$

$$= \underbrace{\delta(t)}_{\text{impulsive term}} + \underbrace{\{-v_C(t)/R\}}_{\text{bounded term}}$$

Note also that the bounded term has no effect on the value of $v_C(0^+)$. Since any bounded voltage source instead of $\oplus$ $v_C(t)$ would yield the same impulsive term, we can replace $\oplus$ $v_C(t)$ with the simplest bounded voltage source, i.e., the short circuit to compute the impulsive term of $i_C(t)$ on $0^- < t < 0^+$.

Example (revisited) For $0^- < t < 0^+$

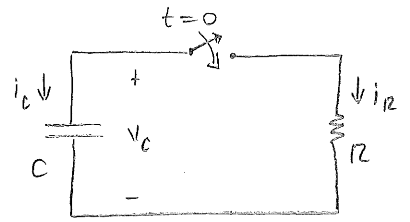


$$i_C(t) = \delta(t) \Rightarrow v_C(0^+) = V_0 + \frac{1}{2}\int_{0^-}^{0^+} \delta(\tau)d\tau$$

$= V_0 + \frac{1}{2}$ Volts as expected.

#9 25 Kasım 2018

Example



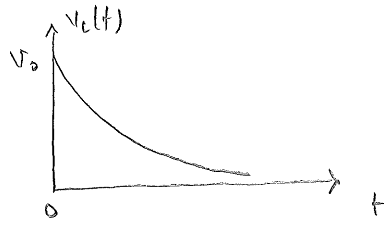
$v_c(0^-) = V_0$

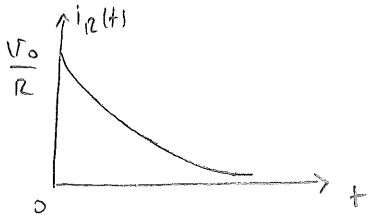
Note that $v_c(0^+) = V_0$ (1) (why?)

For $t>0$

$$\left.\begin{aligned} v_c &= R i_R \\ i_c &= C \frac{dv_c}{dt} \\ i_R + i_c &= 0 \end{aligned}\right\} \quad \frac{d}{dt} v_c(t) = -\frac{1}{RC} v_c(t) \qquad (2)$$

(1) & (2) $\Rightarrow$ $v_c(t) = V_0 e^{-t/RC}$ & $i_R(t) = \frac{V_0}{R} e^{-t/RC}$





Energy displaced at capacitor during $[0,t]$:

$$w_c(0,t) = \frac{1}{2} C v_c(t)^2 - \frac{1}{2} C V_0^2 = \frac{1}{2} C V_0^2 \left[e^{-2t/RC} - 1 \right]$$

Note that $w_c(t) < 0$. Therefore energy is drawn from the capacitor by the circuit.

Energy dissipated at resistor during $[0,t]$:

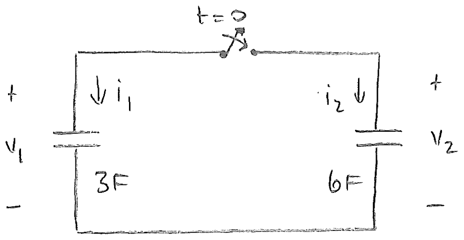
$$w_R(0,t) = \int_0^t \underbrace{i_R(\tau) v_R(\tau)}_{P_R(\tau)} d\tau = \frac{V_0^2}{R} \int_0^t e^{-2\tau/RC} d\tau = \frac{V_0^2}{R} \left\{ -\frac{RC}{2} e^{-2\tau/RC} \right\}_0^t$$

$$= \frac{1}{2} C V_0^2 \left[1 - e^{-2t/RC} \right]$$

Observe $w_c(0,t) + w_R(0,t) = 0$. Hence the energy drawn from the capacitor is dissipated at the resistor.

#10 25 Kasım 2018

Example



$v_1(0^-) = 0$, $v_2(0^-) = 3V$

Find $v_1(0^+)$ and the total stored energy at $t=0^-$ & $t=0^+$.

Sol'n

$v_1(t) = v_1(0^-) + \frac{1}{3}\int_{0^-}^{t} i_1(\tau)d\tau \quad \Rightarrow \quad \int_{0^-}^{t} i_1(\tau)d\tau = 3v_1 \qquad (1)$

$v_2(t) = v_2(0^-) + \frac{1}{6}\int_{0^-}^{t} i_2(\tau)d\tau \quad \Rightarrow \quad \int_{0^-}^{t} i_2(\tau)d\tau = 6(v_2-3) \qquad (2)$

KCL $\Rightarrow$ $i_1 + i_2 = 0$ (3)

(1),(2),(3) $\Rightarrow$ $3v_1 + 6(v_2-3) = 0$ (4)

t>0 KVL $\Rightarrow$ $v_1 = v_2$ (5)

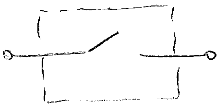
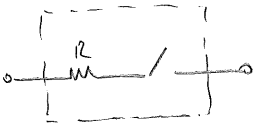
(4) & (5) $\Rightarrow$ $3v_1 + 6(v_1-3) = 0 \Rightarrow \boxed{v_1(t) = 2V}$ for all t>0.

$E(0^-) = \frac{1}{2}C_1v_1(0^-)^2 + \frac{1}{2}C_2v_2(0^-)^2 = 0 + 27 = \boxed{27\,J}$

$E(0^+) = \frac{1}{2}C_1v_1(0^+)^2 + \frac{1}{2}C_2v_2(0^+)^2 = \frac{1}{2}(C_1+C_2)v_1(0^+)^2 = \boxed{18J}$

Question What happened to $E(0^-) - E(0^+) = 9J$?

Answer Dissipated at the switch. To see this:

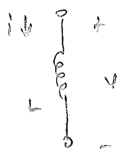
1) Replace switch  with 

2) Compute the energy $W_R(0,T)$ dissipated at R during the interval [0, T]

3) Show that $\lim_{R\to 0} W_R(0,T) = 9J$ for any T>0.

#11 25 Kasım 2018

LTI Inductor



$\phi(t) = L\,i(t)$

$v(t) = \frac{d}{dt}\phi(t)$

$$\boxed{\begin{aligned} v(t) &= L\frac{di(t)}{dt} \\ i(t) &= i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau \end{aligned}}$$

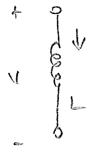
L : inductance, measured in Henries (H)

instantaneous power : $p(t) = v(t)\,i(t)$

energy accumulated during interval $[t_0, t]$: $w(t_0,t) = \frac{1}{2}L\,i(t)^2 - \frac{1}{2}L\,i(t_0)^2$

energy stored at time t : $w(t) = \boxed{\frac{1}{2}L\,i(t)^2}$

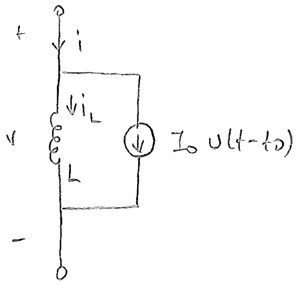
Initial condition models



$i(t) = I_0 + \frac{1}{L}\int_{t_0^-}^{t} v(\tau)\,d\tau$

$i(t_0^-) = I_0$

$(t>t_0)$
$\equiv$

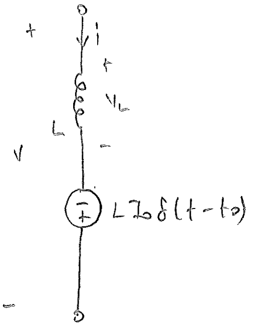


$i(t) = I_0\,u(t-t_0) + i_L(t)$

$= I_0 + \frac{1}{L}\int_{t_0^-}^{t} v(\tau)\,d\tau$ for $t>t_0$

$i_L(t_0^-) = 0$

$(t>t_0)$
$\equiv$



$i(t) = \frac{1}{L}\int_{t_0^-}^{t} v_L(\tau)\,d\tau$

$= \frac{1}{L}\int_{t_0^-}^{t} \{L I_0\,\delta(\tau-t_0) + v(\tau)\}\,d\tau$

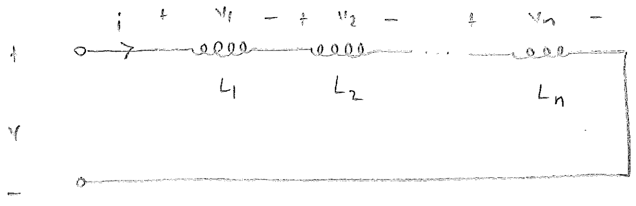
$= I_0\int_{t_0^-}^{t}\delta(\tau-t_0)\,d\tau + \frac{1}{L}\int_{t_0^-}^{t} v(\tau)\,d\tau$

$= I_0 + \frac{1}{L}\int_{t_0^-}^{t} v(\tau)\,d\tau$ for $t>t_0$

$i(t_0^-) = 0$

12   25 Kasım 2018

Inductors in series connection



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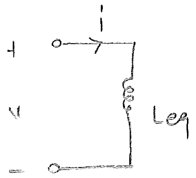


Leq = ?

$i(t_0) = I_0$ $i(t_0) = I_0$

$$v = v_1 + v_2 + \cdots + v_n$$

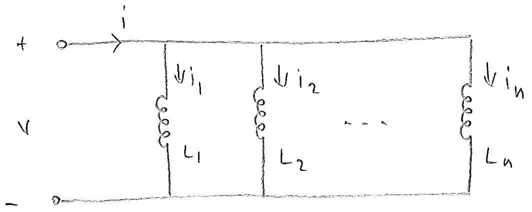
$$= L_1\frac{di}{dt} + L_2\frac{di}{dt} + \cdots + L_n\frac{di}{dt}$$

$$= \underbrace{\{L_1 + L_2 + \cdots + L_n\}}_{L_{eq}}\frac{di}{dt}$$

$$v = L_{eq}\frac{di}{dt}$$

$$\Rightarrow \boxed{L_{eq} = L_1 + L_2 + \cdots + L_n}$$

Inductors in parallel connection



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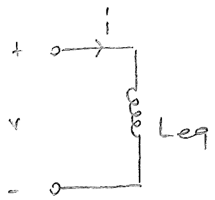


Leq = ?

$i_1(t_0) = I_{10},\ \ldots,\ i_n(t_0) = I_{n0}$

$$i(t_0) = I_{10} + I_{20} + \cdots + I_{n0}$$

$$i(t) = i(t_0) + \frac{1}{L_{eq}}\int_{t_0}^{t} v(\tau)\,d\tau$$

$$i(t) = i_1(t) + i_2(t) + \cdots + i_n(t)$$

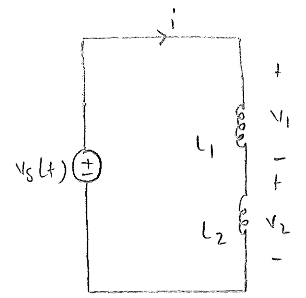
$$= \left\{I_{10} + \frac{1}{L_1}\int_{t_0}^{t} v(\tau)\,d\tau\right\} + \cdots + \left\{I_{n0} + \frac{1}{L_n}\int_{t_0}^{t} v(\tau)\,d\tau\right\}$$

$$= \{I_{10} + \cdots + I_{n0}\} + \underbrace{\left\{\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}\right\}}_{\frac{1}{L_{eq}}}\int_{t_0}^{t} v(\tau)\,d\tau$$

$$\Rightarrow \boxed{L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}\right)^{-1}}$$

#13 25 Kasım 2018

Voltage division



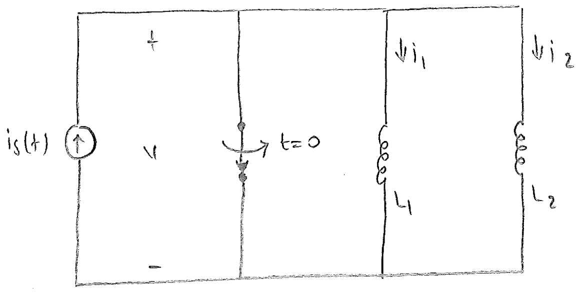
$$\left.\begin{aligned} v_1 &= L_1 \frac{di}{dt} \\ v_2 &= L_2 \frac{di}{dt} \end{aligned}\right\} \quad \frac{v_1}{L_1} = \frac{v_2}{L_2} \quad (1)$$

$$\text{KVL: } \quad v_1 + v_2 = v_s \quad (2)$$

(1) & (2) :

$$\boxed{\begin{aligned} v_1(t) &= \frac{L_1}{L_1+L_2} v_s(t) \\ v_2(t) &= \frac{L_2}{L_1+L_2} v_s(t) \end{aligned}}$$

Current division



$$i_1(0^-) = I_{10} \ , \quad i_2(0^-) = I_{20}$$

$$i_1(t),\ i_2(t) = \ ? \quad \text{for} \quad t>0$$

Since both inductors have the same voltage

$$i_1(t) = i_1(0^-) + \frac{1}{L_1}\int_{0^-}^{t} v(\tau)\,d\tau \quad \Rightarrow \int_{0^-}^{t} v(\tau)\,d\tau = L_1\left(i_1(t) - I_{10}\right)$$

$$i_2(t) = i_2(0^-) + \frac{1}{L_2}\int_{0^-}^{t} v(\tau)\,d\tau \quad \Rightarrow \int_{0^-}^{t} v(\tau)\,d\tau = L_2\left(i_2(t) - I_{20}\right)$$

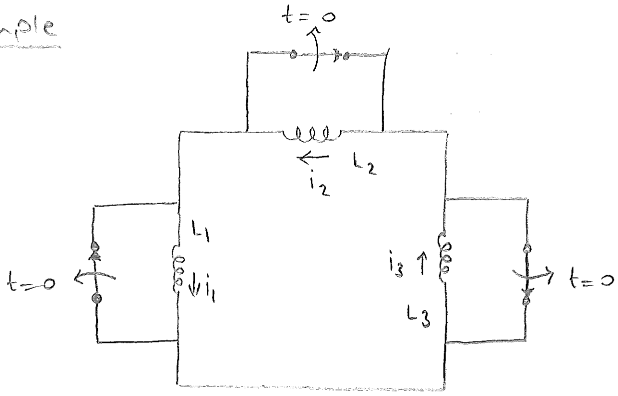
$$\Rightarrow \quad L_1\left(i_1(t) - I_{10}\right) = L_2\left(i_2(t) - I_{20}\right) \quad (1)$$

$$\text{KCL: } \quad i_1(t) + i_2(t) = i_s(t) \quad (2)$$

$$(1)\,\&\,(2) \Rightarrow \quad \boxed{\begin{aligned} i_1(t) &= \frac{L_2}{L_1+L_2} i_s(t) + \frac{L_1 I_{10} - L_2 I_{20}}{L_1+L_2} \\ \& \quad i_2(t) &= \frac{L_1}{L_1+L_2} i_s(t) + \frac{L_2 I_{20} - L_1 I_{10}}{L_1+L_2} \end{aligned}} \quad \text{for } t>0$$

#14 25 Kasım 2018

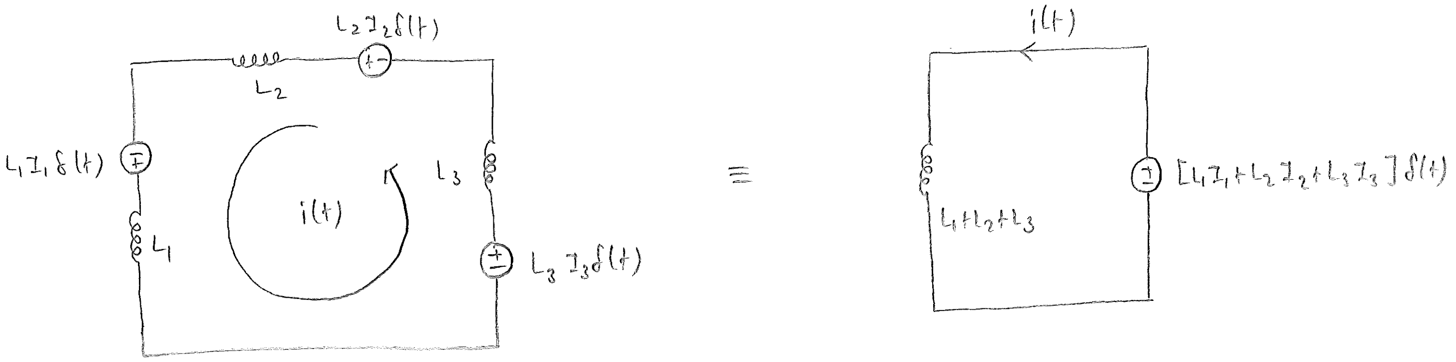
Example



$i_1(0^-) = I_1$, $i_2(0^-) = I_2$, $i_3(0^-) = I_3$

$i_1(0^+) = ?$

Sol'n 1 Use initial condition model



[No initial current on inductors]

$$i(0^+) = \frac{1}{L_1+L_2+L_3}\int_{0^-}^{0^+}[L_1I_1+L_2I_2+L_3I_3]\,\delta(t)\,dt = \frac{L_1I_1+L_2I_2+L_3I_3}{L_1+L_2+L_3}$$

Sol'n 2

$$i(t) = i_k(0^-) + \frac{1}{L_k}\int_{0^-}^{t} v_k(\tau)\,d\tau \qquad k=1,2,3$$

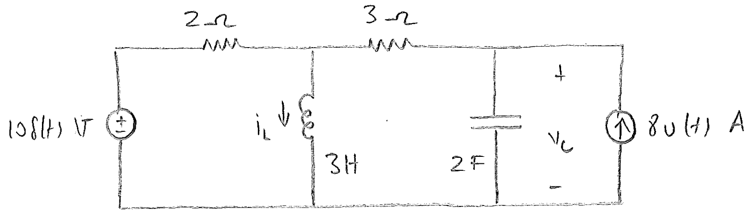
$$\Rightarrow \int_{0^-}^{t} v_k(\tau)\,d\tau = L_k\, i(t) - L_k I_k \qquad k=1,2,3$$

$$\Rightarrow \int_{0^-}^{t} \underbrace{[v_1(\tau)+v_2(\tau)+v_3(\tau)]\,d\tau}_{=0 \text{ by KVL}} = \sum_{k=1}^{3}\left(L_k\, i(t) - L_k I_k\right)$$

$$\Rightarrow \left(\Sigma L_k\right) i(t) = \Sigma L_k I_k \qquad \Rightarrow \qquad i(t) = \frac{\Sigma L_k I_k}{\Sigma L_k}$$

#15   25 Kasım 2018

Example

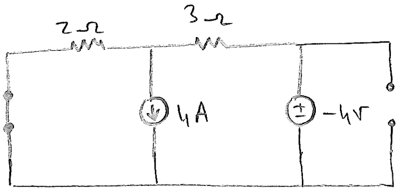


$v_c(0^-) = -4V$

$i_L(0^-) = 4A$

Find all branch voltages & currents at $t=0^-$, $t=0^+$, $t=\infty$.

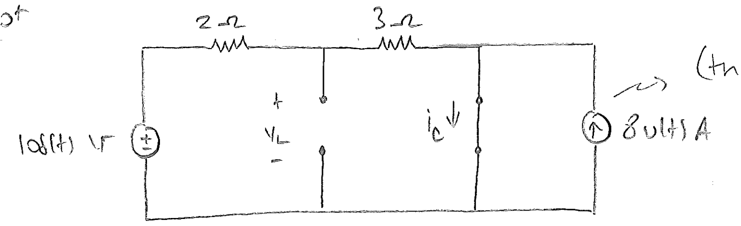
Sol'n $t=0^-$ Solve the following circuit



$0^-<t<0^+$ To determine whether there will be a jump (discontinuity) at the capacitor voltage or inductor current at $t=0$ we need to figure out the impulsive component of the capacitor current and the inductor voltage during $0^-<t<0^+$.

How to compute impulsive components? Assuming the capacitor voltage and inductor current cannot be unbounded, replace the capacitor with any bounded voltage source (e.g. short circuit) and the inductor with any bounded current source (e.g. open circuit) provided that KCL and KVL are not violated.

$0^-<t<0^+$



(this source can be killed since it is bounded therefore has no contribution to impulsive components.)

$$i_c(t) = \frac{10\delta(t)}{5} + 8u(t) = 2\delta(t) + 8u(t)\ A$$

$$v_L(t) = V_{3\Omega}(t) = \frac{3}{3+2} \cdot 10\delta(t) = 6\delta(t)\ V$$

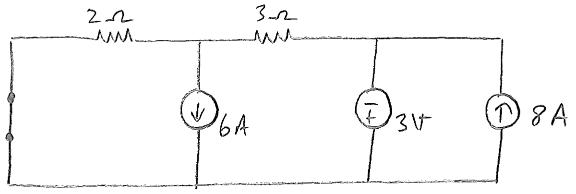
Now we can compute $v_c(0^+)$ and $i_L(0^+)$

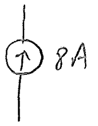
$$v_c(0^+) = v_c(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_c(t)\,dt = -4 + \frac{1}{2}\int_{0^-}^{0^+} (2\delta(t) + \underbrace{8u(t)}_{})\,dt = \boxed{-3V}$$

this term has no effect since bounded

$$i_L(0^+) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{0^+} v_L(t)\,dt = 4 + \frac{1}{3}\int_{0^-}^{0^+} 6\delta(t)\,dt = \boxed{6A}$$

<u>$t=0^+$</u> Solve the following circuit

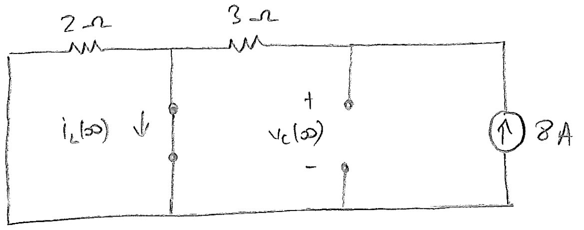


<u>$t=\infty$</u> The only remaining source in the circuit for $t>0$ is  8A DC current source. "Assuming" the circuit reaches the $\underbrace{\text{DC steady state}}_{\text{every voltage and current becomes constant}}$ as $t\to\infty$ we can proceed as follows.

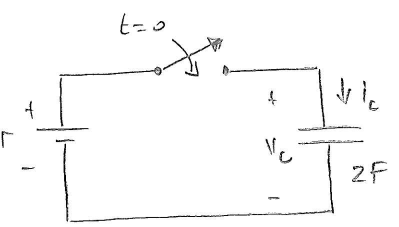
$$i_c(\infty) = C \left.\cancel{\frac{dv_c(t)}{dt}}^{\,0}\right|_{t=\infty} = 0$$ Hence the capacitor behaves as open circuit at DC steady state.

$$v_L(\infty) = L \left.\cancel{\frac{di_L(t)}{dt}}^{\,0}\right|_{t=\infty} = 0$$ Hence the inductor behaves as short circuit at DC steady state.

Solve the following circuit for $t=\infty$ values.

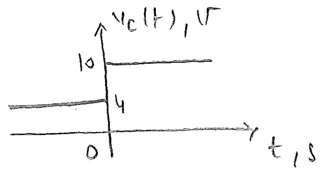


<u>Example</u> compute $i_c(t)$.



$v_c(0^-) = 4V$

<u>Sol'n</u> Note that



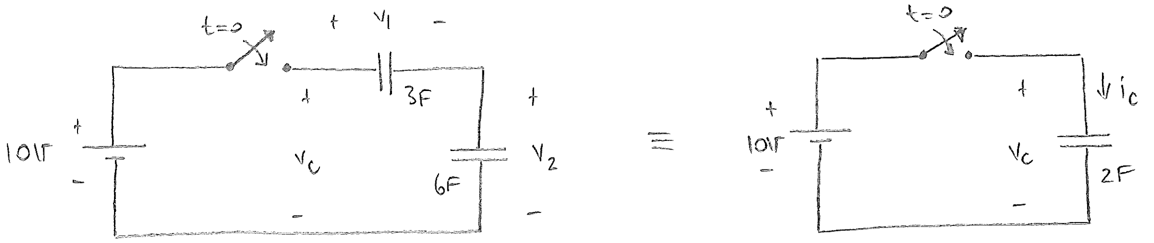
That is, $v_c(t) = 4 + 6u(t)$ V

Hence $i_c(t) = C\frac{d}{dt}v_c(t)$

$$= 2\frac{d}{dt}\{4+6u(t)\} = \boxed{12\delta(t)\ A}$$

#17 25 Kasım 2018

Example



$v_1(0^-) = 3V$, $v_2(0^-) = 1V$

$v_2(t) = ?$ for $t>0$

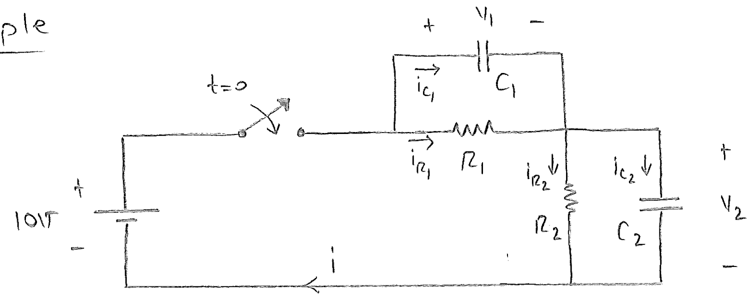
$v_c(0^-) = 3+1 = 4V$

$\Rightarrow i_c(t) = 12\delta(t)$ A

Hence, $v_2(t) = v_2(0^-) + \frac{1}{6}\int_{0^-}^{t} i_c(\tau)\,d\tau$

$$= 1 + \frac{1}{6}\int_{0^-}^{t} 12\delta(\tau)\,d\tau = \boxed{3V} \quad \text{(for } t>0\text{)}$$

Example



$v_1(0^-) = v_2(0^-) = 0$

Find $v_2(0^+)$ & $v_2(\infty)$.

t=0⁺

$$v_1(0^+) = \frac{1}{C_1}\int_{0^-}^{0^+} i_{c_1}\,dt = \frac{1}{C_1}\int_{0^-}^{0^+} (i - i_{R_1})\,dt = \frac{1}{C_1}\int_{0^-}^{0^+} i\,dt - \cancelto{0 \text{ (why?)}}{\frac{1}{C_1}\int_{0^-}^{0^+} \frac{v_1}{R_1}\,dt}$$

likewise, $v_2(0^+) = \frac{1}{C_2}\int_{0^-}^{0^+} i\,dt$

Hence $C_1 v_1(0^+) = C_2 v_2(0^+)$

Also, $v_1(0^+) + v_2(0^+) = 10$

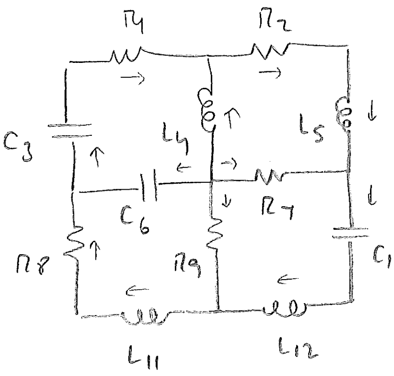
$$\left.\right\} \quad v_2(0^+) = \boxed{\frac{C_1}{C_1+C_2}\,10\,V}$$

t=∞ (DC steady state, capacitors become open circuit)

$$v_2(\infty) = \boxed{\frac{R_2}{R_1+R_2}\,10\,V}$$

#18 25 Kasım 2018

Example Show that all branch currents / voltages remain bounded for $t \geq 0$.



$v_3(0) = 3V$, $v_6(0) = 6V$, $v_{10}(0) = 10V$

$i_4(0) = 4A$, $i_5(0) = 5A$, $i_{11}(0) = 11A$, $i_{12}(0) = 12A$

$R_\ell = \ell\,\Omega$, $C_\ell = \ell F$, $L_\ell = \ell H$ $\qquad \ell = 1, 2, \dots, 12$

Question: How to solve this circuit?

Answer: Trick is not to.

Sol'n Total stored energy at time t, $E(t) = ?$

$$E(t) = E_C(t) + E_L(t) = \sum_{k \in \text{Cap.}} \tfrac{1}{2} C_k v_k(t)^2 + \sum_{k \in \text{Ind.}} \tfrac{1}{2} L_k i_k(t)^2 \qquad (1)$$

$\dot{E}(t) = ?$

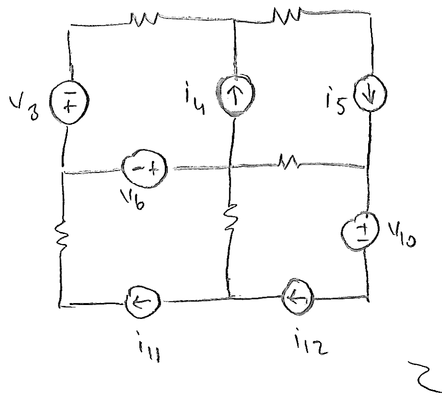
$$\dot{E} = \sum_{k \in \text{Cap.}} \underbrace{C_k \dot{v}_k}_{i_k} v_k + \sum_{k \in \text{Ind.}} \underbrace{L_k \dot{i}_k}_{v_k} i_k = \sum_{k \in \text{Cap., Ind.}} i_k v_k \qquad (2)$$

Tellegen's Thm $\Rightarrow$ $0 = \sum_{\text{all } k} i_k v_k = \sum_{k \in \text{Cap., Ind.}} i_k v_k + \sum_{k \in \text{Res.}} i_k v_k \qquad (3)$

(2)&(3) $\Rightarrow$ $\dot{E} = -\sum_{k \in \text{Res.}} i_k v_k = -\sum_{k \in \text{Res.}} R_k i_k^2 \leq 0 \qquad (4)$

(1)&(4) $\Rightarrow$ $E(t)$ bounded for all t $\Rightarrow$ All cap. volt. & ind. curr. bounded for all t.

Substitution thm. $\Rightarrow$ Replace cap. & ind. with IVS & ICS respectively



In circuit N all currents and voltages are bounded because

bounded input $\Rightarrow$ bounded output (linearity)

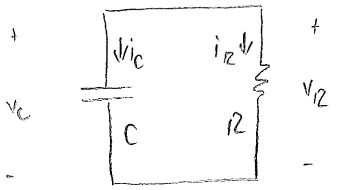
N

↳ LTI res. circuit

#1 3 Aralık 2018

$\boxed{\text{Ch. VI}}$ First Order Circuits

zero input response Response of a circuit to initial conditions only. (No input.)



$v_C(0) = V_0$

$v_C(t) = ?$ for $t \geq 0$ (Notation : $Df := \frac{d}{dt} f(t)$)

$i_C = C D v_C$

$$\Rightarrow D v_C = \frac{1}{C} i_C \overset{\text{KCL}}{=} -\frac{1}{C} i_R \overset{\text{terminal eqn.}}{=} -\frac{1}{C} \frac{v_R}{R} \overset{\text{KVL}}{=} -\frac{1}{RC} v_C$$

$$\Rightarrow D v_C + \frac{1}{RC} v_C = 0 \quad (1)$$

Eqn. (1) is a first-order homogeneous [i.e. right-hand side zero] differential equation. (The right-hand side is zero because we have no input.) Solution to (1) is of the form $v_C(t) = ke^{st}$ where k & s are to be determined.

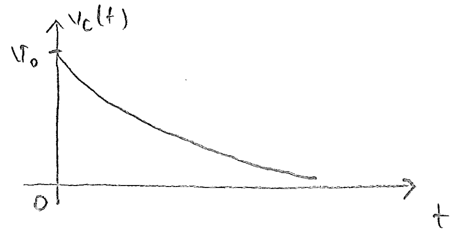
$$\frac{d}{dt}\{ke^{st}\} + \frac{1}{RC}\{ke^{st}\} = 0 \Rightarrow ske^{st} + \frac{1}{RC} ke^{st} = 0$$

$$\Rightarrow \left[s + \frac{1}{RC}\right] ke^{st} = 0 \Rightarrow s = -\frac{1}{RC} \quad : \text{"natural frequency"}$$

Then $v_C(t) = ke^{-t/RC}$ for $t \geq 0$. How about k ?

Initial cond. constraint : $V_0 = v_C(0) = ke^{-t/RC}\Big|_{t=0} = k \Rightarrow k = V_0$

Hence $\boxed{v_C(t) = V_0 e^{-t/RC}}$



Remark $\tau = RC$ is sometimes called the "time constant". Note that the smaller the time constant the faster the response.

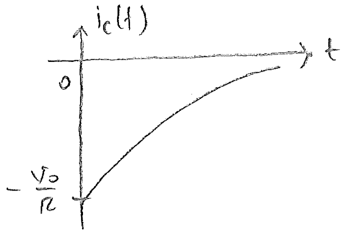
#2 3 Aralık 2018

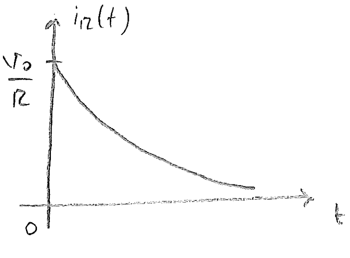
How about $i_C(t)$, $i_R(t)$?

$$i_C(t) = C\frac{d}{dt}v_C(t) = C\frac{d}{dt}\left\{v_o e^{-t/RC}\right\} = -\frac{V_o}{R}e^{-t/RC}$$

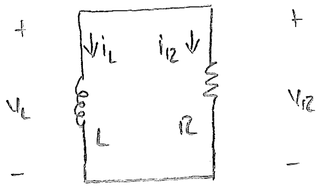
$$i_R(t) = \frac{v_R(t)}{R} = \frac{v_C(t)}{R} = \frac{V_o}{R}e^{-t/RC}$$

Note that $i_C(t)+i_R(t)=0$ (KCL) as expected.





Example $i_L(t) = ?$ for $t\geq 0$



$i_L(0) = I_o$

$$Di_L = \frac{1}{L}v_L = \frac{1}{L}v_R = \frac{1}{L}Ri_R = -\frac{R}{L}i_L$$

$$\Rightarrow Di_L + \frac{R}{L}i_L = 0$$

Natural freq. $s = -\frac{R}{L} \Rightarrow i_L(t) = Ke^{-\frac{R}{L}t}$, $K = ?$

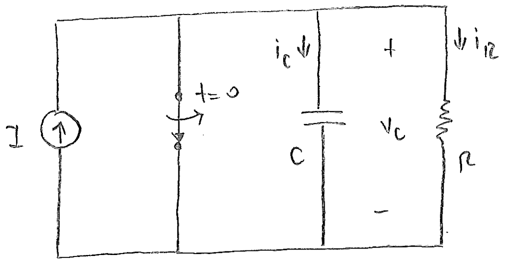
Init. cond. constraint : $i_L(0) = I_o \Rightarrow K = I_o$

$\Rightarrow \boxed{i_L(t) = I_o e^{-\frac{R}{L}t}}$ [Sometimes we write $i_L(t) = I_o e^{-t/\tau}$ where $\tau = \frac{L}{R}$ is the time constant.]

— o —

Zero-state response Response of a circuit with zero init. cond. to an input.

Example [constant input]



Note that $v_C(0^-) = 0$

"zero initial condition"

For $t>0$ we can write

$$\left.\begin{aligned} Dv_C &= \frac{1}{C}i_C \\ &= \frac{1}{C}(I-i_R) \\ &= \frac{1}{C}\left(I-\frac{v_C}{R}\right)\end{aligned}\right\} Dv_C + \frac{1}{RC}v_C = \underbrace{\frac{I}{C}}_{\text{input term}}$$

Solution is of the form: $v_C(t) = \underbrace{v_h(t)}_{\text{homogeneous solution}} + \underbrace{v_p(t)}_{\text{particular sol'n}}$

#3 3 Aralık 2018

to find $v_h(t)$ ignore the righthand side (i.e. consider the homogeneous diff. eqn.)

$$Dv_c + \frac{1}{RC} v_c = 0 \Rightarrow \underbrace{v_h(t) = Ke^{-t/RC}}_{\text{hmg. solution}}$$

hmg. solution (K to be determined later)

Natural freq. $= -\frac{1}{RC}$ (time constant $= RC$)

to find $v_p(t)$: v_p has the same "form" as input [Ex input constant $\Rightarrow$ v_p constant]

Hence let $v_p(t) = A$ (constant)

Substitute $v_p(t)$ in the diff. eqn.

$$\left.\frac{d}{dt} v_p(t) + \frac{1}{RC} v_p(t) = \frac{I}{C}\right|_{v_p(t)=A} \Rightarrow \frac{A}{RC} = \frac{I}{C} \Rightarrow A = IR$$

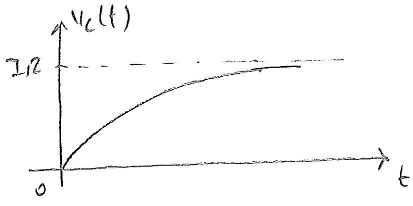
$\Rightarrow v_p(t) = IR$

Hence the overall solution : $v_c(t) = v_h(t) + v_p(t) = Ke^{-t/RC} + IR$ (K yet unknown)

To find K, use the initial cond. constraint :

$$\left.Ke^{-t/RC} + IR\right|_{t=0} = 0 \Rightarrow K = -IR$$

Finally, $v_c(t) = \boxed{IR(1-e^{-t/RC})}$



Remark Note that in our example $\lim_{t\to\infty} v_h(t) = 0$. This is due to that the natural freq. is negative. Such circuits (i.e. circuits whose natural frequencies have negative real parts) are called stable. Otherwise, they are called unstable.

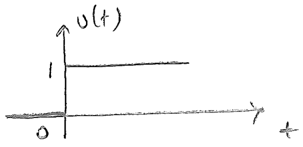
#4 3 Aralık 2018

Step Response

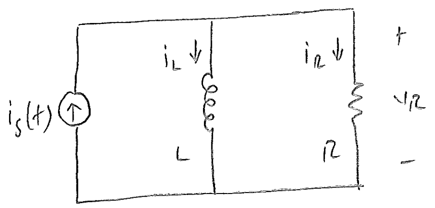
The step response of a circuit is its zero-state response to a unit step excitation. That is, we set

1) Initial conditions to zero

2) input = $u(t)$
↳ $[i_s(t)$ or $v_s(t)]$



Example Find the step response $v_R(t)$



Sol'n Diff eqn. ?

$$\left.\begin{aligned} Di_L &= \frac{1}{L} v_L \\ &= \frac{1}{L} R i_R \\ &= \frac{R}{L}(i_s - i_L) \end{aligned}\right\} \quad Di_L + \frac{R}{L} i_L = \frac{R}{L} i_s \qquad (i_s(t) = u(t))$$

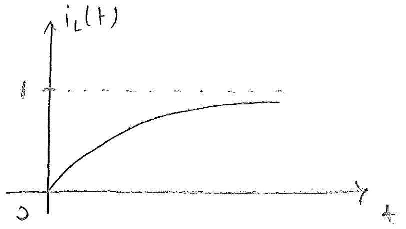
Hence for $t > 0$:

$$\left.\begin{aligned} &Di_L + \frac{R}{L} i_L = \frac{R}{L} \\ &i_L(0) = 0 \end{aligned}\right\} \quad \begin{aligned} &\text{hmg. sol'n} \quad i_h(t) = ke^{-\frac{R}{L}t} \\ &\text{part. sol'n} \quad i_p(t) = A \Rightarrow \frac{R}{L} A = \frac{R}{L} \Rightarrow A = 1 \end{aligned}$$

$$\Rightarrow i_L(t) = i_h(t) + i_p(t) = 1 + ke^{-\frac{R}{L}t}$$

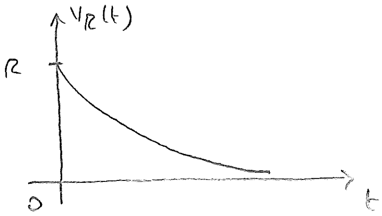
init. cond. constraint : $1 + ke^{-\frac{R}{L}t}\Big|_{t=0} = 0 \Rightarrow k = -1$

$$\Rightarrow \quad i_L(t) = 1 - e^{-\frac{R}{L}t} \quad (t > 0)$$



$v_R(t) = ?$

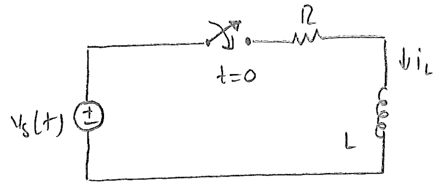
$$\begin{aligned} v_R(t) &= R i_R(t) = R(i_s(t) - i_L(t)) \\ &= R\left[1 - \left(1 - e^{-\frac{R}{L}t}\right)\right] \\ &= Re^{-\frac{R}{L}t} \quad (t > 0) \end{aligned}$$



OR,
$v_R(t) = v_L(t) = LDi_L(t)$

#5   3 Aralık 2018

Example [Sinusoidal input]



Find $i_L(t)$ for $t>0$.

$v_s(t) = V_0 \cos(\omega t + \phi)$

Sol'n $\quad LDi_L = v_L = v_s - Ri_L \quad \Rightarrow \quad Di_L + \frac{R}{L} i_L = \frac{1}{L} v_s$

Hence $\quad Di_L + \frac{R}{L} i_L = \frac{V_0 \cos(\omega t + \phi)}{L}$ [Diff. Eqn.] (Note: natural freq. $s = -\frac{R}{L}$

$i_L(0) = 0$ [Init. Cond.]

Then hmg. sol'n : $i_h(t) = Ke^{-\frac{R}{L}t}$

part. sol'n : $i_p(t) = A\cos\omega t + B\sin\omega t$ (i.e. a sinusoidal function whose freq. ω same as that of the input.)

To be determined: A, B, K (use the diff. eqn. to find A, B ; use the initial cond. constraint to find K)

$Di_p + \frac{R}{L} i_p = \frac{V_0}{L} \cos(\omega t + \phi)$ $\qquad$ $(\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta)$

$$\Rightarrow \frac{d}{dt}\{A\cos\omega t + B\sin\omega t\} + \frac{R}{L}\{A\cos\omega t + B\sin\omega t\} = \frac{V_0}{L}\{\cos\phi\cos\omega t - \sin\phi\sin\omega t\}$$

$$\Rightarrow -A\omega\sin\omega t + B\omega\cos\omega t + \frac{AR}{L}\cos\omega t + \frac{BR}{L}\sin\omega t = \frac{V_0\cos\phi}{L}\cos\omega t - \frac{V_0\sin\phi}{L}\sin\omega t$$

$$\Rightarrow \underbrace{\left\{B\omega + \frac{AR}{L} - \frac{V_0\cos\phi}{L}\right\}}_{=0}\cos\omega t + \underbrace{\left\{-A\omega + \frac{BR}{L} + \frac{V_0\sin\phi}{L}\right\}}_{=0}\sin\omega t = 0$$

$$\Rightarrow \begin{bmatrix} \frac{R}{L} & \omega \\ -\omega & \frac{R}{L} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{V_0}{L}\cos\phi \\ -\frac{V_0}{L}\sin\phi \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \frac{V_0/L}{\frac{R^2}{L^2} + \omega^2} \begin{bmatrix} \frac{R}{L} & -\omega \\ \omega & \frac{R}{L} \end{bmatrix} \begin{bmatrix} \cos\phi \\ -\sin\phi \end{bmatrix}$$

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$$\Rightarrow i_L(t) = i_h(t) + i_p(t) = ke^{-\frac{R}{L}t} + A\cos\omega t + B\sin\omega t$$

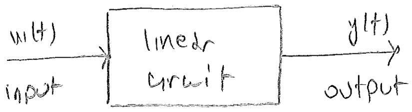
$$i_L(0) = 0 \Rightarrow k = -A$$

Hence $\boxed{i_L(t) = -Ae^{-\frac{R}{L}t} + A\cos\omega t + B\sin\omega t}$ $(t>0)$

Let $R = 3\Omega$, $L = 2H$, $\omega = 2\text{rad/sec}$, $\phi = 0°$, $V_0 = 10V$

$$\Rightarrow A = \frac{6}{5} \text{ \& } B = \frac{8}{5} \Rightarrow i_L(t) = \underbrace{-\frac{6}{5}e^{-\frac{3}{2}t}}_{\text{transient part}} + \underbrace{\frac{6}{5}\cos 2t + \frac{8}{5}\sin 2t}_{\text{(sinusoidal) steady state part}} \text{ Amps}$$

Superposition in linear dynamic circuits



Note that

$$y(t) = y(t, \underset{\text{initial condition (vector)}}{x}, \underset{\text{input signal (vector)}}{w(\cdot)})$$

init. cond. $x = [v_{C_1}(0^-) \cdots v_{C_m}(0^-), i_{L_1}(0^-) \cdots i_{L_n}(0^-)]^T$

input $w(t) = [v_{s_1}(t) \cdots v_{s_p}(t), i_{s_1}(t) \cdots i_{s_q}(t)]^T$

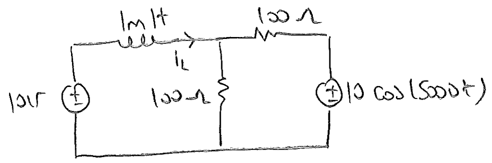
We have :

1) $y(t, \alpha_1 x_1 + \alpha_2 x_2, 0) = \alpha_1 y(t, x_1, 0) + \alpha_2 y(t, x_2, 0)$ superposition w.r.t. init. cond.

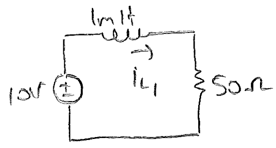
2) $y(t, 0, \beta_1 w_1 + \beta_2 w_2) = \beta_1 y(t, 0, w_1) + \beta_2 y(t, 0, w_2)$ superposition w.r.t. inputs

3) $y(t, \alpha x, \beta w) = \alpha y(t, x, 0) + \beta y(t, 0, w)$ superposition w.r.t. init. cond. & input

Exercise Find the zero-state response

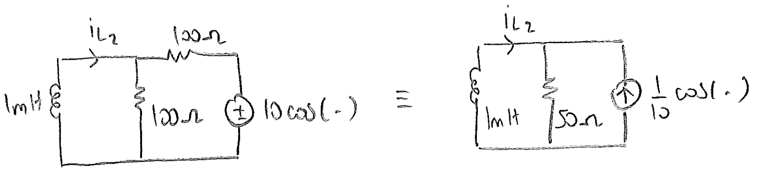


Step 1 Kill AC input to find i_{L_1}



$i_{L_1}(0) = 0$

Step 2 Kill DC input to find i_{L_2}



$i_{L_2}(0) = 0$

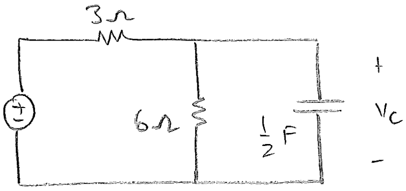
Step 3 Superpose

$$i_L(t) = i_{L_1}(t) + i_{L_2}(t)$$

#7 3 Aralık 2018

Complete response [nonzero input + nonzero init. cond.]

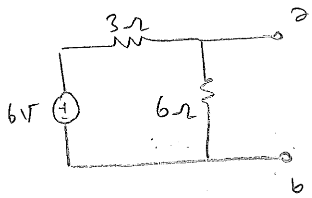
Example



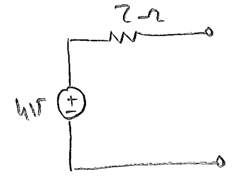
$v_c(0) = 2V$

$v_c(t) = ?$ for $t \geq 0$

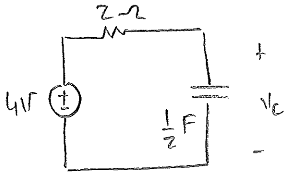
Sol'n



$\equiv$



$\Rightarrow$



$v_c(0) = 2V$

KVL: $0 = -4 + 2i_c + v_c = -4 + 2\left[\frac{1}{2}Dv_c\right] + v_c \Rightarrow Dv_c + v_c = 4$ (diff. eqn.)

hmg. sol'n: $v_h(t) = ke^{-t}$

prt. sol'n: $v_p(t) = 4$

$$\left.\begin{matrix} v_h(t) = ke^{-t} \\ v_p(t) = 4 \end{matrix}\right\} v_c(t) = 4 + ke^{-t} \xrightarrow[(v_c(0)=2V)]{} \boxed{v_c(t) = 4 - 2e^{-t}\ V}$$

Another approach (superposition)

Step 1 Find zero-state resp. $v_{zs}(t)$

$Dv_c + v_c = 4$ with $v_c(0) = 0 \Rightarrow v_{zs}(t) = 4 - 4e^{-t}$ V

Step 2 Find zero-input resp. $v_{zi}(t)$

$Dv_c + v_c = 0$ with $v_c(0) = 2V \Rightarrow v_{zi}(t) = 2e^{-t}$ V

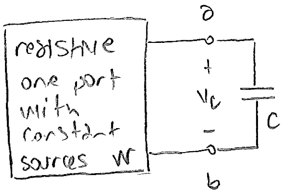
Step 3 Superpose

$v_c(t) = v_{zs}(t) + v_{zi}(t)$

$= 4 - 2e^{-t}$ V

(as expected)

Yet another approach



$v_c(0) = V_0$

What do we know?

→ form of diff. eqn. $Dv_c + \frac{1}{\tau}v_c = \text{constant}$

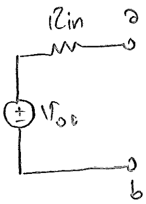
→ form of sol'n $v_c(t) = A + Be^{-t/\tau}$ (1)

How to find A, B, τ ?

$$\left.\begin{matrix} (1) \Rightarrow v_c(\infty) = A \\ (2) \Rightarrow v_c(\infty) = V_{oc} \end{matrix}\right\} \boxed{A = V_{oc}}$$

$(2) \Rightarrow \boxed{\tau = R_{in}C}$

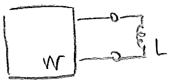
Let $N \equiv$



(2)

$(1) \Rightarrow A + B = v_c(0) \Rightarrow B = v_c(0) - v_c(\infty) \Rightarrow \boxed{B = V_0 - V_{oc}}$

Exercise Work out the dual case



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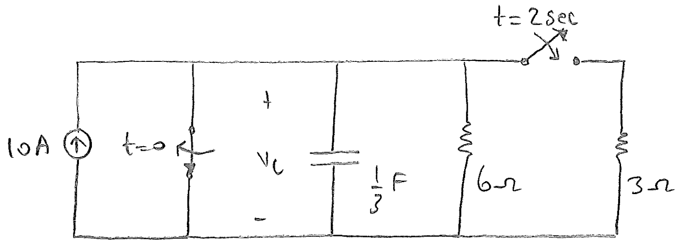
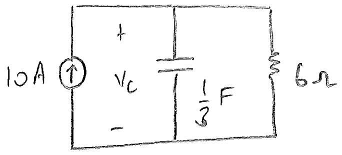
#8 3 Aralık 2018

Example



Find $v_c(t)$ for $t \geq 0$.

Sol'n $\underline{0 \leq t < 2}$



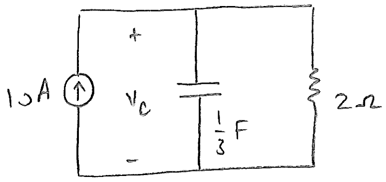
$v_c(0) = 0$

KCL $\Rightarrow$ $\frac{1}{3}Dv_c + \frac{v_c}{6} = 10 \Rightarrow Dv_c + \frac{1}{2}v_c = 30$

$$\left.\begin{aligned} v_h(t) &= Ke^{-t/2} \\ v_p(t) &= 60 \end{aligned}\right\} \begin{aligned} v_c(t) &= 60 + Ke^{-t/2} \\ &= 60 - 60e^{-t/2} \end{aligned} \quad \begin{aligned} &\hookleftarrow v_c(0) = 0 \\ &V \end{aligned}$$

at $t = 2^-$ $v_c(2) = 60\left(1 - \frac{1}{e}\right) =: V_2 \cong 38V$

$\underline{t \geq 2}$



$v_c(2) = V_2$

$$Dv_c + \frac{3}{2}v_c = 30$$

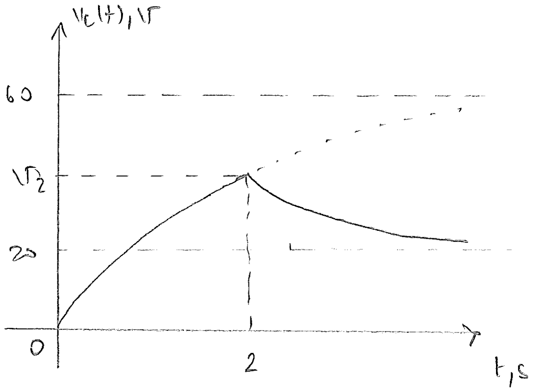
$$\left.\begin{aligned} v_h(t) &= Ke^{-\frac{3}{2}(t-2)} \\ v_p(t) &= 20\,V \end{aligned}\right\} \begin{aligned} v_c(t) &= 20 + Ke^{-\frac{3}{2}(t-2)} \\ &= 20 + [V_2 - 20]e^{-\frac{3}{2}(t-2)} \end{aligned} \quad \begin{aligned} &\hookleftarrow v_c(2) = V_2 \\ &V \end{aligned}$$

OR,

$$\left.\begin{aligned} v_h(t) &= \tilde{K}e^{-\frac{3}{2}t} \\ v_p(t) &= 20V \end{aligned}\right\} \begin{aligned} v_c(t) &= 20 + \tilde{K}e^{-\frac{3}{2}t} \\ &= 20 + [(V_2 - 20)e^3]e^{-\frac{3}{2}t} \end{aligned} \quad \begin{aligned} &\hookleftarrow v_c(2) = V_2 \\ &V \end{aligned}$$

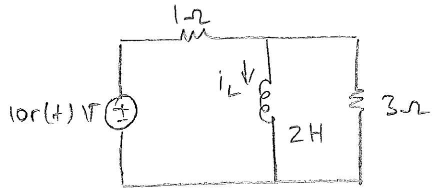
Hence

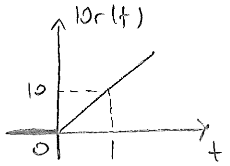
$$v_c(t) = \begin{cases} 60[1 - e^{-t/2}]\ V & \text{for } 0 \leq t < 2 \\ 20 + [V_2 - 20]e^{-\frac{3}{2}(t-2)} & \text{for } t \geq 2 \end{cases}$$



#9 3 Aralık, 2018

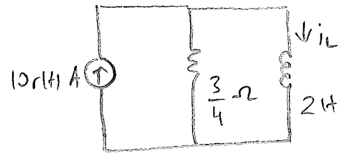
Example [Ramp excitation] Find $i_L(t)$.





$i_L(0) = 0$

Sol'n Equivalent circuit:



KCL: $i_L + \frac{2Di_L}{3/4} = 10r(t) \overset{(t\geq 0)}{\Rightarrow} Di_L + \frac{3}{8}i_L = \frac{15}{4}t$ (1)

$\Rightarrow i_h(t) = Ke^{-\frac{3}{8}t}$ & $i_p(t) = A+Bt$

$i_p(t) = ?$ i_p should solve (1). Hence

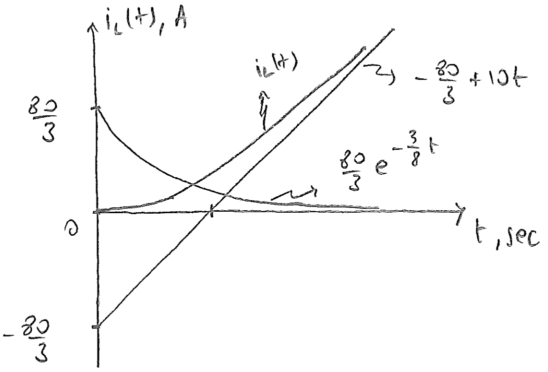
$$\frac{d}{dt}\{A+Bt\} + \frac{3}{8}\{A+Bt\} = \frac{15}{4}t \Rightarrow \underbrace{[B+\frac{3}{8}A]}_{0} + \underbrace{\frac{3B}{8}}_{\frac{15}{4}}t = \frac{15}{4}t$$

$\Rightarrow B = 10$ & $A = -\frac{80}{3}$ $\Rightarrow i_p(t) = -\frac{80}{3} + 10t$ Amps

$\Rightarrow i_L(t) = -\frac{80}{3} + 10t + Ke^{-\frac{3}{8}t}$

$i_L(0) = 0 \Rightarrow K = \frac{80}{3}$

$$\boxed{i_L(t) = -\frac{80}{3} + 10t + \frac{80}{3}e^{-\frac{3}{8}t} \text{ Amps}}$$

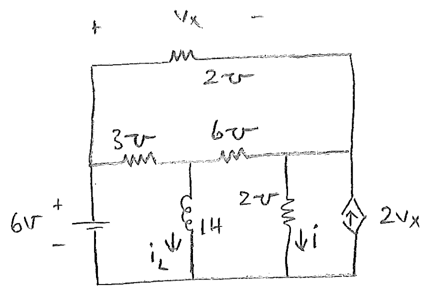


$i_L(t)$ remains positive because $i_L(0) = 0$ &

$Di_L = 10 - 10e^{-\frac{3}{8}t} > 0$ for $t > 0$

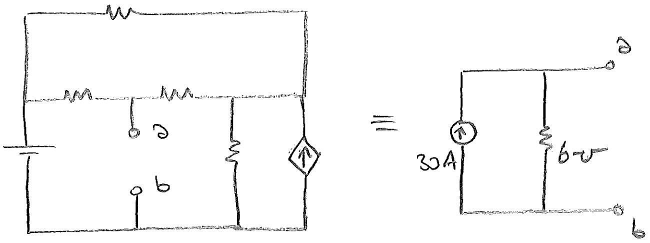
#1 21 Aralık 2015

Example

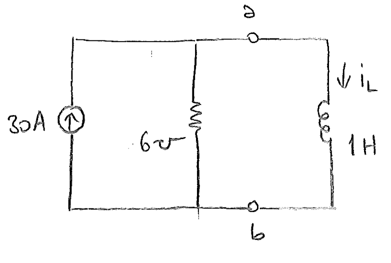


$i_L(0) = 6A$, $i(t) = ?$ for $t \geq 0$

Step1 Find the norton / thevenin equiv. circuit seen by the inductor



Step2 Using the equiv. circuit find $i_L(t)$



$i_L(0) = 6A$

KCL : $-30 + i_L + 6LDi_L = 0$

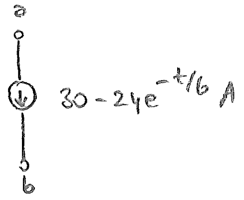
$\Rightarrow Di_L + \frac{1}{6} i_L = 5 \Rightarrow i_L(t) = 30 + Ke^{-t/6}$

$i_L(0) = 6 \Rightarrow K = -24 \Rightarrow i_L(t) = 30 - 24e^{-t/6}$ A

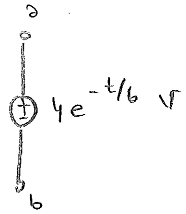
$\Rightarrow v_L(t) = 4e^{-t/6}$ V

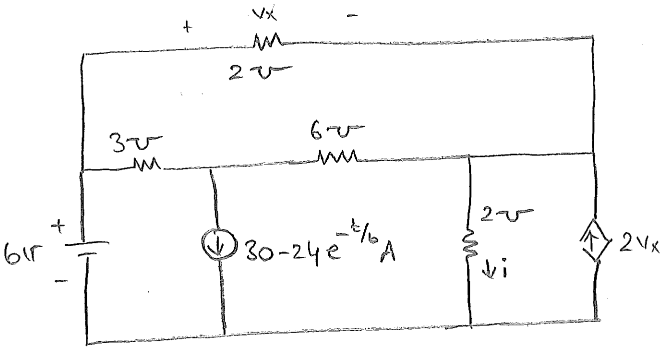
Step3 Substitute

with



or

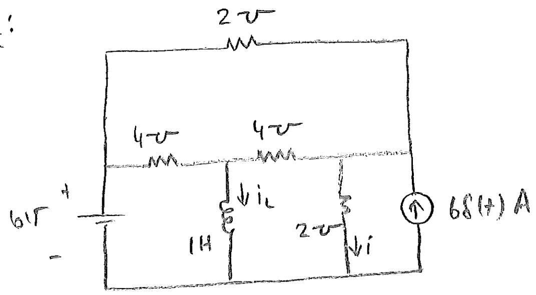




Solve for i

Answer : $\boxed{i(t) = 4 + 4e^{-t/6} \text{ A}}$ for $t \geq 0$

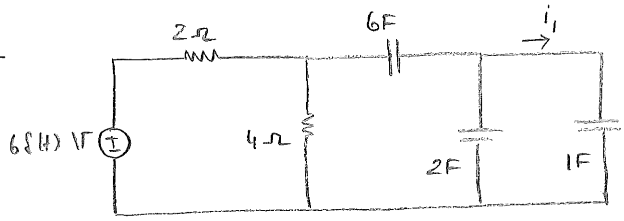
Exercise:



$i_L(0^-) = 6A$, $i(t) = ?$ for $t > 0$

#2 21 Aralık 2015

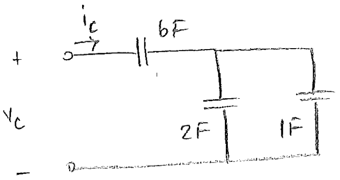
Example



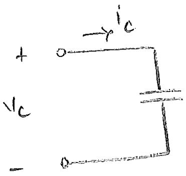
Capacitors initially uncharged.

Find $i_1(t)$ for $t>0$.

Sol'n



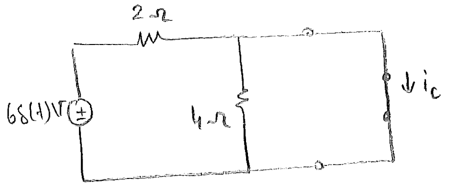
$\equiv$



2F = $(2 /\!/ 1) \gg 6$

$v_c(0^-) = 0$

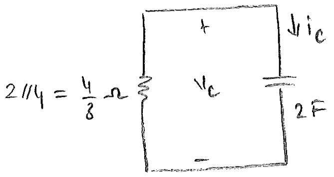
$0^- < t < 0^+$



$$i_c(t) = \frac{6\delta(t)}{2} = 3\delta(t) \text{ A}$$

$$\Rightarrow v_c(0^+) = v_c(0^-) + \frac{1}{2}\int_{0^-}^{0^+} i_c(t)\,dt = \frac{1}{2}\int_{0^-}^{0^+} 3\delta(t)\,dt = \frac{3}{2} \text{ V}$$

$t>0$



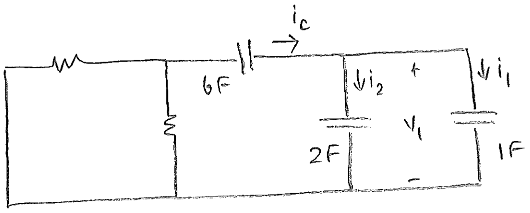
$v_c(0^+) = \frac{3}{2}$ V

$$Dv_c + \frac{3}{8}v_c = 0$$

$$\Rightarrow v_c(t) = \frac{3}{2}e^{-\frac{3t}{8}} \text{ V}$$

$$\Rightarrow i_c(t) = 2Dv_c(t) = -\frac{9}{8}e^{-\frac{3t}{8}} \text{ A}$$

Return to the original configuration:



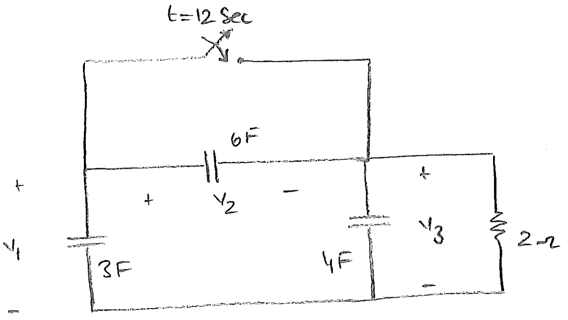
$$\left.\begin{array}{l} i_1 = Dv_1 \\ i_2 = 2Dv_1 \end{array}\right\} \Rightarrow i_2 = 2i_1$$

$$i_c = i_1 + i_2 = 3i_1 \Rightarrow i_1 = \frac{1}{3}i_c$$

$$\boxed{\Rightarrow i_1(t) = -\frac{3}{8}e^{-\frac{3}{8}t} \text{ A}} \quad \text{for } t>0$$

#3 21 Aralık 2015

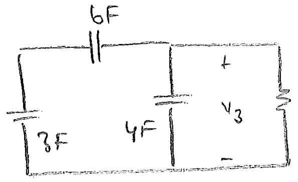
Example



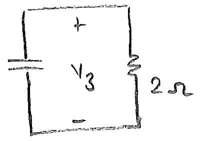
$v_1(0) = 5V$, $v_2(0) = 1V$

$v_3(t) = ?$ for $t \geq 0$

$0 \leq t < 12$



$\equiv$ $(6 \gg 3) \parallel 4 = 6F$



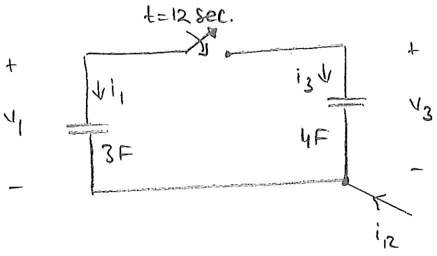
$v_3(0) = v_1(0) - v_2(0)$
$= 4V$

$\Rightarrow v_3(t) = v_3(0) e^{-t/RC}$, $\boxed{v_3(t) = 4e^{-t/12} \; V}$

how about $v_1(t)$, $v_2(t)$?

$$\left.\begin{aligned} & v_1 - v_2 = v_3 \\ & v_1(t) = v_1(0) + \frac{1}{3}\int_0^t i_1(\tau)d\tau \\ & v_2(t) = v_2(0) + \frac{1}{6}\int_0^t -i_1(\tau)d\tau \end{aligned}\right\} \quad \left.\begin{aligned} 3(v_1(t) - v_1(0)) &= 6(v_2(0) - v_2(t)) \\ &= 6(v_2(0) - v_1(t) + v_3(t)) \end{aligned}\right\} \quad \begin{aligned} & v_1(t) = \frac{7}{3} + \frac{2}{3}v_3(t) \\ & v_2(t) = \frac{7}{3} - \frac{1}{3}v_3(t) \end{aligned}$$

$12^- < t < 12^+$



$v_1(12^-) \neq v_3(12^-)$ but $v_1(12^+) = v_3(12^+)$ $\Rightarrow$ impulsive current

$$v_1(12^+) = v_1(12^-) + \frac{1}{3}\int_{12^-}^{12^+} i_1(t)dt$$

$$v_3(12^+) = v_3(12^-) + \frac{1}{4}\int_{12^-}^{12^+} (-i_1(t) - \underset{\text{bounded}}{i_R(t)})\,dt = v_3(12^-) - \frac{1}{4}\int_{12^-}^{12^+} i_1(t)dt$$

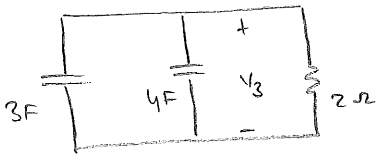
$v_1(12^-) = \frac{7}{3} + \frac{8}{3e} \; V$

$v_3(12^-) = \frac{4}{e} \; V$

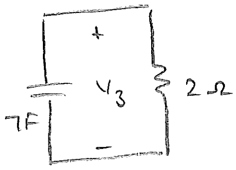
$$\Rightarrow \int i_1 = 3(v_3(12^+) - v_1(12^-)) = 4(v_3(12^-) - v_3(12^+))$$

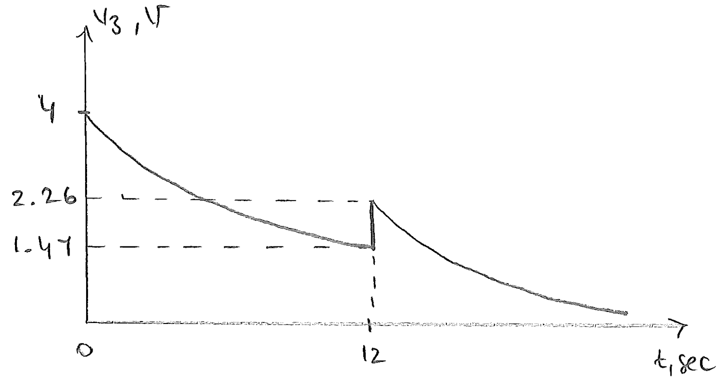
$$\Rightarrow v_3(12^+) = \frac{C_1 v_1(12^-) + C_3 v_3(12^-)}{C_1 + C_3} = \frac{1}{7}\left\{7 + \frac{8}{e} + \frac{16}{e}\right\} \cong 2.26V$$

$t > 12$



$=$



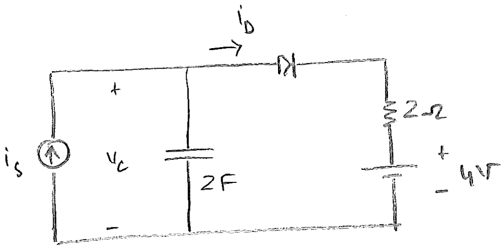


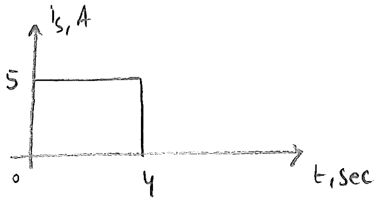
$v_3(12^+) = 2.26V$

$$\Rightarrow \boxed{v_3(t) = 2.26 e^{-(t-12)/14} \; V}$$

#4 21 Aralık 2015

Example

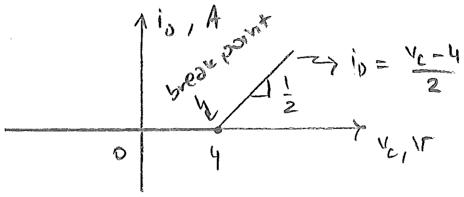


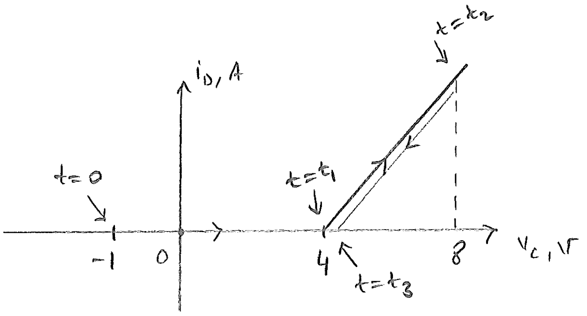


$v_c(0) = -1V$

$v_c(t) = ?$

Sol'n obtain $i_D - v_c$ char.





$t_1 = 2$ sec

$t_2 = 4$ sec

$t_3 = \infty$

<u>$0 \le t < t_1$</u> $\quad i_D = 0 \Rightarrow i_c = i_s \Rightarrow 2Dv_c = 5 \Rightarrow \boxed{v_c(t) = -1 + \frac{5}{2}t \ V}$

<u>$t_1 = ?$</u> $\quad t_1 = \min\{t_a, t_b\}$ where $\underbrace{t_a = 4 \text{ sec}}_{\text{when } i_s(t) \text{ becomes zero}}$ & $\underbrace{v_c(t_b) = 4V}_{\text{the break point of the } i_D-v_c \text{ curve}}$

$\Rightarrow -1 + \frac{5}{2}t_b = 4 \Rightarrow t_b = 2 \text{ sec} \Rightarrow t_1 = 2 \text{ sec} \Rightarrow v_c(t_1) = 4V$

<u>$t_1 \le t < t_2$</u> $\quad i_s = i_c + i_D \Rightarrow 5 = 2Dv_c + \frac{v_c - 4}{2} \Rightarrow Dv_c + \frac{1}{4}v_c = \frac{7}{2}$

$$\Rightarrow \boxed{v_c(t) = 14 - 10e^{-(t-2)/4} \ V}$$

<u>$t_2 = ?$</u> $\quad t_2 = 4$sec (when $i_s(t)$ becomes zero) $\Rightarrow v_c(t_2) = 14 - 10e^{-\frac{1}{2}} \cong 8V$

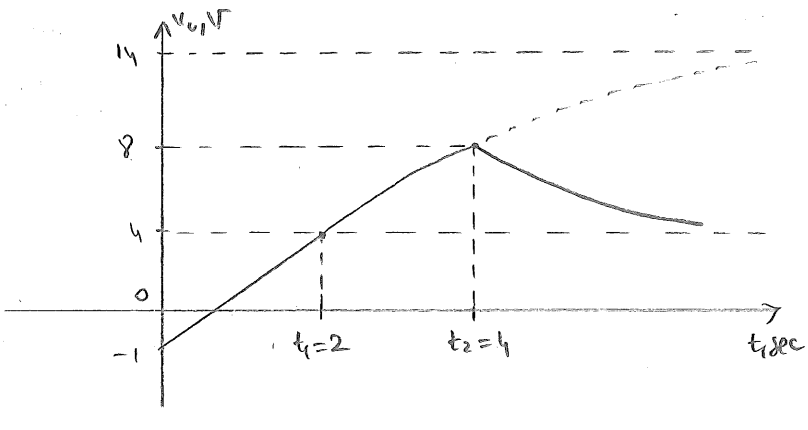
<u>$t_2 \le t < t_3$</u> $\quad i_s = 0 \Rightarrow i_c + i_D = 0 \Rightarrow 2Dv_c + \frac{v_c - 4}{2} = 0 \Rightarrow Dv_c + \frac{1}{4}v_c = 1$

$$\Rightarrow \boxed{v_c(t) = 4 + (v_c(4) - 4)e^{-\frac{1}{4}(t-4)} \ V}$$

<u>$t_3 = ?$</u> Note that $v_c(t) > 4$ for $t > t_2$

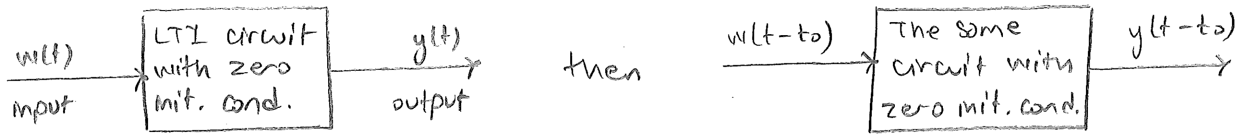
$\Rightarrow$ we never exactly reach the break point

$\Rightarrow t_3 = \infty$

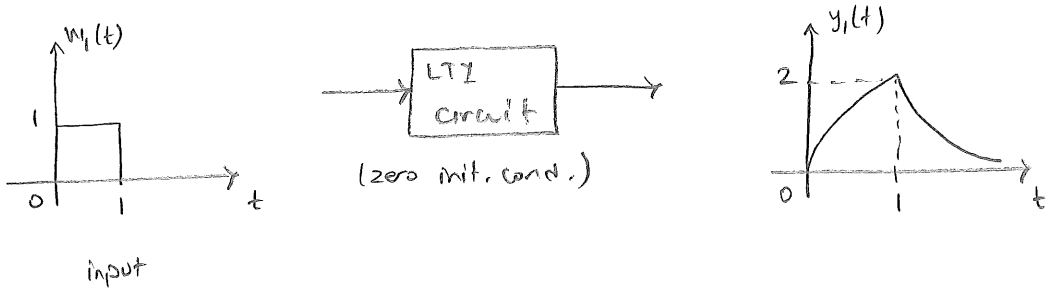


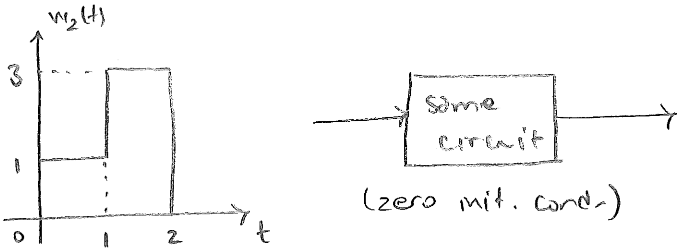
Time-invariance of the zero-state response

Let

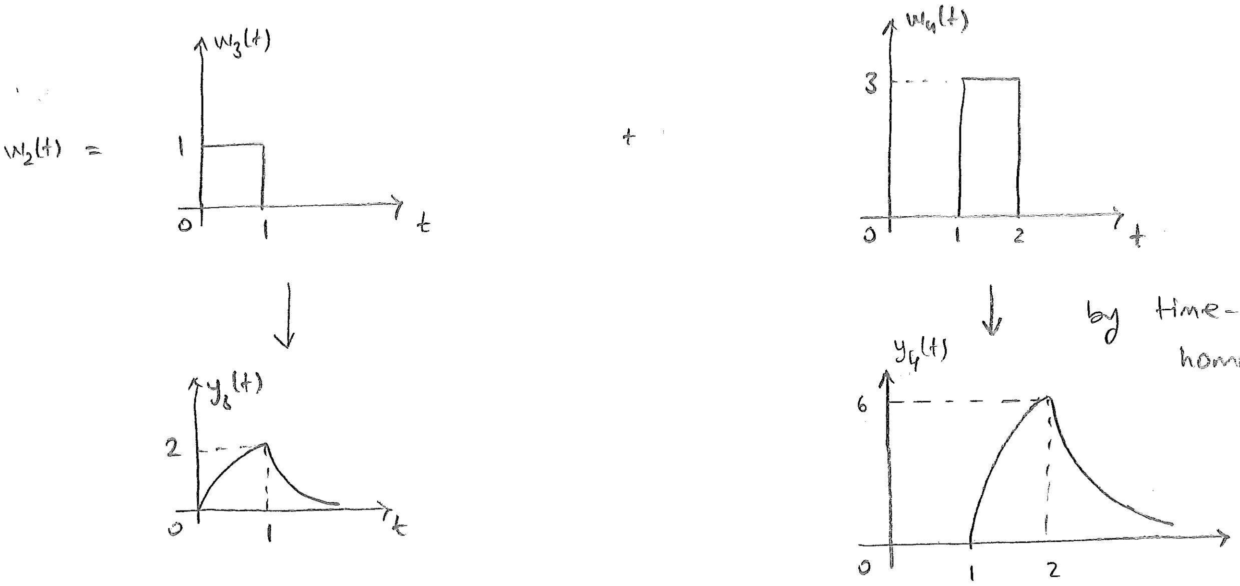


Example [time-invariance]



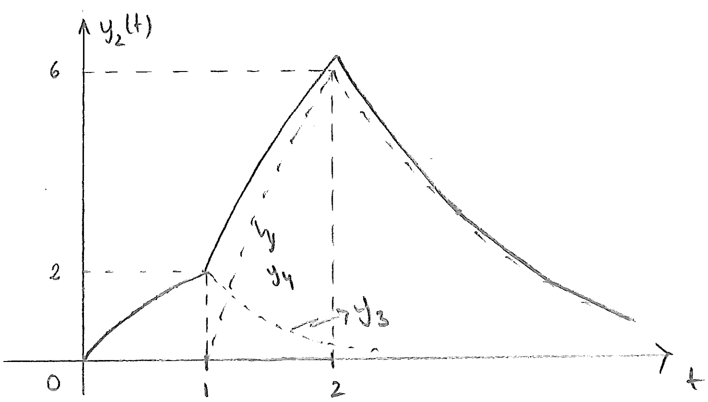


$y_2(t) = ?$



by time-invariance & homogeneity

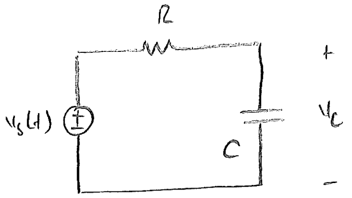
By superposition : $y_2(t) = y_3(t) + y_4(t)$



#6 21 Aralık 2015

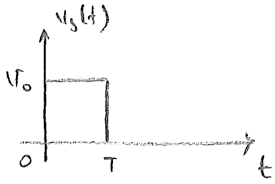
Pulse Response

Ex



$v_c(0)=0$

Find $v_c(t)$ for



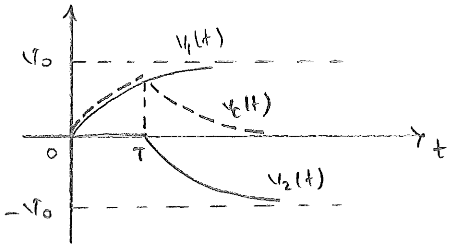
Note that $v_s(t) = V_0 u(t) - V_0 u(t-T)$

Since the initial condition is zero we can apply superposition:

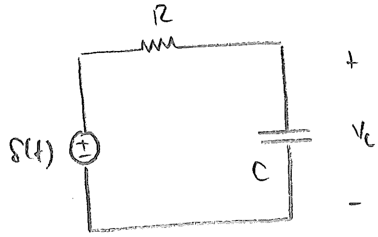
response due to $V_0 u(t)$: $v_1(t) = V_0(1-e^{-t/RC})\,u(t)$

response due to $-V_0 u(t-T)$: $v_2(t) = -v_1(t-T) = -V_0(1-e^{-(t-T)/RC})\,u(t-T)$

$v_c(t) = v_1(t) + v_2(t)$



Impulse response $\overset{\text{def}}{=}$ zero state response for unit impulse $\delta(t)$ input.



$v_c(0^-)=0$

Diff. eqn ?

$\delta(t) = Ri_c + v_c = RCDv_c + v_c$

$\Rightarrow Dv_c + \frac{1}{RC}v_c = \frac{1}{RC}\delta(t)$ (1)

In this case [that is, init. cond. = 0 & input = $\delta(t)$] the solution $v_c(t)$ to diff. eqn. (1) is called the impulse response. Impulse response is usually denoted by $h(t)$.

Consider (1). Due to $\delta(t)$ function $v_c(0^+)$ may not equal $v_c(0^-)=0$. Let $V_0 = v_c(0^+)$

$$\left.\begin{aligned}&\text{Then, for } t>0,\quad v_c(t) = V_0 e^{-t/RC}\\ &\text{Also, for } t<0,\quad v_c(t) = 0\end{aligned}\right\}\ \text{Hence we can write } v_c(t) = V_0 e^{-t/RC}u(t)$$

Question : $V_0 = ?$

Answer : Place $v_c(t) = V_0 e^{-t/RC}u(t)$ into the diff. eqn. (1).

#7 21 Aralık 2015

$$\Rightarrow \frac{d}{dt}\left\{ V_0 e^{-t/RC} u(t) \right\} + \frac{1}{RC} V_0 e^{-t/RC} u(t) = \frac{1}{RC}\delta(t)$$

$$\Rightarrow \cancel{-\frac{V_0}{RC} e^{-t/RC} u(t)} + \underbrace{V_0 e^{-t/RC}\delta(t)}_{} + \cancel{\frac{1}{RC} V_0 e^{-t/RC} u(t)} = \frac{1}{RC}\delta(t)$$

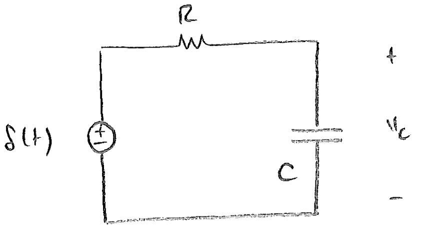
$$= V_0 e^{-t/RC}\Big|_{t=0} \cdot \delta(t)$$

$$= V_0 \delta(t) \quad \text{(sifting property)}$$

$$\Rightarrow V_0\delta(t) = \frac{1}{RC}\delta(t) \Rightarrow V_0 = \frac{1}{RC} .$$

Therefore the impulse response is $\boxed{h(t) = \frac{1}{RC} e^{-t/RC} u(t)}$

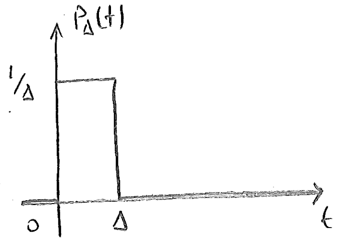
Impulse Response by limit approach



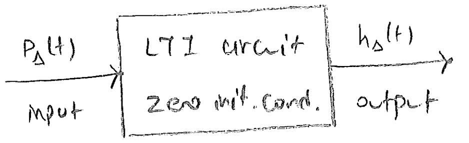
$v_c(0^-) = 0$

Approximate the unit impulse $\delta(t)$ by the pulse function:

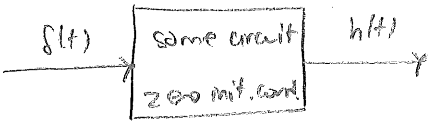
$$P_\Delta(t) = \frac{u(t) - u(t-\Delta)}{\Delta}$$



Let $h_\Delta(t)$ denote the response due to $P_\Delta(t)$



&



Since $\delta(t) = \lim_{\Delta \to 0} P_\Delta(t)$, we should have $h(t) = \lim_{\Delta \to 0} h_\Delta(t)$

$h_\Delta(t) = ?$

$$h_\Delta(t) = \frac{1}{\Delta}\left(1 - e^{-t/RC}\right) u(t) - \frac{1}{\Delta}\left(1 - e^{-(t-\Delta)/RC}\right) u(t-\Delta)$$

Clearly, $h_\Delta(t) = 0$ for $t<0$. How about for $t > \Delta$?

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for $t > \Delta$ $\quad h_\Delta(t) = \frac{1}{\Delta}(1-e^{-t/RC}) - \frac{1}{\Delta}(1-e^{\frac{\Delta}{RC}}\, e^{-t/RC})$

$$= \frac{e^{\Delta/RC}-1}{\Delta} \cdot e^{-t/RC}$$

$$\Rightarrow \lim_{\Delta\to 0} h_\Delta(t) = \underbrace{\left\{\lim_{\Delta\to 0}\frac{e^{\Delta/RC}-1}{\Delta}\right\}}_{?} e^{-t/RC} = \frac{1}{RC}e^{-t/RC}$$

$$e^{\Delta/RC} = 1 + \frac{\Delta}{RC} + \frac{1}{2}\left(\frac{\Delta}{RC}\right)^2 + \frac{1}{3!}\left(\frac{\Delta}{RC}\right)^3 + \dots$$

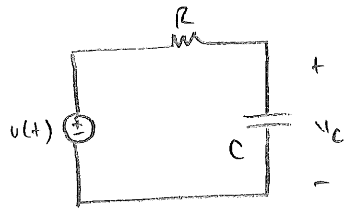
$$\frac{e^{\Delta/RC}-1}{\Delta} = \frac{1}{RC} + \underbrace{\frac{1}{2}\frac{\Delta}{(RC)^2} + \frac{1}{3!}\frac{\Delta^2}{(RC)^3} + \dots}_{\text{these vanish as } \Delta\to 0}$$

Therefore $h(t) = \lim_{\Delta\to 0} h_\Delta(t) = \begin{cases} \frac{1}{RC}e^{-t/RC} & \text{for } t>0 \\ 0 & \text{for } t<0 \end{cases}$

$\Rightarrow \boxed{h(t) = \frac{1}{RC}e^{-t/RC}\, u(t)}$ as expected.

— o —

Let's find the step response of the same circuit. [Let s(t) denote the step response.]



$v_c(0^-) = 0$

For $t>0$ $\quad Dv_c + \frac{1}{RC}v_c = \frac{1}{RC} \Rightarrow v_c(t) = 1-e^{-t/RC}$

For $t<0$ $\quad v_c(t) = 0$

Hence $\boxed{s(t) = (1-e^{-t/RC})\, u(t)}$

Recall : $\delta(t) = Du(t)$

Claim : $h(t) = Ds(t)$. Proof : $\frac{d}{dt}s(t) = \underbrace{\frac{1}{RC}e^{-t/RC}u(t)}_{h(t)} + \underbrace{(1-e^{-t/RC})\delta(t)}_{=(1-e^{-t/RC})\big|_{t=0}\delta(t) = 0\,\delta(t) = 0}$.

☒

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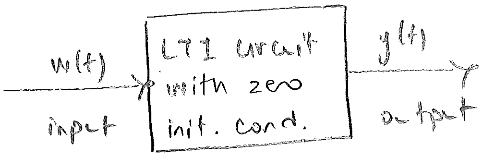
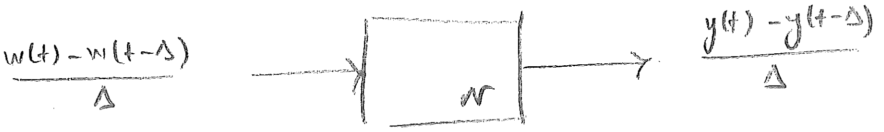
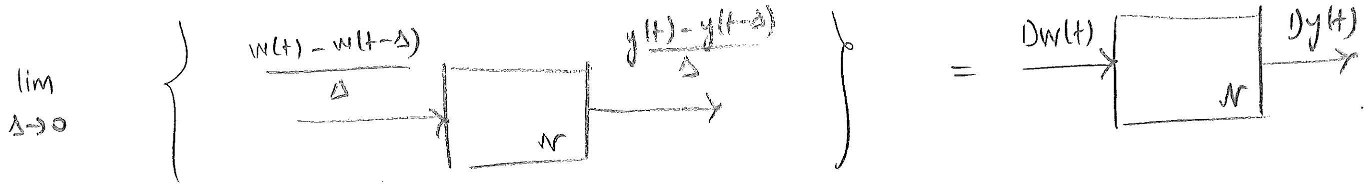
This is true in general. Let



Then by time-invariance & homogeneity



Take the limit $\Delta \to 0$

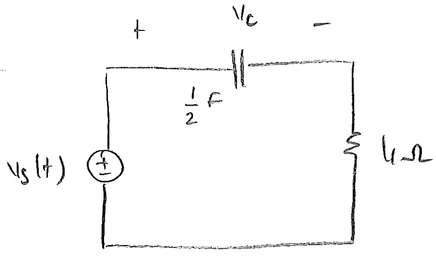


Exercise: For the previous circuit compute the ramp response and verify

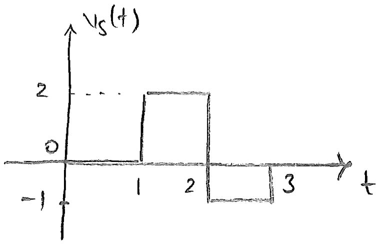
$$\frac{d}{dt}\left\{\begin{matrix}\text{ramp}\\ \text{response}\end{matrix}\right\} = \left\{\begin{matrix}\text{step}\\ \text{response}\end{matrix}\right\}.$$

[Ramp response $\overset{\text{def}}{=}$ zero-state response to unit ramp r(t) excitation.]

Example



$v_c(0)=0$



$v_c(t) = ?$

Sol'n step response $s(t) = (1-e^{-t/2})\,u(t)$ V

input: $v_s(t) = 2u(t-1) - 3u(t-2) + u(t-3)$ V

Therefore: $v_c(t) = 2s(t-1) - 3s(t-2) + s(t-3)$.

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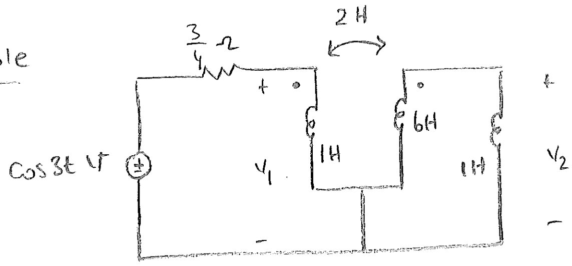
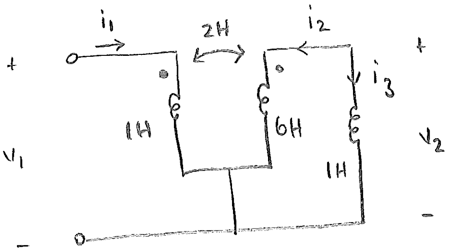
Example



Assume zero init. conditions.

Find $v_1(t)$, $v_2(t)$.

Step1 Find equiv. inductance



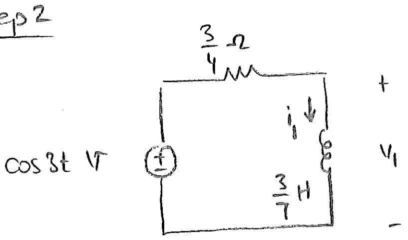
$v_1 = Di_1 + 2Di_2$ (1)

$v_2 = 6Di_2 + 2Di_1$

$v_2 = Di_3 = -Di_2$ } ⇒ $-Di_2 = 6Di_2 + 2Di_1$ ⇒ $Di_2 = -\frac{2}{7}Di_1$ (2)

(1) & (2) ⇒ $v_1 = Di_1 - \frac{4}{7}Di_1$ ⇒ $v_1 = \frac{3}{7}Di_1$ ⇒ $\boxed{L_{eq} = \frac{3}{7}H}$

Step2



$\cos 3t = \frac{3}{4}i_1 + \frac{3}{7}Di_1$

$\Rightarrow Di_1 + \frac{7}{4}i_1 = \frac{7}{3}\cos 3t$

$i_h(t) = Ke^{-\frac{7}{4}t}$ (nat. freq. $s = -\frac{7}{4}$)

$i_p(t) = A\cos 3t + B\sin 3t$

$Di_p + \frac{7}{4}i_p = \frac{7}{3}\cos 3t$ ⇒ $-3A\sin 3t + 3B\cos 3t + \frac{7}{4}\{A\cos 3t + B\sin 3t\} = \frac{7}{3}\cos 3t$

$$\Rightarrow \underbrace{\left\{-3A + \frac{7B}{4}\right\}}_{=0}\sin 3t + \underbrace{\left\{3B + \frac{7A}{4}\right\}}_{=\frac{7}{3}}\cos 3t = \frac{7}{3}\cos 3t$$

$$\Rightarrow \begin{bmatrix} -3 & 7/4 \\ 7/4 & 3 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 7/3 \end{bmatrix}$$ compute A & B

Then $i_1(t) = i_h(t) + i_p(t)$, $i_1(0) = 0$ } $K = -A$ ⇒ $i_1(t) = A\cos 3t + B\sin 3t - Ae^{-\frac{7}{4}t}$ Amps

$v_1(t) = \frac{3}{7}Di_1(t)$ ⇒ $\boxed{v_1(t) = \frac{3}{7}\left\{-3A\sin 3t + 3B\cos 3t + \frac{7}{4}Ae^{-\frac{7}{4}t}\right\} V}$

$v_2 = -Di_2$, $Di_2 = -\frac{2}{7}Di_1$ } $v_2 = \frac{2}{7}Di_1$ & $v_1 = \frac{3}{7}Di_1$ ⇒ $\boxed{v_2(t) = \frac{2}{3}v_1(t)}$

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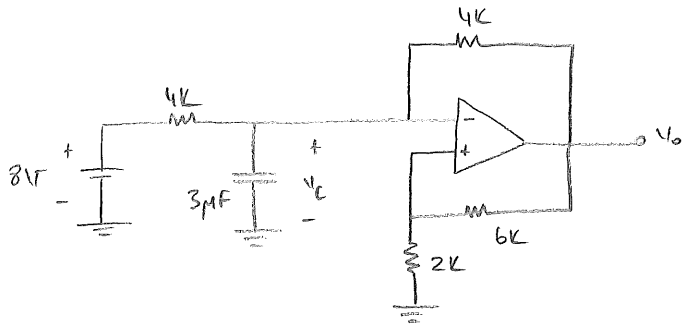
Example



$v_c(0) = 0$

$v_o(0) = 0$

$E_s = 16V$

$v_c(t)$, $v_o(t) = ?$

Node eqn. $\frac{v_c - 8}{4k} + 3\mu Dv_c + \frac{v_c - v_o}{4k} = 0 \Rightarrow Dv_c + \frac{500}{3}v_c = \frac{250}{3}(v_o + 8)$ (1)

Also, $v_+ = \frac{v_o}{4}$ & $v_- = v_c$

Initially, $|v_o(0)| < 16$. Hence OP-AMP is in linear region.

linear : $v_+ = v_-$, $|v_o| < 16$ V

$v_+ = v_- \Rightarrow v_o = 4v_c$

(1) $\Rightarrow Dv_c + \frac{500}{3}v_c = \frac{250}{3}(4v_c + 8) \Rightarrow Dv_c - \frac{500}{3}v_c = \frac{2000}{3}$ & $v_c(0) = 0$

$\Rightarrow \boxed{v_c(t) = -4 + 4e^{\frac{500}{3}t} \text{ V}}$ & $\boxed{v_o(t) = -16 + 16e^{\frac{500}{3}t} \text{ V}}$ for $0 \le t < t_s$

$t_s = ?$ $v_o(t_s) = -16 + 16e^{\frac{500}{3}t_s} = 16 \Rightarrow \boxed{t_s = \frac{3}{500}\ln 2 \text{ sec}}$ & $v_c(t_s) = 4$ V

at $t = t_s$ OPAMP leaves the linear region!

+sat region ? $v_+ > v_-$, $v_o = 16V \Rightarrow v_+ = 4V$

(1) $\Rightarrow Dv_c + \frac{500}{3}v_c = 2000$, $v_c(t_s) = 4V$

$\Rightarrow v_c(t) = 12 - 8e^{-\frac{500}{3}(t - t_s)}$ V (2)

However, (2) $\Rightarrow v_c(t) > 4$ for $t > t_s \Rightarrow v_- > 4 \Rightarrow v_- > v_+ \Rightarrow$ +sat impossible.

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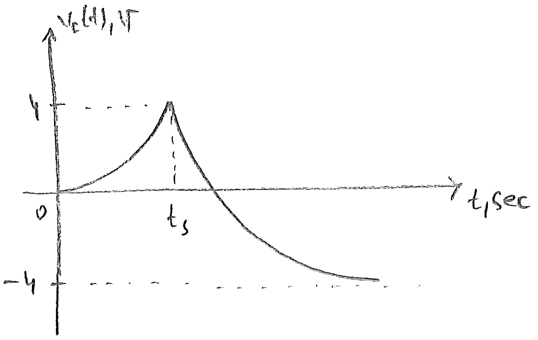
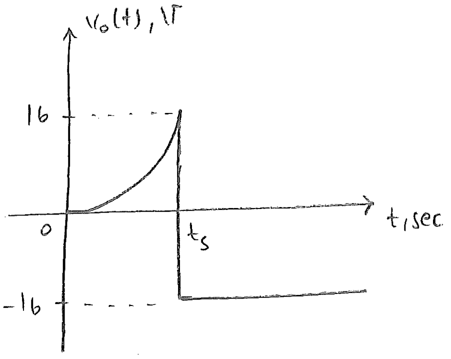
- sat region $v_+ < v_-$, $v_o = -16V \Rightarrow v_+ = -4V$

$$(1) \Rightarrow \left.\begin{matrix} Dv_c + \frac{500}{3} v_c = -\frac{2000}{3} \\ v_c(t_s) = 4V \end{matrix}\right\} \quad \boxed{v_c(t) = -4 + 8e^{-\frac{500}{3}(t-t_s)} \; V} \quad \& \quad \boxed{v_o(t) = -16V}$$

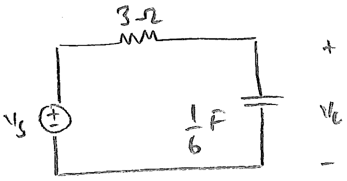
for $t > t_s$





Note that $v_+ < v_-$ for $t > t_s$.

Example [Exponential Input]



Find $v_c(t)$ for

a) $\alpha = -3$

b) $\alpha = -2$

$v_c(0) = 6V$, $v_s(t) = e^{\alpha t}$ V

Sol'n   Diff eqn. $Dv_c + \frac{1}{RC} v_c = \frac{1}{RC} v_s \Rightarrow Dv_c + 2v_c = 2e^{\alpha t}$

Nat. freq. $s = -\frac{1}{RC} = -2$

a) $\alpha = -3$ $Dv_c + 2v_c = 2e^{-3t}$

$$\left.\begin{matrix} v_p(t) = Ae^{-3t} \\ v_h(t) = Ke^{-2t} \end{matrix}\right\} \begin{matrix} Dv_p + 2v_p = 2e^{-3t} \Rightarrow -3Ae^{-3t} + 2Ae^{-3t} = 2e^{-3t} \Rightarrow A = -2 \\ \Rightarrow v_c(t) = v_h(t) + v_p(t) = Ke^{-2t} - 2e^{-3t} , \; v_c(0) = 6 \Rightarrow K = 8 \end{matrix}$$

$$\Rightarrow \boxed{v_c(t) = 8e^{-2t} - 2e^{-3t} \; V}$$

b) $\underline{\alpha = -2}$ $\quad Dv_c + 2v_c = 2e^{-2t}$

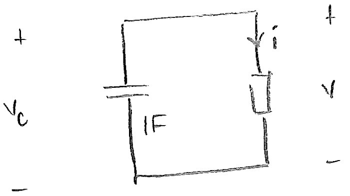
Since α coincides with nat. freq. $s = -2$, $v_p(t) = Ate^{-2t}$

$$Dv_p + 2v_p = 2e^{-2t} \Rightarrow \left\{ Ae^{-2t} - \cancel{2Ate^{-2t}} \right\} + \cancel{2Ate^{-2t}} = 2e^{-2t} \Rightarrow A = 2$$

$$\Rightarrow v_c(t) = v_h(t) + v_p(t) = ke^{-2t} + 2te^{-2t} \quad , \quad v_c(0) = 6 \Rightarrow k = 6$$

$$\Rightarrow \boxed{v_c(t) = 6e^{-2t} + 2te^{-2t} \ V}$$

Example [LTV resistor]



$$v = \frac{1}{1 + \frac{1}{2}\cos t}\, i$$

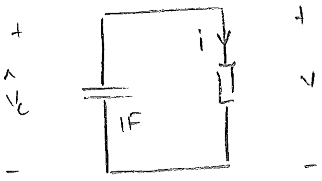
$v_c(0) = 1V$, $v_c(t) = ?$

Sol'n $\quad Dv_c = i_c = -i = -\left(1 + \frac{1}{2}\cos t\right) v_c \Rightarrow \frac{dv_c}{v_c} = -\left[1 + \frac{1}{2}\cos t\right] dt$

$$\Rightarrow \int \frac{dv_c}{v_c} = -\int \left[1 + \frac{1}{2}\cos t\right] dt \Rightarrow \ln v_c = -\left(t + \frac{1}{2}\sin t\right) + \text{constant}$$

$$\Rightarrow v_c(t) = ke^{-\left(t + \frac{1}{2}\sin t\right)} \quad , \quad v_c(0) = 1 \Rightarrow k = 1 \Rightarrow \boxed{v_c(t) = e^{-\left[t + \frac{1}{2}\sin t\right]} \ V}$$

How about ?



$$v = \frac{1}{1 + \frac{1}{2}\cos t}\, i$$

$\hat{v}_c(\pi) = 1V$, $\hat{v}_c(t) = ?$ for $t \geq \pi$

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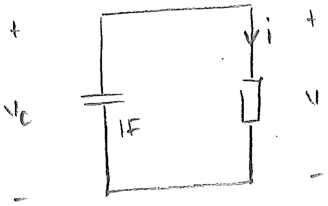
Sol'n $\hat{v}_c(t) = \hat{k}\, e^{-(t+\frac{1}{2}\sin t)}$, $\hat{v}_c(\pi) = 1 \Rightarrow \hat{k} = e^{\pi}$

$$\Rightarrow \hat{v}_c(t) = e^{\pi} e^{-[t+\frac{1}{2}\sin t]} = \boxed{e^{-[(t-\pi)+\frac{1}{2}\sin t]} \quad V}$$

Remark Note that the time-invariance property is lost. That is, $\hat{v}_c(t) \neq v_c(t-\pi)$

Example [Nonlinear resistor]



$i = v^3$

$v_c(0) = V_0 > 0$, $v_c(t) = ?$ for $t \geq 0$

Sol'n $Dv_c = i_c = -i = -v_c^3$

$$\Rightarrow -\frac{dv_c}{v_c^3} = dt \Rightarrow \frac{1}{2} v_c^{-2} = t + \text{constant} \Rightarrow v_c(t) = \frac{1}{\sqrt{2t+k}}$$

$$v_c(0) = V_0 \Rightarrow \frac{1}{\sqrt{k}} = V_0 \Rightarrow k = \frac{1}{V_0^2}$$

$$\Rightarrow \boxed{v_c(t) = \frac{V_0}{\sqrt{1+2V_0^2 t}}} \text{ for } t \geq 0$$

Remark Note that the homogeneity property is lost. That is,

Let $v_c(0) = V_0$ & $\hat{v}_c(0) = \lambda V_0$. But $\hat{v}_c(t) \neq \lambda v_c(t)$.

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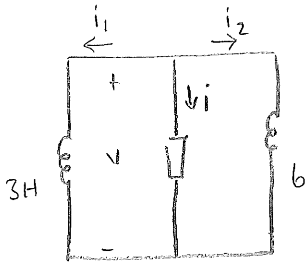
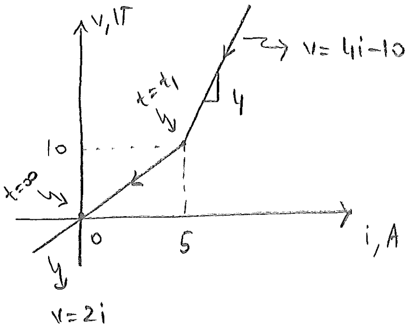
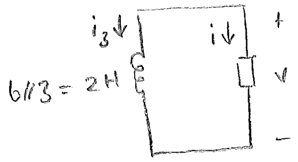
Example





$i_1(0) = 3A$, $i_2(0) = -10\,A$

$i_1(t)$, $i_2(t) = ?$



$i_3(0) = i_1(0) + i_2(0) = -7A$

0≤t<t₁

$i(0) = 7A \Rightarrow v = 4i - 10$

$2Di_3 = v = 4(-i_3) - 10 \Rightarrow Di_3 + 2i_3 = -5$

$$\Rightarrow \boxed{i_3(t) = -\frac{5}{2} - \frac{9}{2}e^{-2t}\ A}$$

$t_1 = ?$ $\quad i(t_1) = 5 \Rightarrow i_3(t_1) = -5 \Rightarrow -5 = -\frac{5}{2} - \frac{9}{2}e^{-2t_1} \Rightarrow \boxed{t_1 = \frac{1}{2}\ln\frac{9}{5}\ \text{sec}}$

$t \geq t_1$ $\quad v = 2i \Rightarrow 2Di_3 = v = -2i_3 \Rightarrow Di_3 + i_3 = 0 \Rightarrow i_3(t) = i_3(t_1)e^{-(t-t_1)}$

$$\Rightarrow \boxed{i_3(t) = -5e^{-(t-t_1)}\ A}$$

$$\Rightarrow i_3(t) = \begin{cases} -\frac{5}{2} - \frac{9}{2}e^{-2t}\ A & \text{for} \quad 0 \leq t < t_1 \\ -5e^{-(t-t_1)}\ A & \text{for} \quad t \geq t_1 \end{cases}$$

$i_1(t), i_2(t) = ?$

$$\left.\begin{aligned} i_1(t) &= i_1(0) + \frac{1}{3}\int_0^t v(\tau)d\tau \\ i_2(t) &= i_2(0) + \frac{1}{6}\int_0^t v(\tau)d\tau \end{aligned}\right\} \begin{aligned} &3(i_1(t) - 3) = 6(i_2(t) + 10) &(1) \\ &\text{Also,}\quad i_1 + i_2 = i_3 &(2) \end{aligned}$$

$(1)\&(2) \Rightarrow 3i_1 - 9 = 6i_3 - 6i_1 + 60 \Rightarrow 9i_1 = 6i_3 + 69 \Rightarrow$

$$\boxed{\begin{aligned} i_1(t) &= \frac{2}{3}i_3(t) + \frac{23}{3}\,A \\ \&\quad i_2(t) &= \frac{1}{3}i_3(t) - \frac{23}{3}\,A \end{aligned}}$$

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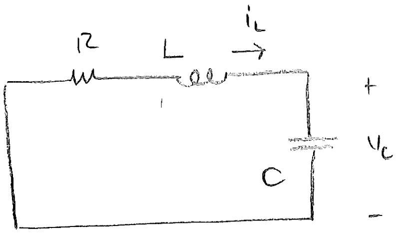
Ch. III

Second Order Circuits

zero-input response

series RLC circuit



$i_L(0) = I_0$

$v_C(0) = V_0$

formulation variable: v_C

KVL : $v_R + v_L + v_C = 0$

$$\Rightarrow R i_C + L D i_C + v_C = 0 \qquad i_C = C D v_C$$

$$\Rightarrow RCDv_C + LCD^2 v_C + v_C = 0$$

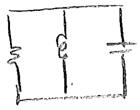
$$\Rightarrow \boxed{D^2 v_C + \frac{R}{L} D v_L + \frac{1}{LC} v_C = 0}$$ Diff. eqn.

init. cond. $v_C(0)$, $Dv_C(0) = ?$

$\rightarrow v_C(0) = V_0$ (given)

$\rightarrow Dv_C(0) = \frac{1}{C} i_L(0) = \frac{I_0}{C}$

Exercise : Work out the dual case (parallel RLC circuit)



formulation variable : i_L

$v_R + v_L + v_C = 0$

$$\Rightarrow R i_L + L D i_L + v_C(0) + \frac{1}{C}\int_0^t i_L(\tau)\, d\tau = 0 \qquad D$$

$$\Rightarrow RDi_L + LD^2 i_L + \frac{1}{C} i_L = 0$$

$$\Rightarrow \boxed{D^2 i_L + \frac{R}{L} D i_L + \frac{1}{LC} i_L = 0}$$ Diff. eqn.

init. cond. $i_L(0)$, $Di_L(0) = ?$

$\rightarrow i_L(0) = I_0$

$$\rightarrow Di_L(0) = \frac{1}{L} v_L(0) = \frac{1}{L}\left(-Ri_L(0) - v_C(0)\right)$$

$$= -\frac{R}{L} I_0 - \frac{1}{L} V_0$$

Characteristic polynomial : $d(s) = s^2 + \frac{R}{L} s + \frac{1}{LC}$ (char. eqn. $d(s) = 0$)

In general, characteristic eqn. can be written as

$$\boxed{s^2 + 2\alpha s + \omega_0^2 = 0}$$

Then the roots are $s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$

Def Roots of chr. poly. = natural frequencies of the circuit

$\omega_0 = \frac{1}{\sqrt{LC}}$ is called the resonant frequency

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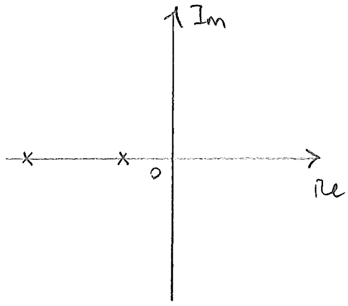
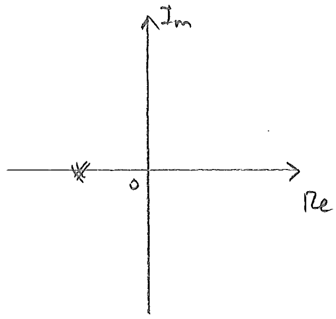
Solution of the circuit depends on the locations of the nat. frequencies on complex plane. Four cases are possible (for passive circuits)



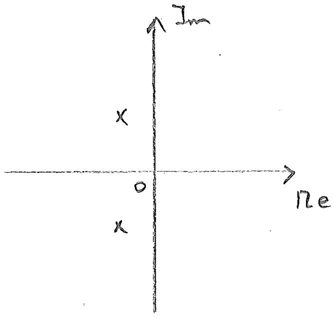
$s_1 < 0$, $s_2 < 0$, $s_1 \neq s_2$

(overdamped)



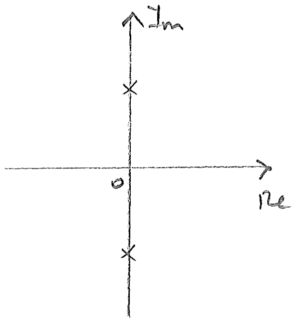
$s_1 = s_2 < 0$

(critically damped)



$s_1 = s_2^*$, $Re(s_1) < 0$

(underdamped)



$S_{1,2} = \mp j w_0$

(lossless)

Case I : $\alpha > w_0 > 0$ (circuit is overdamped)

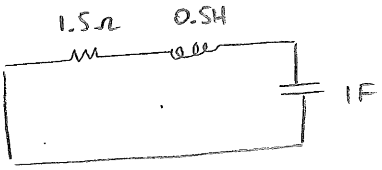
$$S_{1,2} = -\alpha \mp \underbrace{\sqrt{\alpha^2 - w_0^2}}_{\alpha_d} = -\alpha \mp \alpha_d$$

(the natural frequencies are negative, real, distinct)

$$\Rightarrow \boxed{v_c(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}}$$

Ex : $R = \frac{3}{2} \Omega$, $L = \frac{1}{2} H$, $C = 1F$

$v_c(0^-) = 2V$, $i_L(0^-) = 10A$



$\Rightarrow D^2 v_c + 3 D v_c + 2 v_c = 0$

$v_c(0^-) = 2V$, $D v_c(0^-) = \frac{i_L(0^-)}{C} = 10V/s$

$\Rightarrow$ charac. poly. $d(s) = s^2 + 3s + 2 = (s+1)(s+2)$

natural frequencies : $s_{1,2} = -1, -2$

$\Rightarrow v_c(t) = k_1 e^{-t} + k_2 e^{-2t}$

$$\left.\begin{array}{l} v_c(0) = k_1 + k_2 = 2 \\ D v_c(0) = -k_1 - 2k_2 = 10 \end{array}\right\} \left.\begin{array}{l} k_1 = 14 \\ k_2 = -12 \end{array}\right\} \boxed{v_c(t) = 14 e^{-t} - 12 e^{-2t} \ V}$$

for $t \geq 0$

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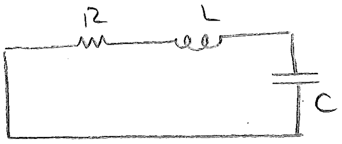
Case 2 : $\alpha = w_o > 0$ (critically damped)



$$\frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

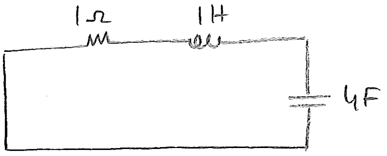
char. poly : $d(s) = s^2 + 2\alpha s + w_o^2 = (s+\alpha)^2$

natural freq : $s_{1,2} = -\alpha, -\alpha$

$$\Rightarrow \boxed{v_c(t) = k_1 e^{-\alpha t} + k_2 t e^{-\alpha t}}$$

Ex : $R = 1\Omega$, $L = 1H$, $C = 4F$

$v_c(0^-) = 2V$, $i_L(0^-) = 4A$



$$\Rightarrow D^2 v_c + D v_c + \frac{1}{4} v_c = 0$$

$v_c(0^-) = 2V$, $Dv_c(0^-) = 1V/s$

$\Rightarrow$ char. poly. $d(s) = s^2 + s + \frac{1}{4} = (s+\frac{1}{2})^2$

nat. freq: $s_{1,2} = -\frac{1}{2}, -\frac{1}{2}$

$$\Rightarrow v_c(t) = k_1 e^{-\frac{t}{2}} + k_2 t e^{-\frac{t}{2}}$$

$v_c(0) = k_1 = 2$

$Dv_c(0) = -\frac{1}{2} k_1 + k_2 = 1 \Rightarrow k_2 = 2$

$$\boxed{v_c(t) = 2e^{-t/2} + 2te^{-t/2} \; V}$$ for $t \geq 0$

Case 3 : $w_o > \alpha > 0$ (underdamped)

char. poly. $d(s) = s^2 + 2\alpha s + w_o^2$

natural freq. $s_{1,2} = -\alpha \mp j w_d$ where $w_d = \sqrt{w_o^2 - \alpha^2}$

$$\Rightarrow \boxed{v_c(t) = k_1 e^{-\alpha t} \sin w_d t + k_2 e^{-\alpha t} \cos w_d t}$$

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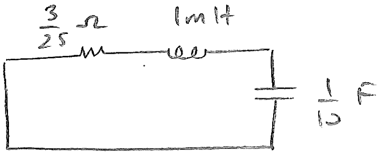
Ex: $R = \frac{3}{25}\,\Omega$, $L = 1mH$, $C = \frac{1}{10}F$

$v_c(0^-) = 3V$, $i_L(0^-) = 14A$



$\Rightarrow D^2 v_c + 120 D v_c + 10^4 v_c = 0$

$v_c(0^-) = 3V$, $Dv_c(0^-) = 140 V/s$

$\Rightarrow d(s) = s^2 + 120s + 10^4 = (s+60)^2 + 80^2$

nat. freq. $s_{1,2} = -60 \mp j80$

$\Rightarrow v_c(t) = k_1 e^{-60t} \sin 80t + k_2 e^{-60t} \cos 80t$

$$\left.\begin{array}{l} v_c(0) = k_2 = 3 \\ Dv_c(0) = 80k_1 - 60k_2 = 140 \Rightarrow k_1 = 4 \end{array}\right\} \quad \boxed{v_c(t) = 4e^{-60t}\sin 80t + 3e^{-60t}\cos 80t \; V}$$

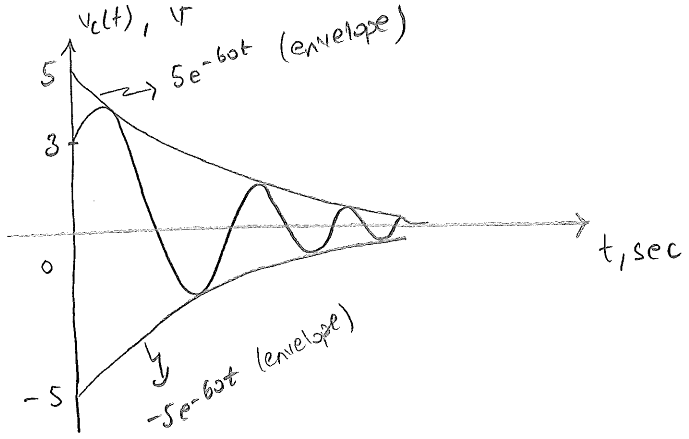
for $t \geq 0$

Note that we can write

$$v_c(t) = e^{-60t}\{4\sin 80t + 3\cos 80t\}$$

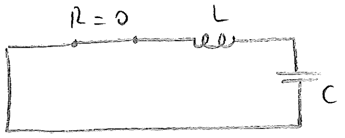
$$= \sqrt{3^2+4^2}\, e^{-60t} \Big\{ \underbrace{\frac{4}{\sqrt{3^2+4^2}}}_{\cos\phi} \sin 80t + \underbrace{\frac{3}{\sqrt{3^2+4^2}}}_{\sin\phi} \cos 80t \Big\} \qquad \left(\phi = \arctan\frac{3}{4}\right)$$

$$= 5e^{-60t}\sin(80t + \phi)$$



-1 27 Aralık 2012

Case 4: $w_0 > \alpha = 0$ (lossless)



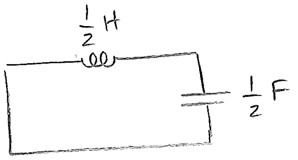
diff. eqn. $D^2 v_c + \frac{1}{LC} v_c = 0$

char. poly. $d(s) = s^2 + \frac{1}{LC} = s^2 + w_0^2$

nat. freq. $s_{1,2} = \mp j w_0 = \mp j \frac{1}{\sqrt{LC}}$

$\Rightarrow \boxed{v_c(t) = k_1 \sin w_0 t + k_2 \cos w_0 t}$ (sustaining oscillations)

Ex: $R = 0$, $L = \frac{1}{2} H$, $C = \frac{1}{2} F$



$v_c(0^-) = 10V$, $i_L(0^-) = 10A$

$\Rightarrow D^2 v_c + 4 v_c = 0$

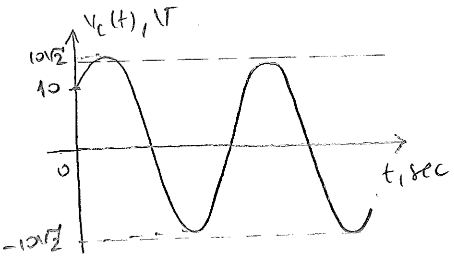
$v_c(0^-) = 10V$, $D v_c(0^-) = 20V/s$

$\Rightarrow d(s) = s^2 + 4$, nat. freq. $s_{1,2} = \mp j2$

$\Rightarrow v_c(t) = k_1 \sin 2t + k_2 \cos 2t$

$$\left.\begin{aligned} v_c(0) &= k_2 = 10 \\ D v_c(0) &= 2k_1 = 20 \Rightarrow k_1 = 10 \end{aligned}\right\}$$

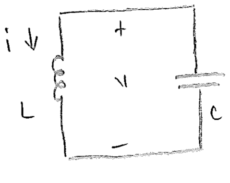
$$\begin{aligned} v_c(t) &= 10 \sin 2t + 10 \cos 2t \\ &= 10\sqrt{2} \left\{ \frac{1}{\sqrt{2}} \sin 2t + \frac{1}{\sqrt{2}} \cos 2t \right\} \\ &= 10\sqrt{2} \left\{ \cos \frac{\pi}{4} \sin 2t + \sin \frac{\pi}{4} \cos 2t \right\} \\ &= \boxed{10\sqrt{2} \sin \left(2t + \frac{\pi}{4}\right) V} \end{aligned}$$ for $t \geq 0$



— o —

Why "lossless"?

Total stored energy $E(t) = \frac{1}{2} L i(t)^2 + \frac{1}{2} C v(t)^2$



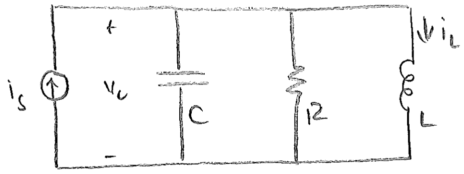
$$\frac{dE}{dt} = L i D i + C v D v = i \underbrace{(L D i)}_{v} + v \underbrace{(C D v)}_{-i} = iv - vi = 0$$

$\Rightarrow E(t) = E(0)$ for all $t \geq 0$ $\Rightarrow$ Electrical energy is conserved.

#1 24 Aralık 2014

zero-state response

For the following circuit find the step response $i_L(t)$



By definition :

$\rightarrow i_s(t) = u(t)$ (unit step input)

$\rightarrow i_L(0^-) = 0$, $v_c(0^-) = 0$ (zero init. cond.)

$$i_s = i_c + i_R + i_L$$

$$= CDv_L + \frac{1}{R} v_L + i_L$$

$$= CD(LDi_L) + \frac{1}{R}(LDi_L) + i_L = LCD^2 i_L + \frac{L}{R} Di_L + i_L$$

$$\boxed{D^2 i_L + \frac{1}{RC} Di_L + \frac{1}{LC} i_L = \frac{1}{LC} i_s}$$ Diff. eqn.

$$\left.\begin{aligned} &i_L(0^-) = 0 \\ &Di_L(0^-) = \frac{v_L(0^-)}{L} = \frac{v_c(0^-)}{L} = 0 \end{aligned}\right\}$$ init. cond.

Example : Let $R = 1\Omega$, $C = 1F$, $L = 1H$.

For $t > 0$ $D^2 i_L + Di_L + i_L = 1$

$i_L(0^+) = 0$, $Di_L(0^+) = 0$ (WHY?)

Char. poly. $d(s) = s^2 + s + 1 = (s + \frac{1}{2})^2 + \frac{3}{4}$

nat. freq. $s_{1,2} = -\frac{1}{2} \mp j\frac{\sqrt{3}}{2}$ (underdamped)

$\Rightarrow$ hmg sol'n: $i_h(t) = k_1 e^{-t/2} \sin\frac{\sqrt{3}}{2}t + k_2 e^{-t/2} \cos\frac{\sqrt{3}}{2}t$

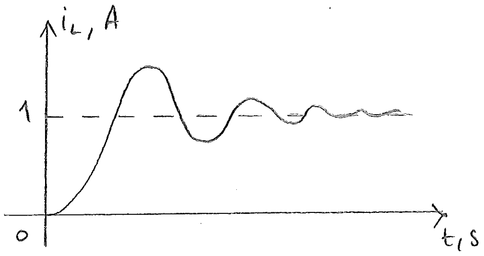
particular sol'n: $i_p(t) = A$ (constant) $\Rightarrow$ $D^2 i_p + Di_p + i_p = 1 \Big|_{i_p = A}$ $\Rightarrow$ $A = 1$

$$\Rightarrow i_L(t) = i_h(t) + i_p(t) = k_1 e^{-t/2} \sin\frac{\sqrt{3}}{2}t + k_2 e^{-t/2} \cos\frac{\sqrt{3}}{2}t + 1$$

init. cond: $i_L(0) = 0 \Rightarrow k_2 + 1 = 0 \Rightarrow k_2 = -1$

$Di_L(0) = 0 \Rightarrow \frac{\sqrt{3}}{2} k_1 - \frac{1}{2} k_2 = 0 \Rightarrow k_1 = -\frac{1}{\sqrt{3}}$

$$\Rightarrow i_L(t) = \underbrace{-\frac{1}{\sqrt{3}} e^{-t/2} \sin\frac{\sqrt{3}}{2}t - e^{-t/2} \cos\frac{\sqrt{3}}{2}t}_{\text{transient part}} \underbrace{+1}_{\text{steady state part}} \text{ Amps for } t > 0$$

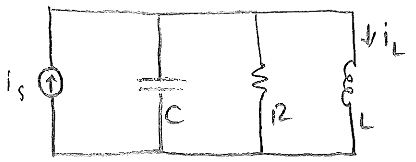


Hence, the step response is :

$$\boxed{s(t) = \left[-\frac{1}{\sqrt{3}} e^{-t/2} \sin\frac{\sqrt{3}}{2}t - e^{-t/2} \cos\frac{\sqrt{3}}{2}t + 1\right] u(t) \text{ Amps}}$$

#2 24 Aralık 2014

Impulse response ?



By definition:

→ $i_s(t) = \delta(t)$ (unit impulse input)

→ $i_L(0^-) = 0$, $v_c(0^-) = 0$ (zero init. cond.)

Diff. eqn. $D^2 i_L + \frac{1}{RC} D i_L + \frac{1}{LC} i_L = \frac{1}{LC} i_s \Rightarrow D^2 i_L + \frac{1}{RC} D i_L + \frac{1}{LC} i_L = \frac{1}{LC}\delta(t)$

Example: Consider the same circuit (R = 1Ω, C = 1F, L = 1H)

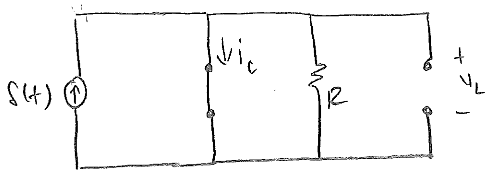
For $t>0$

$$\left.\begin{aligned} & D^2 i_L + D i_L + i_L = 0 \\ & i_L(0^+) = ? \quad D i_L(0^+) = ? \end{aligned}\right\} (*)$$

For $0^- < t < 0^+$

$$i_L(0^+) = i_L(0^-) + \underbrace{\frac{1}{L}\int_{0^-}^{0^+} \overset{\text{bounded}}{v_c(t)}\,dt}_{=0} = i_L(0^-) = 0$$

Also,



$i_c(t) = \delta(t)$

$\Rightarrow v_c(0^+) = v_c(0^-) + \frac{1}{C}\int_{0^-}^{0^+}\delta(t)\,dt = v_c(0^-) + \frac{1}{C}$

$\Rightarrow v_c(0^+) = 1V$

$\Rightarrow D i_L(0^+) = \frac{v_c(0^+)}{L} = 1 A/s$

Return to (*)

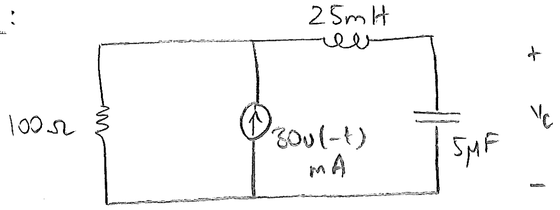
$d(s) = s^2 + s + 1 = (s+\frac{1}{2})^2 + \frac{3}{4} \Rightarrow s_{1,2} = -\frac{1}{2} \mp j\frac{\sqrt{3}}{2}$

$$\left.\begin{aligned} & \Rightarrow i_L(t) = k_1 e^{-t/2}\sin\frac{\sqrt{3}}{2}t + k_2 e^{-t/2}\cos\frac{\sqrt{3}}{2}t \\ & i_L(0) = 0 \Rightarrow k_2 = 0 \\ & D i_L(0) = 1 \Rightarrow \frac{\sqrt{3}}{2}k_1 = 1 \Rightarrow k_1 = \frac{2}{\sqrt{3}} \end{aligned}\right\} i_L(t) = \frac{2}{\sqrt{3}} e^{-t/2}\sin\frac{\sqrt{3}}{2}t \text{ Amps for } t>0$$

Hence, the impulse response is: $\boxed{h(t) = \left(\frac{2}{\sqrt{3}} e^{-t/2}\sin\frac{\sqrt{3}}{2}t\right)u(t) \text{ Amps}}$

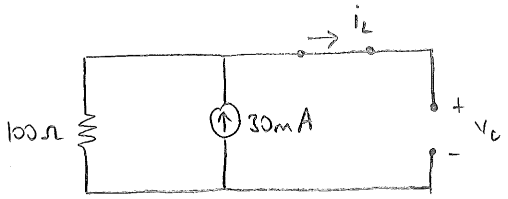
Exercise: Verify $h(t) = \frac{d}{dt}s(t)$.

Example:



$v_c(t) = ?$ for $t \geq 0$.

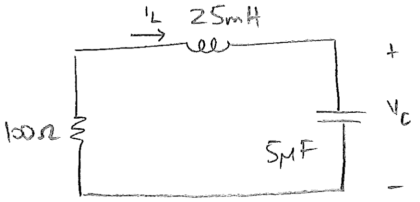
For $t<0$ (assume that the circuit is at DC steady state)



$v_c(0^-) = 3V$

$i_L(0^-) = 0$

for $t>0$



$$D^2 v_c + \frac{R}{L} Dv_c + \frac{1}{LC} v_c = 0$$

$$\Rightarrow D^2 v_c + 4000 Dv_c + 8\times10^6 v_c = 0 \quad \text{(diff. eqn.)}$$

init. cond. $v_c(0^+) = v_c(0^-) = 3V$ | $Dv_c(0^+) = \frac{i_c(0^+)}{C} = \frac{i_L(0^+)}{C} = 0$

$i_L(0^+) = i_L(0^-) = 0$

char. poly. $d(s) = s^2 + 4000s + 8\times10^6 = (s+2000)^2 + 2000^2$

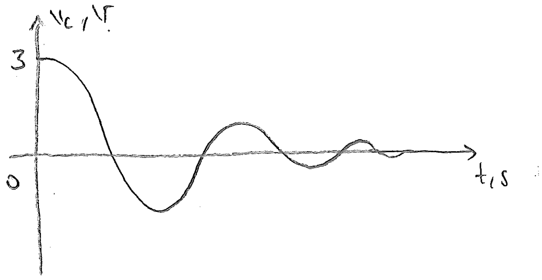
nat. freq. $s_{1,2} = -2000 \mp j2000$ (underdamped)

$$\Rightarrow v_c(t) = e^{-2000t}(k_1 \sin 2000t + k_2 \cos 2000t)$$

$v_c(0) = 3 \Rightarrow k_2 = 3$

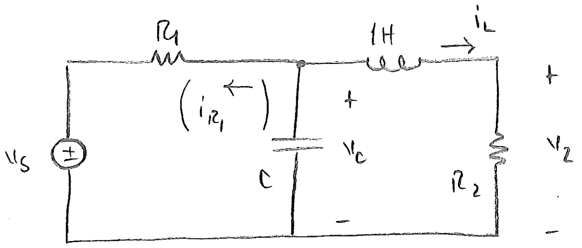
$Dv_c(0) = 0 \Rightarrow -2000k_2 + 2000k_1 = 0 \Rightarrow k_1 = 3$

$$\boxed{\Rightarrow v_c(t) = 3e^{-2000t}(\sin 2000t + \cos 2000t) \quad V}$$



#4 24 Aralık 2014

Example



Design the circuit so that the step response for v_2 is (for $t>0$)

$$v_2(t) = \frac{3}{4} + (A+Bt)e^{-4t} \ V. \quad A, B = ?$$

Sol'n First obtain the diff. eqn. (in terms of i_L)

$$0 = i_{R_1} + i_C + i_L$$

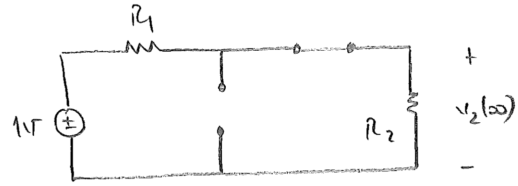
$$= \frac{v_C - v_s}{R_1} + CDv_C + i_L \qquad v_C = LDi_L + R_2 i_L$$

$$= \frac{1}{R_1}\left\{LDi_L + R_2 i_L - v_s\right\} + CD\left\{LDi_L + R_2 i_L\right\} + i_L$$

$$= LCD^2 i_L + \left\{\frac{L}{R_1} + R_2 C\right\} Di_L + \left\{1 + \frac{R_2}{R_1}\right\} i_L - \frac{1}{R_1} v_s$$

$$\Rightarrow \boxed{D^2 i_L + \left(\frac{1}{R_1 C} + \frac{R_2}{L}\right) Di_L + \left(1 + \frac{R_2}{R_1}\right)\frac{1}{LC} i_L = \frac{1}{R_1 LC} v_s}$$

t=∞



$$v_2(\infty) = \frac{3}{4} \Rightarrow \frac{R_2}{R_1+R_2} = \frac{3}{4} \Rightarrow \boxed{R_2 = 3R_1}$$

$$\Rightarrow \text{Char. poly.} \quad d(s) = s^2 + \left(\frac{1}{R_1 C} + 3R_1\right)s + \frac{4}{C} = (s+4)^2 = s^2 + 8s + 16$$

$$\Rightarrow \boxed{C = \frac{1}{4}F} \quad \& \quad \frac{4}{R_1} + 3R_1 = 8 \Rightarrow 3R_1^2 - 8R_1 + 4 = 0 \Rightarrow (3R_1 - 2)(R_1 - 2) = 0$$

$\Rightarrow$ $R_1 = \frac{2}{3}\ \Omega$ & $R_2 = 2\ \Omega$

OR

$R_1 = 2\ \Omega$ & $R_2 = 6\ \Omega$

#5 24 Aralık 2014

A, B = ? zero state ⇒ $i_L(0^-) = 0$, $v_c(0^-) = 0$ (by def.)

Then we have $i_L(0^+) = 0$ & $v_c(0^+) = 0$ (WHY?)

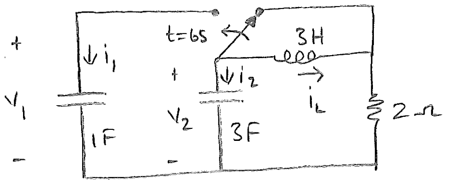
$$v_2(0) = R_2 i_L(0) = 0$$

$$Dv_2(0) = R_2 D i_L(0) = \frac{R_2}{L} v_L(0) = \frac{R_2}{L}\left(v_c(0) - R_2 i_L(0)\right) = 0$$

Now, $v_2(0) = 0 \Rightarrow \frac{3}{4} + A = 0 \Rightarrow \boxed{A = -3/4}$

$Dv_2(0) = 0 \Rightarrow -4A + B = 0 \Rightarrow \boxed{B = -3}$

Example:



$v_1(0) = 18V$, $v_2(0) = -10e\ V$, $i_L(0) = -2A$

$v_2(t) = ?$ for $t \geq 0$

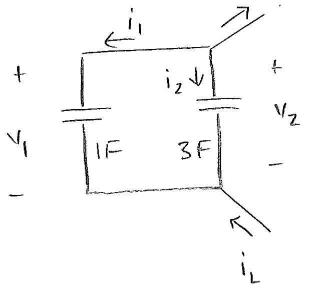
$0 \leq t < 6$

$Dv_1 = i_1 = 0 \Rightarrow v_1(t) = 18V$

$Di_L = \frac{1}{3} v_L = 0 \Rightarrow i_L(t) = -2A$

$Dv_2 = \frac{1}{3} i_2 = \frac{1}{3}\left\{-\frac{v_2}{2}\right\} \Rightarrow Dv_2 + \frac{1}{6} v_2 = 0 \Rightarrow v_2(t) = v_2(0) e^{-t/6} \Rightarrow \boxed{v_2(t) = -10e \cdot e^{-t/6}\ V}$

$6^- < t < 6^+$



$$v_1(6^+) = v_1(6^-) + \frac{1}{1}\int_{6^-}^{6^+} i_1(\tau)\,d\tau$$

$$v_2(6^+) = v_2(6^-) + \frac{1}{3}\int_{6^-}^{6^+} i_2(\tau)\,d\tau$$

$$= v_2(6^-) + \frac{1}{3}\int_{6^-}^{6^+} [-i_1(\tau) \underbrace{-i_L(\tau)}_{\text{bounded}}]\,d\tau$$

$$= v_2(6^-) - \frac{1}{3}\int_{6^-}^{6^+} i_1(\tau)\,d\tau$$

$v_1(6^+) = v_2(6^+) \Rightarrow \int_{6^-}^{6^+} i_1(\tau)\,d\tau = v_2(6^+) - v_1(6^-)$

$= 3(v_2(6^-) - v_2(6^+))$

$$v_2(6^+) = \frac{v_1(6^-) + 3v_2(6^-)}{4}$$

$$= \frac{18 + 3(-10e \cdot e^{-6/6})}{4} = -3V$$

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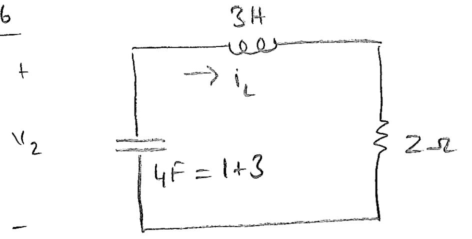
Also, $i_L(6^+) = i_L(6^-) = -2A$ (WHY?)

$t > 6$



$i_L(6^+) = -2A$, $v_2(6^+) = -3V$

$$D^2 v_2 + \frac{R}{L} D v_2 + \frac{1}{LC} v_2 = 0$$

$$\Rightarrow D^2 v_2 + \frac{2}{3} D v_2 + \frac{1}{12} v_2 = 0$$

$$v_2(6) = -3V$$

$$Dv_2(6) = \frac{i_C(6)}{4} = -\frac{i_L(6)}{4} = \frac{1}{2} V/s$$

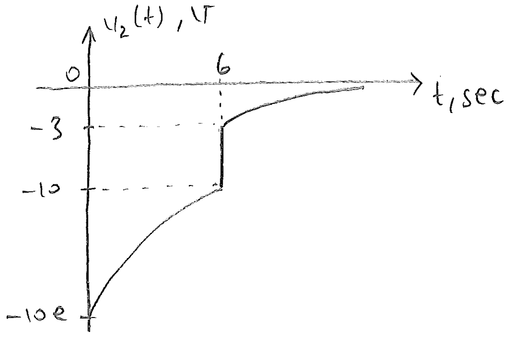
Char. poly. $d(s) = s^2 + \frac{2}{3} s + \frac{1}{12} = (s+\frac{1}{2})(s+\frac{1}{6})$

nat. freq. $s_{1,2} = -\frac{1}{2}, -\frac{1}{6}$ (overdamped)

$$\Rightarrow v_2(t) = k_1 e^{-(t-6)/2} + k_2 e^{-(t-6)/6}$$

$$\left.\begin{aligned} v_2(6) = -3 \quad &\Rightarrow \quad k_1 + k_2 = -3 \\ Dv_2(6) = \frac{1}{2} \quad &\Rightarrow \quad -\frac{1}{2}k_1 - \frac{1}{6}k_2 = \frac{1}{2} \end{aligned}\right\} \quad k_1 = 0, \; k_2 = -3$$

$$\Rightarrow \boxed{v_2(t) = -3e^{-(t-6)/6} \; V}$$



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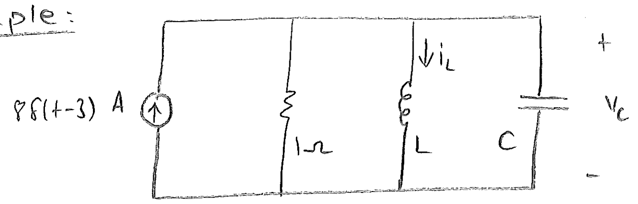
Example:



$v_c(3^+) = 1V$ and the response is critically damped.

a) determine L & C.

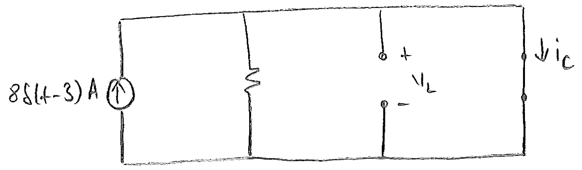
b) find $v_c(t)$ for $t \geq 0$.

$i_L(0) = 0, \quad v_c(0) = 0$

$0 \leq t < 3$

Everything is zero. $\Rightarrow$ $\boxed{v_c(t) = 0}$

$3^- < t < 3^+$

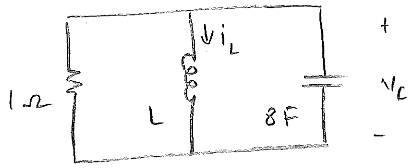


$i_c(t) = 8\delta(t-3)$

$$\Rightarrow v_c(3^+) = v_c(3^-) + \frac{1}{C}\int_{3^-}^{3^+} 8\delta(t-3)dt = \frac{8}{C} = 1$$

$$\Rightarrow \boxed{C = 8F}$$

$t > 3$



$v_c(3^+) = 1V, \quad i_L(3^+) = 0$

$$D^2 v_c + \frac{1}{RC} Dv_c + \frac{1}{LC} v_c = 0$$

$$\Rightarrow D^2 v_c + \frac{1}{8} Dv_c + \frac{1}{8L} v_c = 0$$

char. poly. $d(s) = s^2 + \frac{1}{8}s + \frac{1}{8L} = (s+\alpha)^2 = s^2 + 2\alpha s + \alpha^2$

$$\Rightarrow 2\alpha = \frac{1}{8} \Rightarrow \alpha = \frac{1}{16}$$

$$\Rightarrow \frac{1}{8L} = \alpha^2 = \frac{1}{256} \Rightarrow \boxed{L = 32H}$$

$$\Rightarrow D^2 v_c + \frac{1}{8} Dv_c + \frac{1}{256} v_c = 0$$

$v_c(3) = 1V$

$Dv_c(3) = \frac{1}{8} i_c(3) = \frac{1}{8}\left(-i_L(3) - i_R(3)\right) = \frac{1}{8}\left(-i_L(3) - \frac{v_c(3)}{1}\right) = -\frac{1}{8} V/s$

char. poly. $d(s) = (s + \frac{1}{16})^2$

nat. freq. $s_{1,2} = -\frac{1}{16}, -\frac{1}{16}$

$v_c(t) = \left(k_1 + k_2(t-3)\right)e^{-\frac{1}{16}(t-3)}$

$\left.\begin{array}{l} v_c(3) = 1 \\ Dv_c(3) = -\frac{1}{8} \end{array}\right\} \Rightarrow k_1 = 1, \quad k_2 = -\frac{1}{16}$

$$\Rightarrow \boxed{v_c(t) = \left[1 - \frac{1}{16}(t-3)\right]e^{-\frac{1}{16}(t-3)} \ V}$$ for $t > 3s$.

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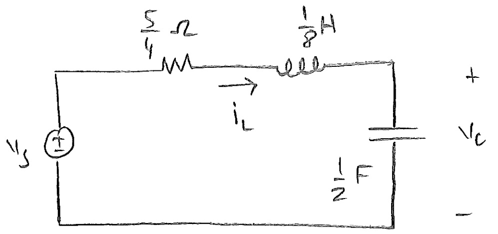
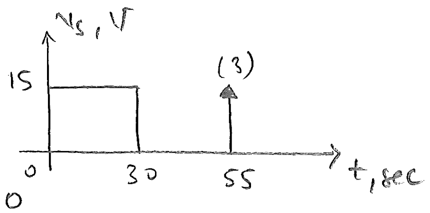
Example:



$v_c(0^-) = -3V$

$i_L(0^-) = 0$

$v_c(t) = ?$ for $t \geq 0$



Diff. eqn. $D^2 v_c + \frac{R}{L} D v_c + \frac{1}{LC} v_c = \frac{1}{LC} v_s$

$\Rightarrow D^2 v_c + 10 D v_c + 16 v_c = 16 v_s$, $v_c(0^-) = -3V$, $D v_c(0^-) = \frac{i_L(0^-)}{C} = 0$

zero-input response : $v_{zi}(t)$

$$\left.\begin{array}{l} D^2 v_c + 10 D v_c + 16 v_c = 0 \\ v_c(0^-) = -3V \ , \ D v_c(0^-) = 0 \end{array}\right\} \begin{array}{l} \text{char. poly. } d(s) = s^2 + 10s + 16 = (s+2)(s+8) \\ \Rightarrow v_c(t) = k_1 e^{-2t} + k_2 e^{-8t} \quad \text{(overdamped)} \end{array}$$

$$\left.\begin{array}{lll} v_c(0) = -3 & \Rightarrow & k_1 + k_2 = -3 \\ D v_c(0) = 0 & \Rightarrow & -2k_1 - 8k_2 = 0 \end{array}\right\} \left.\begin{array}{l} k_1 = -4 \\ k_2 = 1 \end{array}\right\} \Rightarrow \boxed{v_{zi}(t) = -4e^{-2t} + e^{-8t} \ V}$$

step response : $s(t)$

$$\left.\begin{array}{l} D^2 v_c + 10 D v_c + 16 v_c = 16 u(t) \\ v_c(0^-) = 0 \ , \ D v_c(0^-) = 0 \end{array}\right] \left.\begin{array}{l} v_p(t) = 1 \\ v_h(t) = k_3 e^{-2t} + k_4 e^{-8t} \end{array}\right\} v_c(t) = 1 + k_3 e^{-2t} + k_4 e^{-8t}$$

$$\left.\begin{array}{lll} v_c(0^+) = 0 & \Rightarrow & 1 + k_3 + k_4 = 0 \\ D v_c(0^+) = 0 & \Rightarrow & -2k_3 - 8k_4 = 0 \end{array}\right\} \left.\begin{array}{l} k_3 = -\frac{4}{3} \\ k_4 = \frac{1}{3} \end{array}\right\} \Rightarrow \boxed{s(t) = \left[1 - \frac{4}{3}e^{-2t} + \frac{1}{3}e^{-8t}\right] u(t) \ V}$$

impulse response : $h(t) = \frac{d}{dt} s(t)$

$$\Rightarrow \boxed{h(t) = \left[\frac{8}{3}e^{-2t} - \frac{8}{3}e^{-8t}\right] u(t) \ V}$$

zero state response : $v_{zs}(t)$

$v_s(t) = 15u(t) - 15u(t-30) + 3\delta(t-55) \Rightarrow \boxed{v_{zs}(t) = 15s(t) - 15s(t-3) + 3h(t-55)}$

Finally, overall response : $\boxed{v_c(t) = v_{zi}(t) + v_{zs}(t)}$

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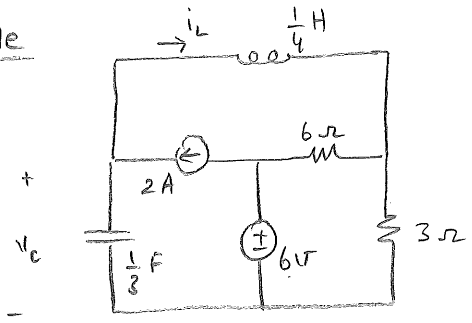
Example



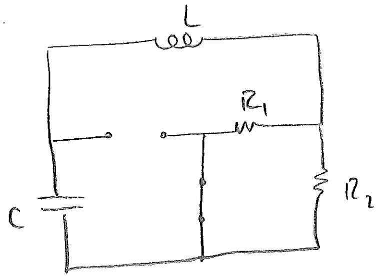
$v_c(0)=9V$, $i_L(0)=0$. Find $v_c(t)$ for $t\geq 0$.

Sol'n Classical way:
1) Derive the diff. eqn.
2) Obtain init. cond. $v_c(0)=9V$ (given)

$$Dv_c(0)=\frac{1}{C}i_c(0)=\frac{1}{C}(i_s(0)-i_L(0))=6V/s$$

3) Solve for $v_c(t)$.

Another way: find the natural freq. first. Characteristic poly. does not depend on the inputs. Hence, kill the indep. sources:



$\Rightarrow$ series RLC $\Rightarrow$ $d(s)=s^2+\frac{R}{L}s+\frac{1}{LC}$ where $R=R_1//R_2$

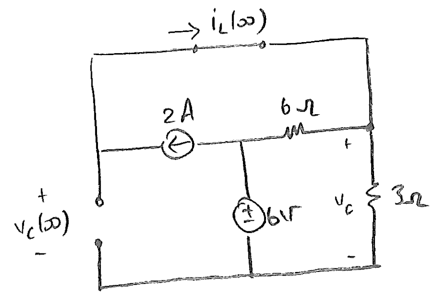
$\Rightarrow d(s)=s^2+8s+12=(s+2)(s+6)$

$\Rightarrow \boxed{s_{1,2}=-2,-6}$

Then, Notice that the sources are DC. Hence the particular sol'n v_p will be constant.

$$\Rightarrow v_c(t)=\underbrace{k_1e^{-2t}+k_2e^{-6t}}_{v_h(t)}+\underbrace{A}_{v_p(t)}$$

Note that $A=v_c(\infty)$. Hence check the steady state circuit:



$$\Rightarrow \frac{v_c}{3}+\frac{v_c-6}{6}-2=0$$

$$\Rightarrow \boxed{v_c(\infty)=6V}$$

$$\Rightarrow v_c(t)=k_1e^{-2t}+k_2e^{-6t}+6$$

Finally, apply the init. cond constraints

$v_c(0)=9V \Rightarrow k_1+k_2+6=9$

$Dv_c(0)=6\frac{V}{s} \Rightarrow -2k_1-6k_2=6$

$k_1=6,\ k_2=-3$

$$\Rightarrow \boxed{v_c(t)=6e^{-2t}-3e^{-6t}+6\ V}$$