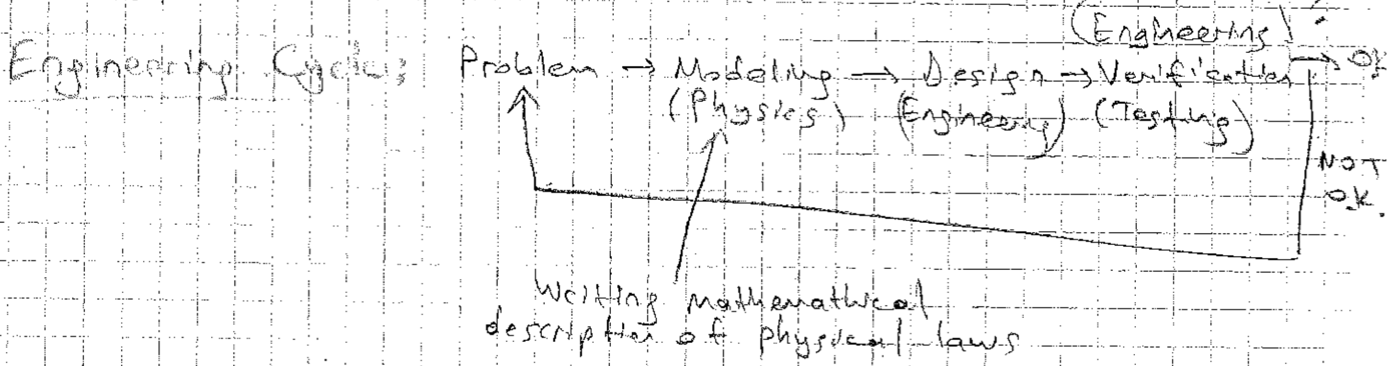
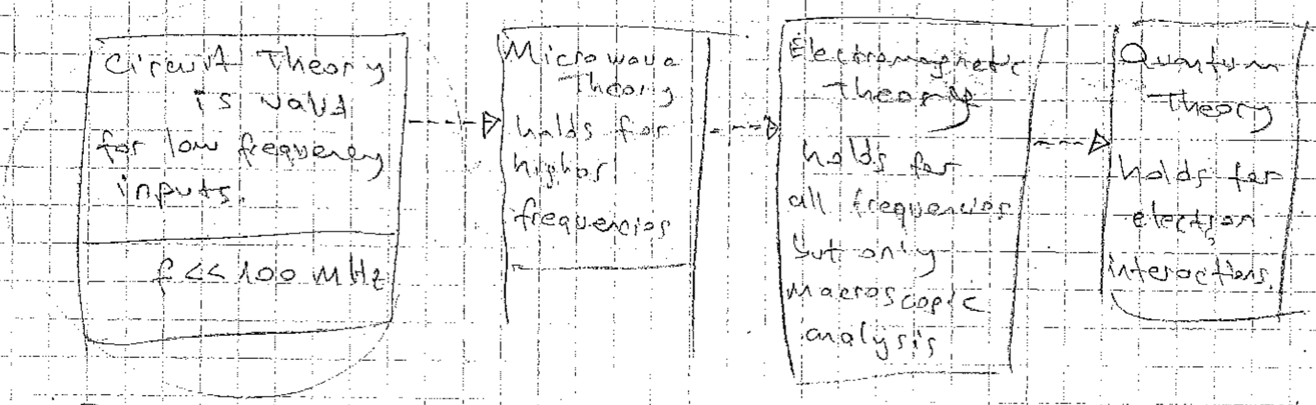
EE 201 Section 03 ([illegible])

Goals: Analyzing / designing systems/equipment such that we have something valuable that is making human life safer, more convinient, healthier etc, and of value to the general public and science.

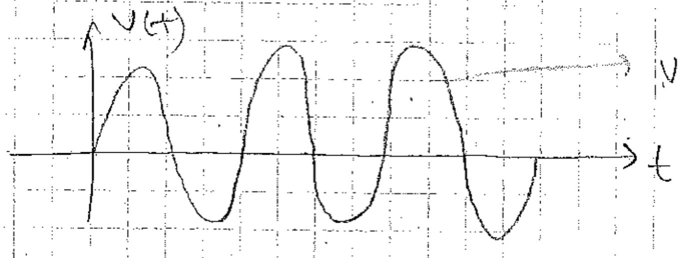
Engineering Cycle:



Modeling: Modelling requires expressing physical laws or objects in terms of mathematical equations. All models are somewhat approximate, but if the model works, we, as engineers are happy and enjoy the results.



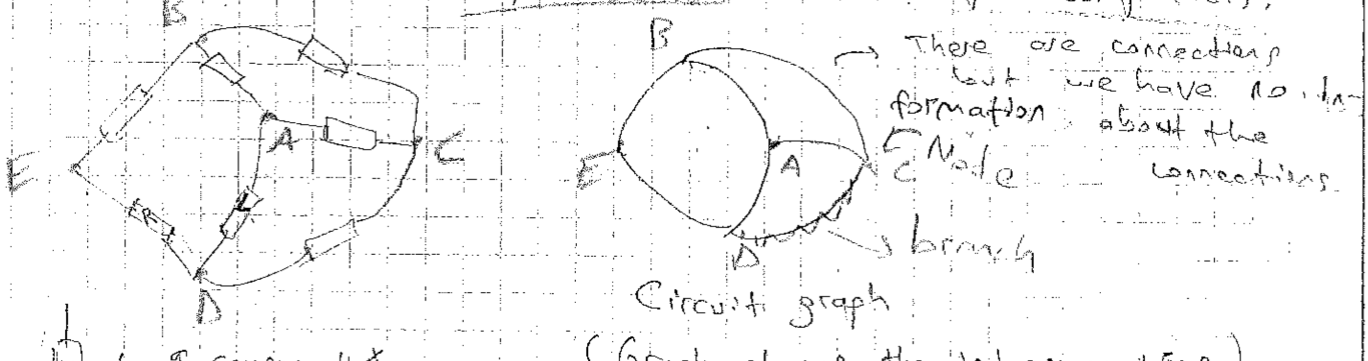
Frequency of power line in Turkey is 50 Hz.



$V(t) = 220 \cos(2\pi \cdot 50t + \phi_0)$ Volts (RMS)

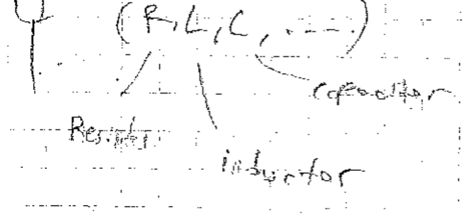
frequency = 50 Hz

EE 201 deals with lumped circuits with lumped components.



Circuit graph

(Graph shows the interconnections between nodes)



Node: $\{A, B, C, D, E\}$

Lumped Circuit: If the propagation delay in the circuit is negligable in comparison to the changes in the changes in the circuit we call such circuits lumped circuits.

voltage and current changes

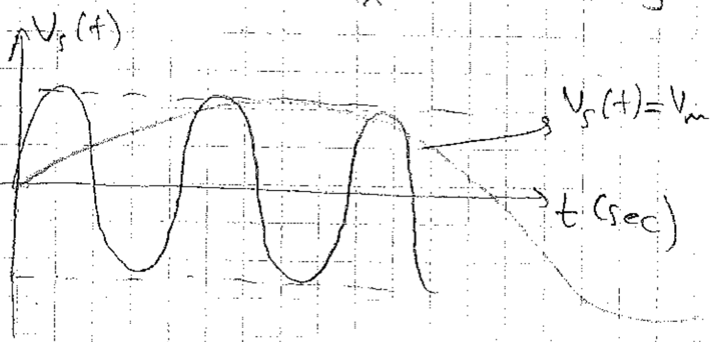
In other words the components are assumed to be instantaneously informed about the circuit variable variations (current/voltage) in the circuit.

If $\lambda >>$ (circuit diameter) then circuit is assumed to be lumped circuit.

$$\lambda = \frac{c}{f_{max}}$$ → c: speed of light (3×10^8 m/sec)

wavelength

f_{max}: max operating frequency



$V_s(t) = V_m \cos(2\pi f t + \phi_0)$ Volts

f: frequency (Hz)

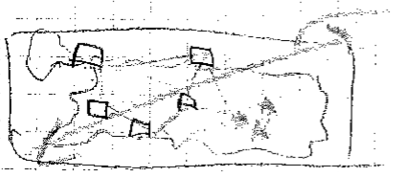
DC circuits are low frequency.

low frequency approximation of EMT

If lumped circuit condition ($\lambda >>$ (circuit diameter)) is not satisfied, we call such circuits distributed circuits. (They are the topic of 4th year courses).

Ex: ① High frequency circuits are distributed circuits.

② Turkey's A.C. Power Grid f: 50 Hz (In Turkey)



Circuit diameter ≈ 1500 km

$$\lambda = \frac{c}{f} = \frac{3\cdot10^8 \, m/\cancel{s}}{\frac{10}{2} \, 1/\cancel{s}}$$

$= 6\cdot10^6$

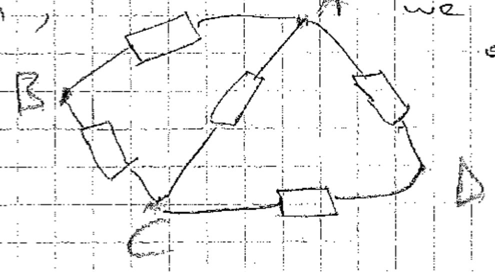
$= 6000$ km

1500 km is not much smaller than 6000 km. There should be at least 1 order of magnitude for us to use lumped theory.

Lumped Components: Component dimensions are negligable in effect in circuit analysis. That is, we never need the physical dimensions such as volume of the component, base area of the components in circuit analysis. Lumped components can be considered as points (dimensionless) in 3-D space.

As a summary, In EE 201 and EE 202 we focus on circuits whose physical attributes does not affect the circuit analysis results.

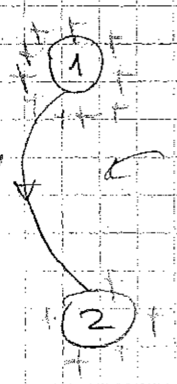
For this reason, we never indicate the length of connections or size of components in this course.



Circuit Variables:

Charge, Current, Voltage → ?

Charge: Unit: Coulumb (C) Charges are countable.



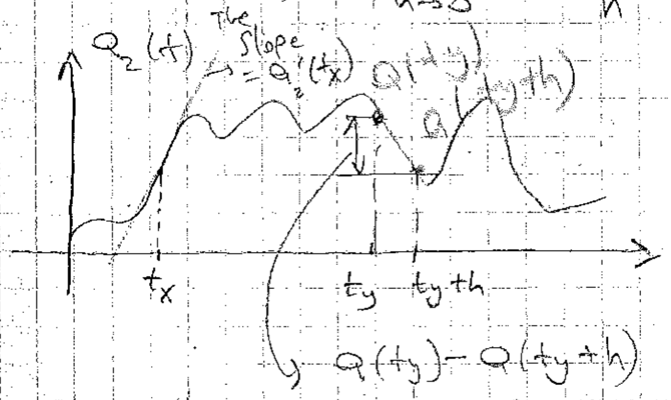
Current: Ampere ← Charge transfer from ① to ②

$Q_2(t)$: Total amount of charge in ② at time "t".

I can transfer charges from ① to ② at a rate 1 C/sec.

Current at time t ⟶ $\frac{Q_2(t+h) - Q_2(t)}{h}$

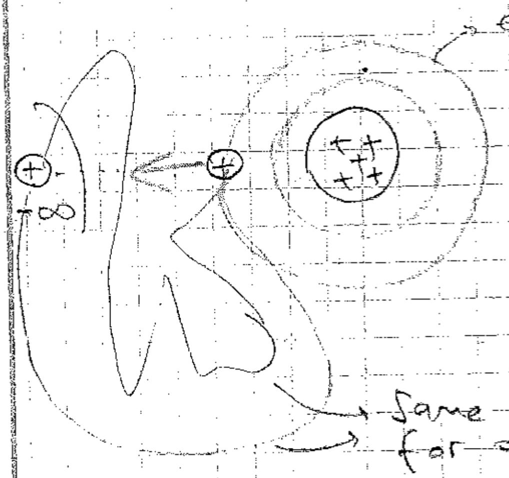
$$i(t_x) = \lim_{h \to 0} \frac{Q(t_x+h) - Q(t_x)}{h} = Q'(t_x) = \left. \frac{dQ(t)}{dt} \right|_{t=t_x}$$



As a summary, current is rate of transfer of charge from ① to ②.

Voltage: Voltage difference is the potential difference between two points in space.

Volts

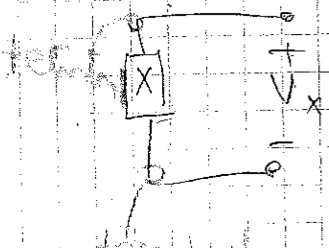


$$\boxed{W = q \cdot \Delta V}$$

Work ↙ Potential difference

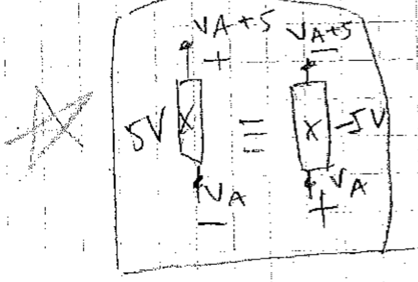
It is path-independent.

Note that voltage difference (or voltage itself) is always referenced to another point.

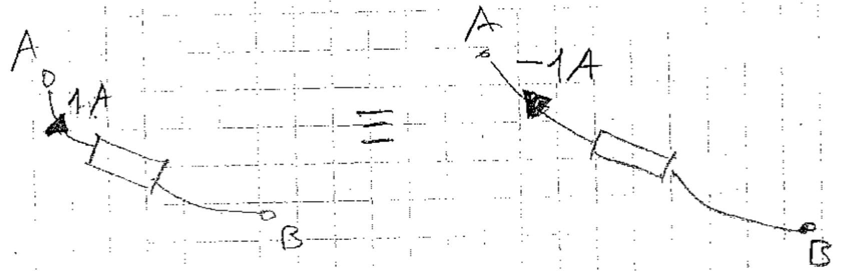


→ V_x is the voltage difference between component terminals.

Also, voltage has a polarity that + and − terminals are important.



Similarly current has a direction and magnitude indicating transfer rate.



You may consider that current direction shows the direction of moving positive charges.

1A = 1 Coulumb/sec transfer rate.

since $i(t) = \lim_{h \to 0} \frac{Q(t+h) - Q(t)}{h}$

Work and Power

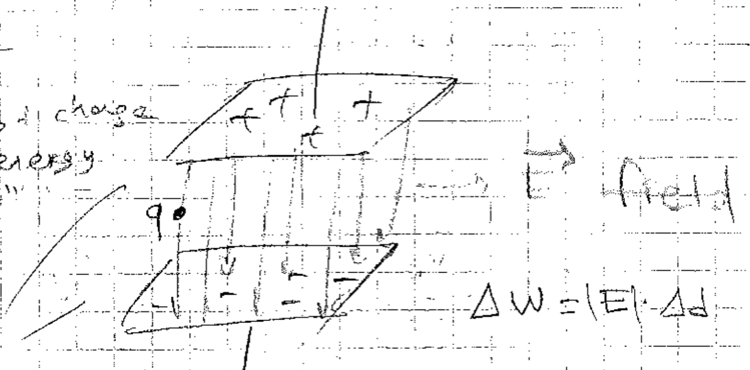
$\underset{\uparrow \text{ Work}}{W} \triangleq$ Energy required to do something

$$\boxed{W = q \cdot \Delta V} \quad \text{Jouleg}$$

Electrostatics fact

$$\underset{\uparrow\ \text{Power } \sim\sim\text{at time "t"}\sim\sim}{p(t)} = \frac{d\,w(t)}{dt}$$

The rate of change of work/energy at time "t"



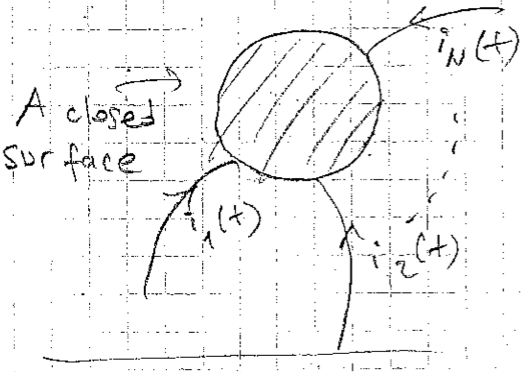
$$\Delta W = |E| \cdot \Delta d$$

$$p(t) = \frac{d}{dt}(q \cdot \Delta V) = \left(\frac{d}{dt} q(t)\right) \Delta V = i(t) \cdot \Delta V$$

$$\boxed{\underset{\uparrow\ \text{Power}}{p(t)} = i(t) \cdot v(t)} \quad \text{Watts}$$

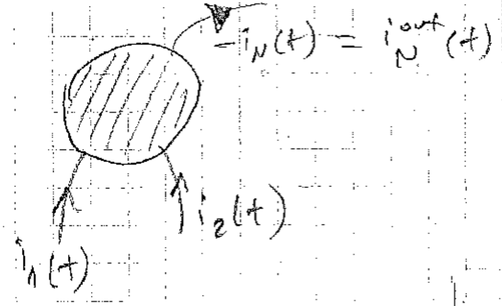
Conservation Laws and Kirchoff's Voltage/Current Laws

① Conservation of Charge



KCL (Kirchoff's Current Law):

$$\boxed{\sum_{k=1}^{N} i_k(t) = 0}$$



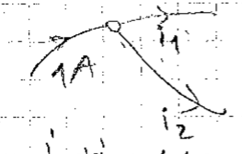
$$\sum_{k=1}^{N-1} i_k(t) + \underbrace{i_N(t)}_{-i_N^{out}(t)} = 0$$

$$\longrightarrow \sum_{k=1}^{N-1} i_k(t) = i_N^{out}(t)$$

or

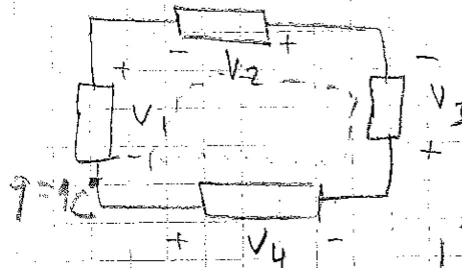
$$\boxed{\sum_{k \in \{\text{Incoming Currents}\}} i_k(t) = \sum_{k \in \{\text{Outgoing Currents}\}} i_k(t)}$$

Sum of incoming currents is equal to sum of outgoing currents.



$$i_1 + i_2 = 1A$$

② ~~Conservation of Energy and KVL~~



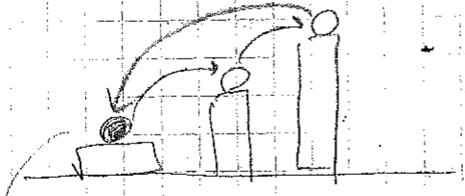
(or energy difference)

Work done by a moving 1C charge in the loop

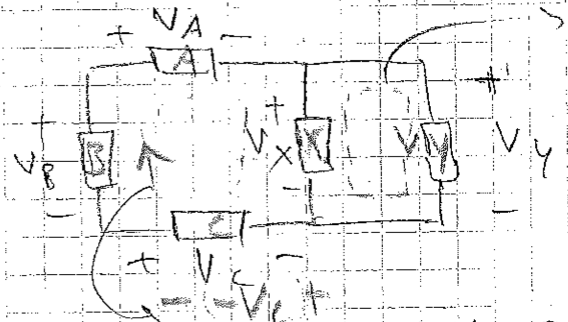
$$0 = qV_1 + qV_2 + qV_3 + qV_4$$

$$\boxed{\sum_{k \in \text{branches in a loop}} V_k = 0}$$

KVL



At the end W=0



$$V_Y - V_X = 0$$

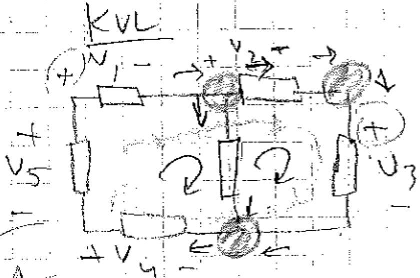
$$\boxed{V_Y = V_X}$$

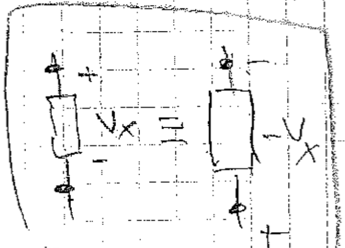
$$V_A + V_X - V_C - V_B = 0$$

Last lecture:

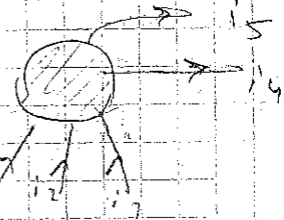
Conservation Laws / KCL / KVL

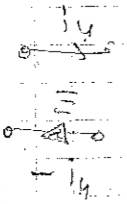
KVL





KCL





$$i_1 + i_2 + i_3 = i_4 + i_5$$

$$i_1 + i_2 + i_3 - i_4 - i_5 = 0$$

Summation of voltage drops across the loop:

$$V_1 + V_2 + V_3 - V_4 - V_5 = 0$$

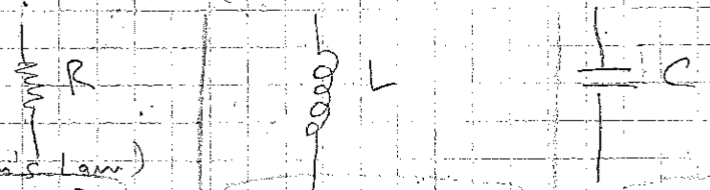
Write the sign of the component to the equation, while traveling in the loop.

We have 3 loops. Conservation of energy is true for all 3 loops.

KVL, KCL is true for all loops, nodes in the circuit.

Circuit Components

Basic Components. R, L, C

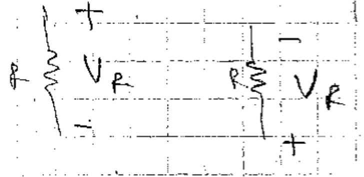


(Ohm's Law)

$$\boxed{V_R = i_R R}$$

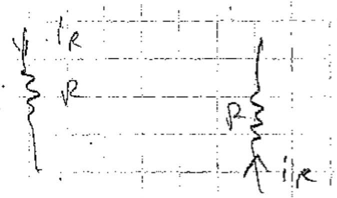
$$V_L(t) = L \frac{d i_L(t)}{dt} \qquad i_c(t) = C \frac{d V_c(t)}{dt}$$

Unit: Ω (ohm)

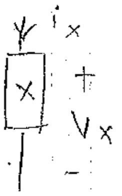


→ which terminal is at higher potential?

We have a convention (accepted rule)

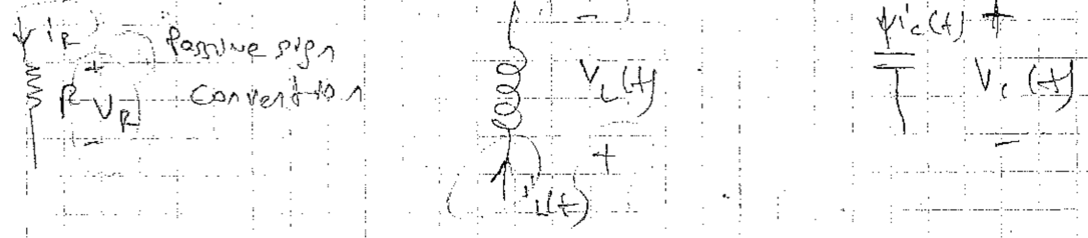


Passive sign convention for circuit components

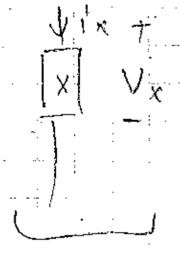


Passive sign convention assumes i_x enters into $\textcircled{+}$ polarity of V_x voltage.

All circuit components (R, L, C etc.) have terminal equations mapping currents to branch voltages (and vice versa) written according to the passive sign convention.

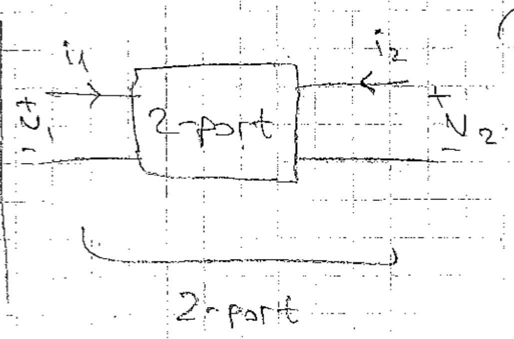


Some definitions



1-Port

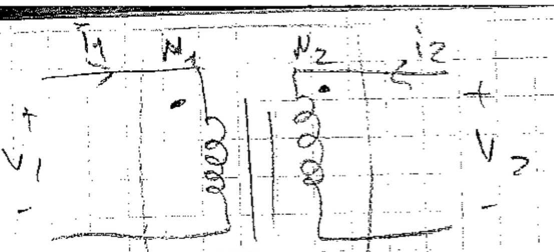
Ex: R, L, C

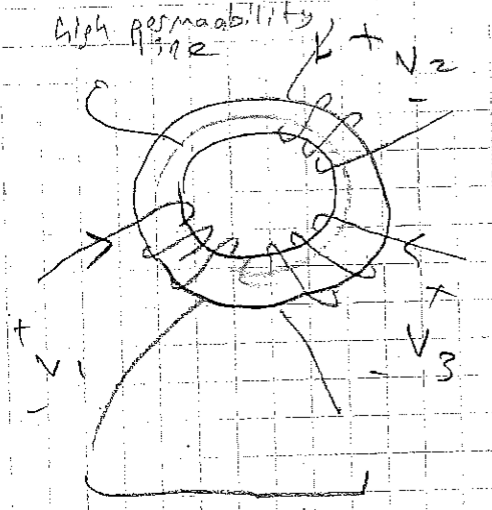


→ Degrees of freedom: 2

2-port

Ex: transformer / amplifier equivalent circuits, ...

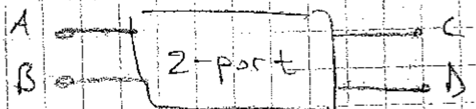




3-port

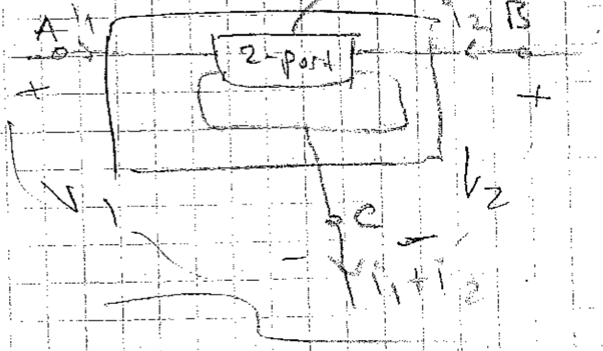
Terminals.

1-port with
2 terminals (A,B)



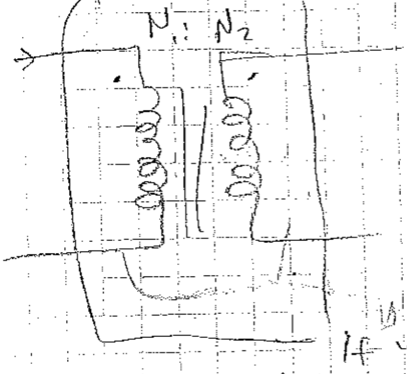
2-port with 4 terminals

2-port with 4 terminals



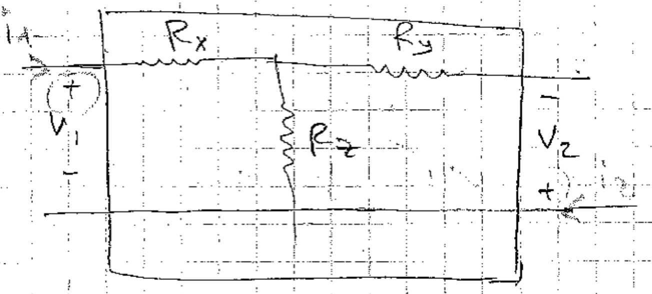
KCL KVL can be applied here as well

2-port with 3 terminals



If we connect them there is no decoupling.

Passive sign convention for 2-ports



(i_1, V_1) and (i_2, V_2)

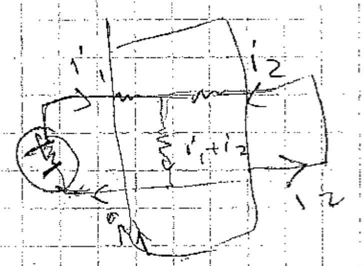
directions and polarity obey the passive sign convention.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}}_{\underline{\underline{R}}} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \longrightarrow \underline{\underline{V}} = \underline{\underline{R}}\, i$$

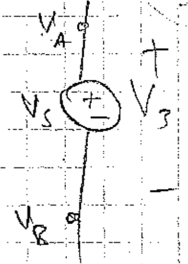
If we know any 2 we can find the 3rd one.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

vectors

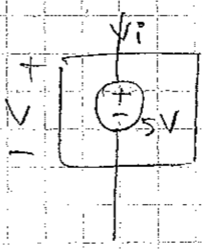


Ideal Voltage / Current Source

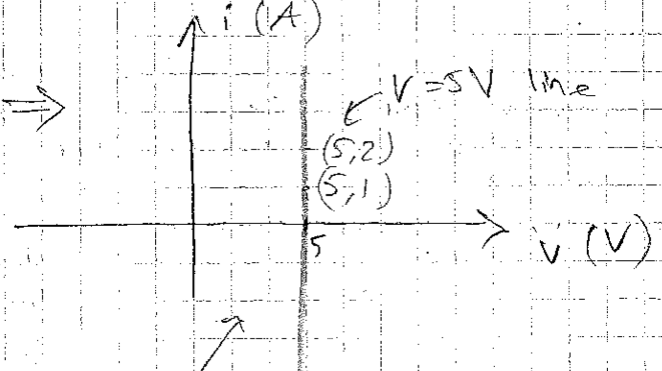


Always : $\boxed{V_A - V_B = V_S}$

Ex :



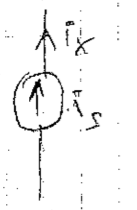
$\Longrightarrow$



Locus of allowed (i, v) pairs

geometrik yer

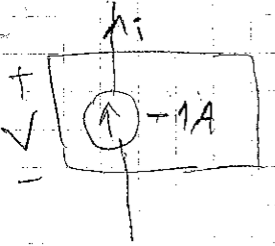
Current Source :

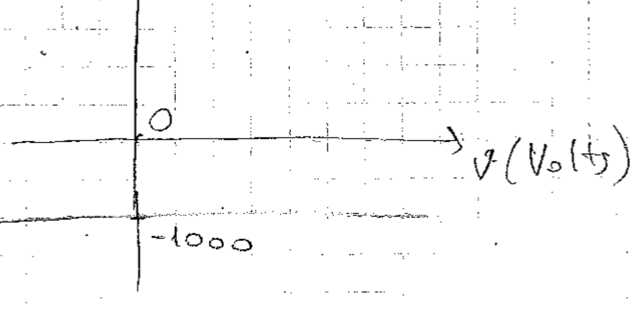


Always : $\boxed{i_x = i_s}$

Ideal current source provides i_s amount of Amperes as branch current no matter what.

Ex:



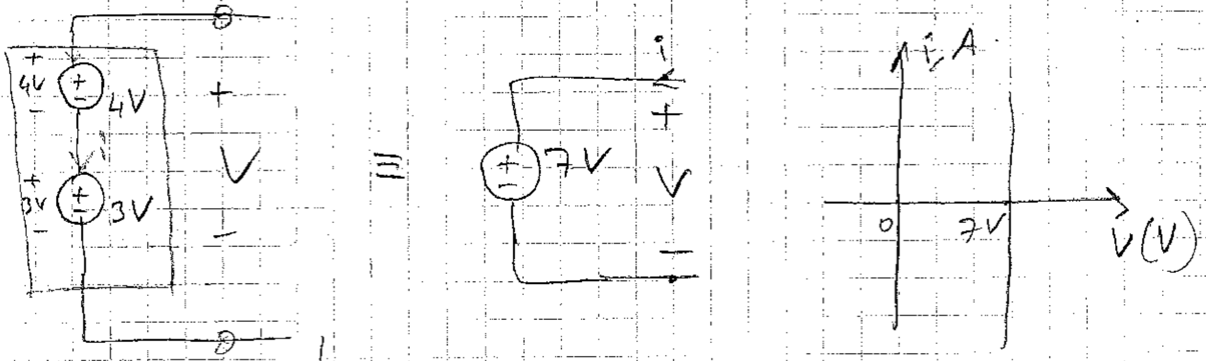


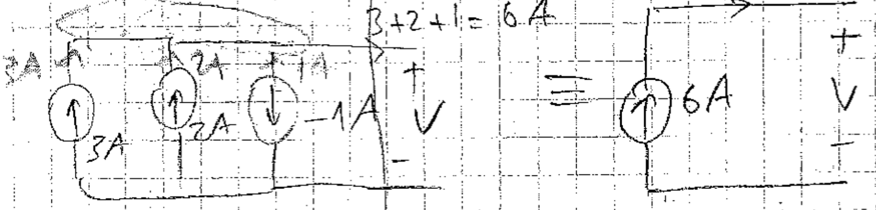
Caution: Ideal current source of i_s Amperes does NOT mean that the branch voltage is 0 Volts.

If this were the case the power of the ideal current source would be

$P = V i_s = 0$ watts all the time!!

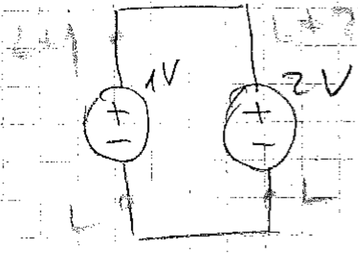
Ex:

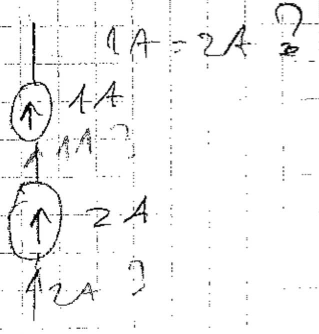




Pathological Combination

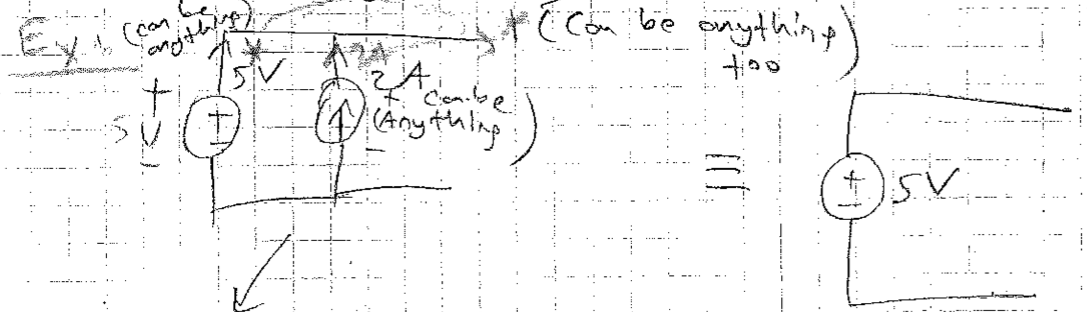
Some source combinations are not possible with ideal sources. These combinations are called pathological combinations.



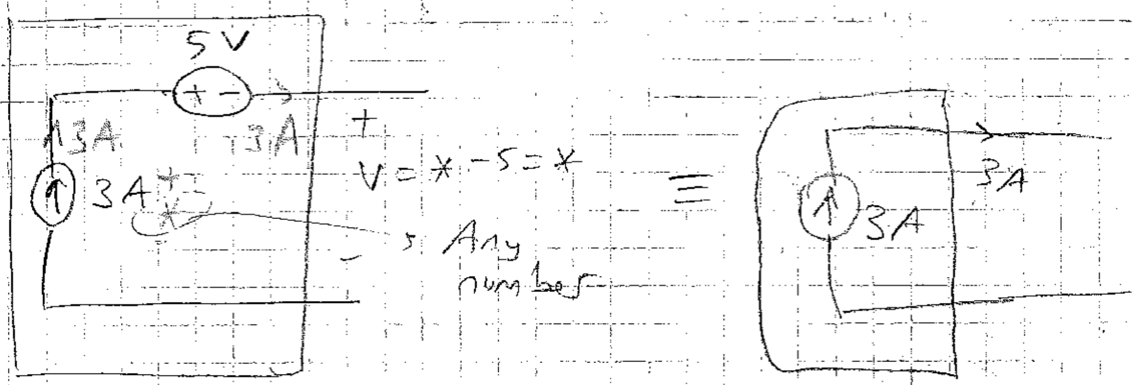


Pathological connection → X + 2 = X

Ex:



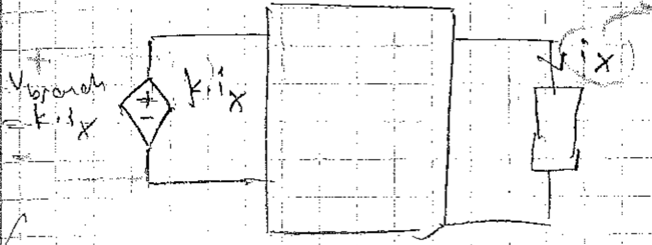
Not a pathological connection.



V = * − 5 = *

→ Any number

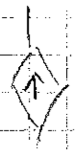
Dependent Sources or Dependent Components



→ controlling variable for dependent source.

Dependent sources are denoted with ◇ (diamond) symbols.

is dependent voltage source.



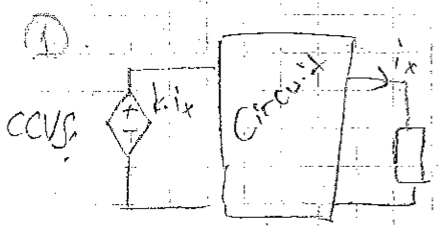
is dependent current source.

k is a scalar proportionality constant. (for this case k has units Ω (ohms))

→ An example of current controlled voltage source. (CCVS)

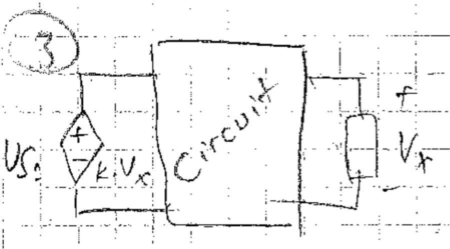
4 Types

① CCVS:



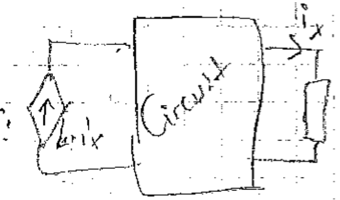
k: Ω

③ VCVS:



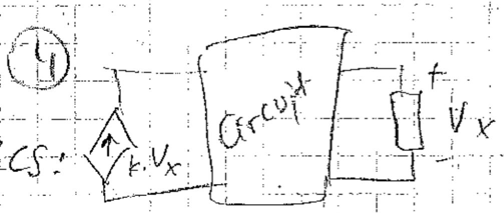
k: Unitless

② CCCS:



k: Unitless

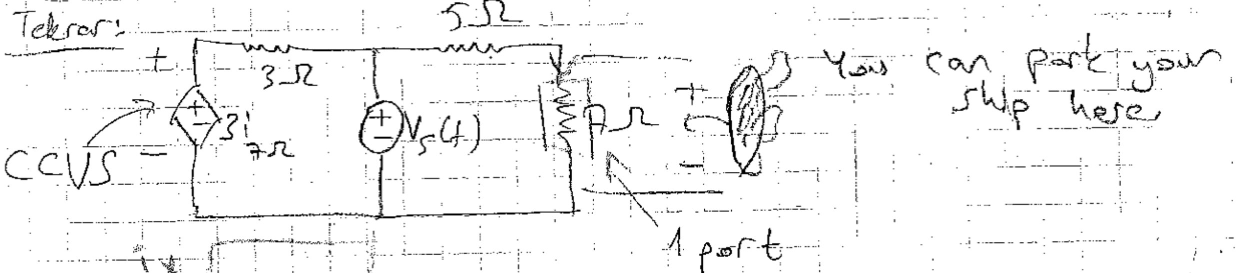
④ VCCS:

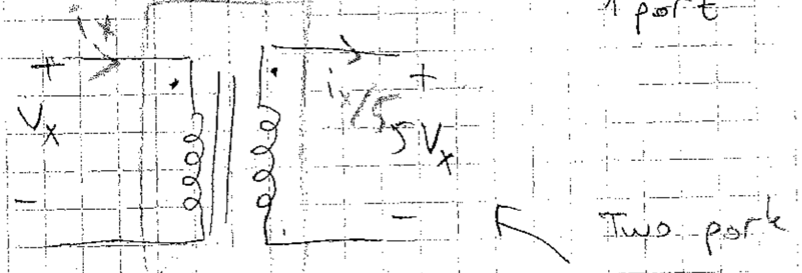


k: $\frac{1}{\Omega}$ = ℧ = Siemens = mho

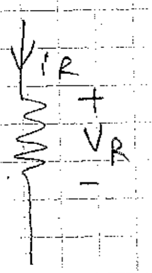
Circuit components (continued):

Tekrar:





Resistors:

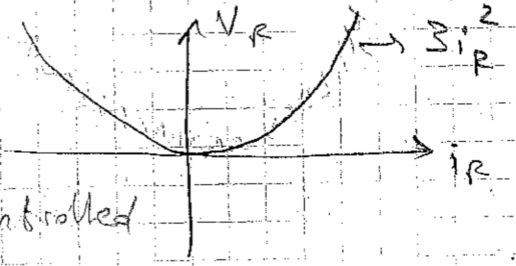


$V_R = func_A(i_R)$ → $V_R = R \cdot i_R$

or → $V_R = R \cdot i_R^2$

$i_R = func_B(V_R)$

↳ $i_R = \tanh(V_R)$

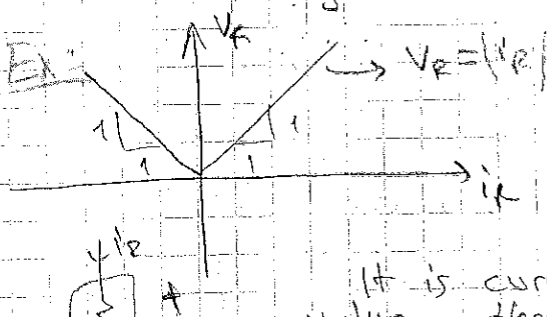


Defn: Voltage controlled and current controlled components:

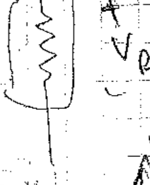
Voltage controlled: If branch current of a component can be written as a function of its branch voltage then component is voltage controlled.

Current controlled: Branch current uniquely specifies the branch voltage.

Ex:

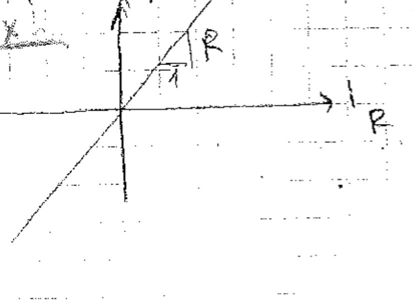


A resistor whose $(V_R - i_R)$ characteristic is given. Determine if the resistor is voltage/current controlled.



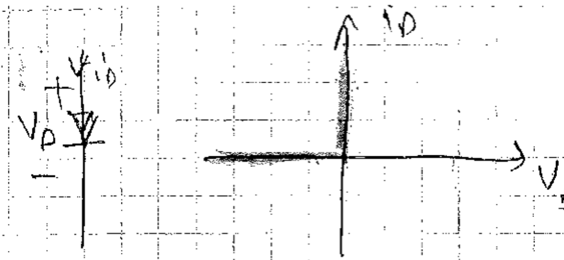
It is current controlled since for every current value, there is a unique voltage.

Ex:



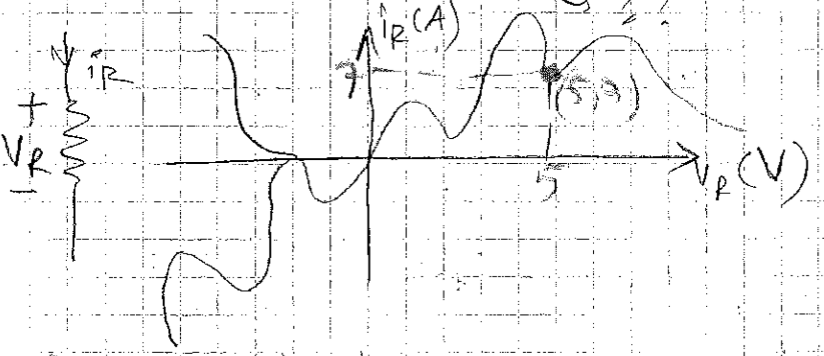
→ Both voltage and current controlled.

Ex:



→ Neither voltage nor current controlled.

In EE 201/202 resistors are components which map branch currents to branch voltages (current controlled resistor) or branch voltages to branch currents (voltage controlled resistor) or they just have a locus of operating points in the (i, v) plane associated to the component (neither voltage nor current controlled)
↳ ??



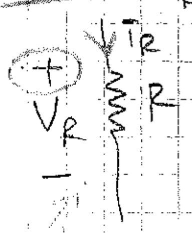
Diode is also a resistor (non-linear resistor)

Passive and Active Components

A component is passive if it cannot deliver energy/power to the other components; hence it only absorbs or stores energy.

A component which is not passive → active component.

Ex: $V_R = R \cdot i_R$



$$\rightarrow \underbrace{P_R(t)}_{} = V_R(t)\, i_R(t) = R \cdot \left(i_R(t)\right)^2$$

Power absorbed or delivered by the component at time "t" = Instantaneous power

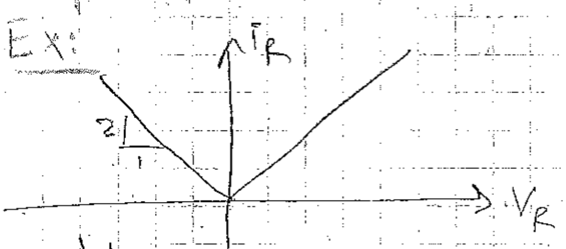
Since $P_R(t) = R\,(i_R(t))^2 \geq 0$

↳ Component absorbs power. $(R > 0)$

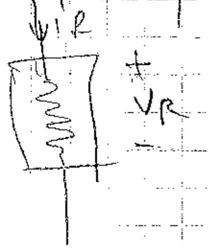
↳ Component is passive.

Passive sign component

Ex:



→ Does the (i, v) char. belong to a passive or active component?



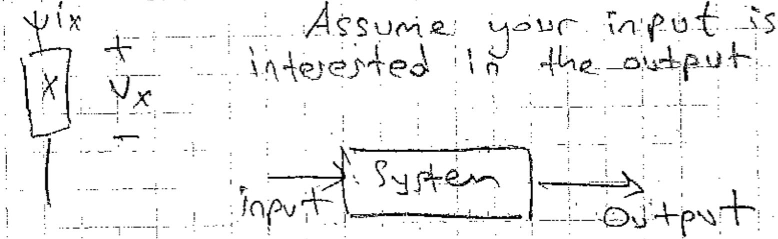
Check the power of the resistor and see whether power of resistor can be negative valued or not.

$$P_R = i_R \cdot V_R$$

If $V_R = -5$ Volts → $i_R = 10$ Amperes from the (i, v) chart

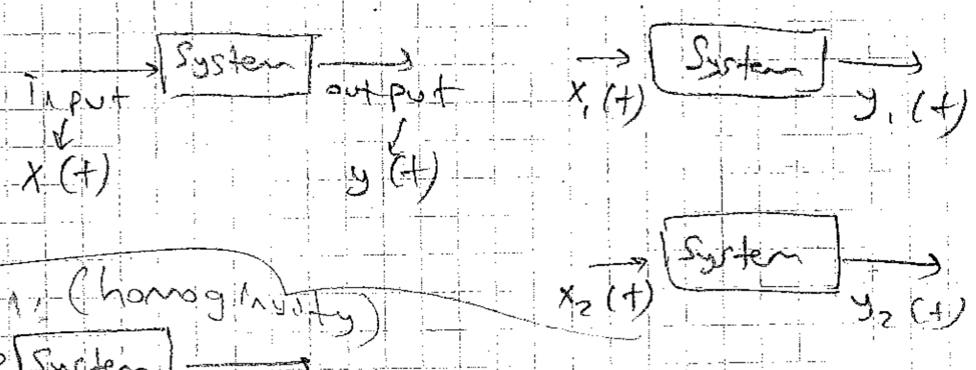
then $P_R = (-5) \cdot 10 = -50$ Watts → At this operating point, the component delivers power to the other components.

Linear and non linear components

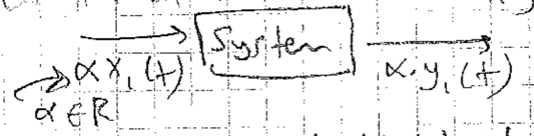


Assume your input is i_x and you are interested in the output V_x.

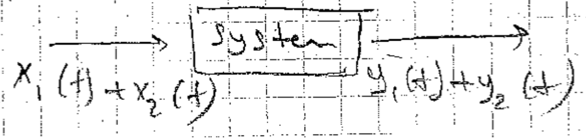
If system satisfies the following conditions it is said to be a linear system:



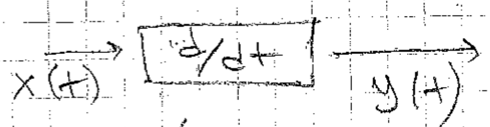
Condition 1: (homogenuity)



Condition 2 (additivity)



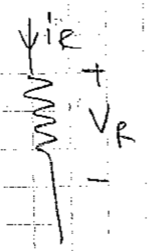
Ex:



It is a linear system.

The system takes derivatives of input.

Ex:



$V_R = R \cdot i_R$ (Ohm's Law)

If input: i_R, output: V_R, then is the simple resistor linear or not?

$V_R(t) = R \cdot i_R(t)$ (Not derivative)

① Homogenuity: $i_R'(t) = \alpha \cdot i_R(t)$

$$V_R'(t) = R\,(i_R'(t)) = R\,\alpha\, i_R(t)$$
$$= \alpha \cdot V_R(t)$$

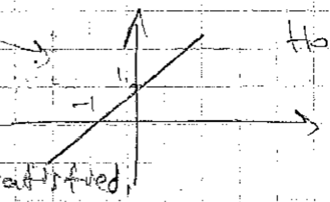
② Additivity can be checked similarly.

A simple resistor is a linear component.

or not linear)

Ex: $V_x(t) = (i_x(t))^2 \longrightarrow$ X (Both conditions are not satisfied)

$V_x(t) = i_x(t) + 1 \longrightarrow$ X



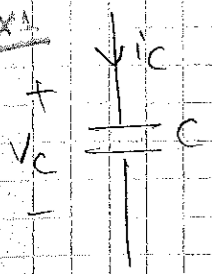
Homogeneity not satisfied.

$V_x(t) = i_x(t) \frac{d i_x(t)}{dt} \longrightarrow$ X Linearity / Homogeneity are not satisfied.

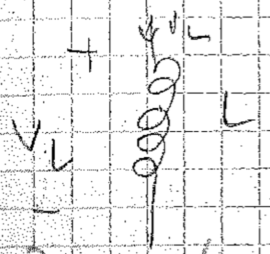
$2 \cdot V_x \neq 4 \cdot V_x$

Hint: If the system is linear zero input should result in zero output.

Ex:



$$I_C(t) = C \frac{dV_C(t)}{dt}$$ → This is linear. ✓



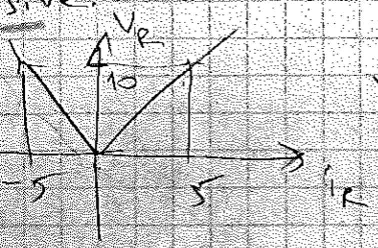
$$V_L(t) = L \frac{di_L(t)}{dt}$$ → Linear ✓.

Passive / active discussion:

If a resistor ever operates in power delivery region, then it is active.

That is, if a resistor always absorbs ($p(t) \geq 0$) power → passive.

Ex:



→ Active.

Then if a memoryless component has an i-V characteristic limited to 1st and 3rd quadrants → passive component.

If it has a single operating point in 2nd or 4th quadrants then it is an active component.

Memoryless Component: A component whose branch current and voltage depend only on instanteneous values of the branch variables.

Ex:

$V_x(t) = R \cdot I_x(t) \longrightarrow$ ✓ Since, $V_x(t_0)$ depends only on the input at time t_0.

$V_L(t) = L \frac{di_L(t)}{dt} \longrightarrow$ Not memoryless (Only t_0 and $i_L(t_0)$ won't be enough)

$V_C(t) = \frac{1}{C} \int_{-\infty}^{t} i_C(\tau) d\tau \longrightarrow$ X Not memoryless / a component with memory

Time invariant / time varying components

Time invariant: The mathematical rule connecting input and output that does not depend on time → Time-invariant system.

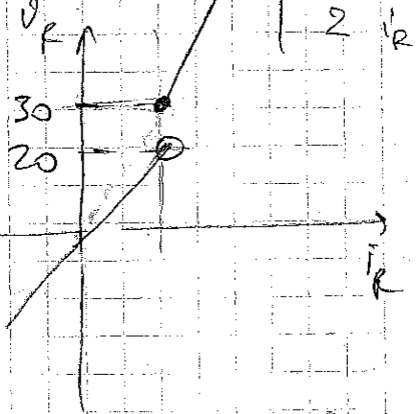
Time varying: Not time-invariant.

Ex: $v_R(t) = \begin{cases} 3\, i_R(t) & t \geq 10 \\ 2\, i_R(t) & t < 10 \end{cases}$ → Time-varying

Then mathematical rule connecting input $i_R(t)$ to output $V_R(t)$ depends on time, ie there is an if statement in the mathematical rule depending on time.

Ex:

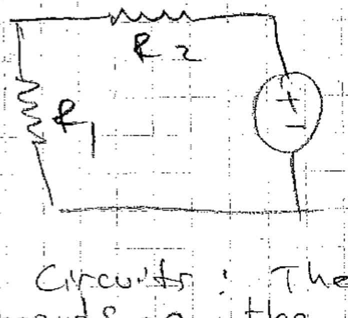
$$V_R(t) = \begin{cases} 3\, i_R(t), & i_R(t) \geq 10 \\ 2\, i_R(t), & i_R(t) < 10 \end{cases}$$



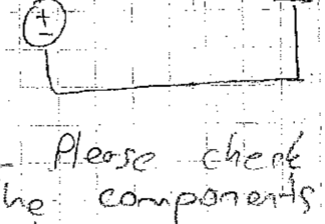
→ Passive, non-linear, time-invariant, current controlled.

Dynamic and Resistive Circuits

If a circuit is composed purely of memoryless (resistors) components → resistive circuits.



Dynamic Circuits: The circuit contains components which depends on the rate of change of circuit variables → dynamic circuits

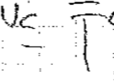


$i_c(t) = C \frac{dV_c(t)}{dt}$

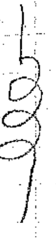
HW: Please check ZPS-I and examine problems on the components, their classification and simple KVL / KCL applications.

$\boxed{Q = V \cdot C}$

In ZPS-I,



$Q_c(t) = C \cdot V_c^2(t)$ $\xrightarrow{d/dt}$ $i_c(t)$ → $C \frac{dV_c^2(t)}{dt}$



$\phi(t) = L\cdot i(t)$

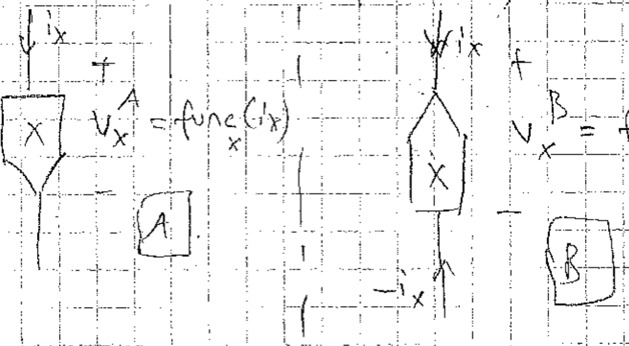
$\hookrightarrow \frac{d}{dt} \rightarrow V_c(t) = L\cdot\frac{di(t)}{dt}$

Components:

- Linear / Non-linear
- Memoryless / Dynamic
- Time-invariant / Time-varying
- Passive / Active

} Classification of Components

Unilateral / Bilateral Component:



If component orientation (which side is up) does not matter → bilateral component,
If it matters → unilateral component.

Then, for the same i_x, if I get the same V_x for components shown in $\boxed{A}$ and $\boxed{B}$,

From Fig. $\boxed{B}$, $\rightarrow V_x^B = -func_x(-i_x)$
Fig. $\boxed{A}$, $\rightarrow V_x^A = func_x(i_x)$

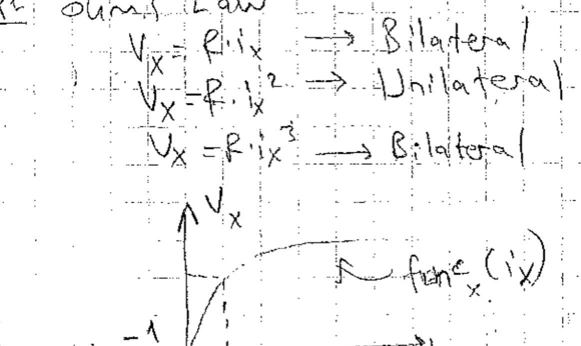
So, $V_x^A = V_x^B \ (\forall i_x)$
$\Longleftrightarrow func_x(i_x) = -func_x(-i_x)$

↓

Bilateral !!!

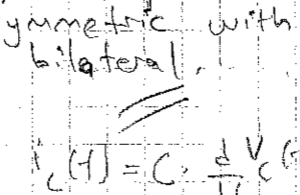
Ex: ohm's Law
$V_x = R\cdot i_x \rightarrow$ Bilateral
$V_x = R\cdot i_x^2 \rightarrow$ Unilateral
$V_x = R\cdot i_x^3 \rightarrow$ Bilateral

A component whose (i, V) char. is symmetric with respect to origin is bilateral.

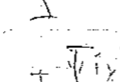


↖ Bilateral

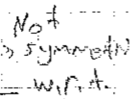
$i_c(t) = C\cdot\frac{d}{dt}V_c(t) \rightarrow$ Bilateral



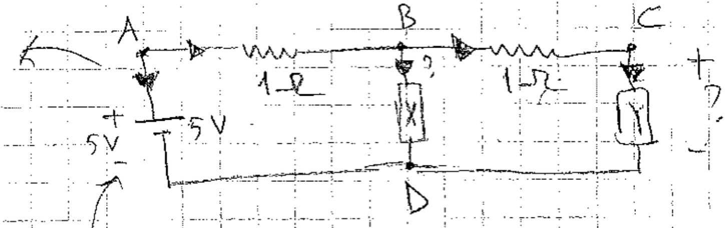
} Bilateral



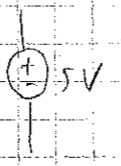
} Unilateral

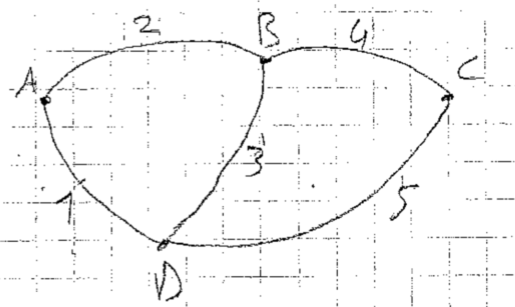


Circuits / Electrical Networks



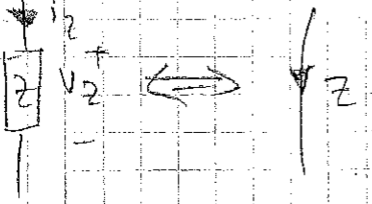
Notation for DC sources =

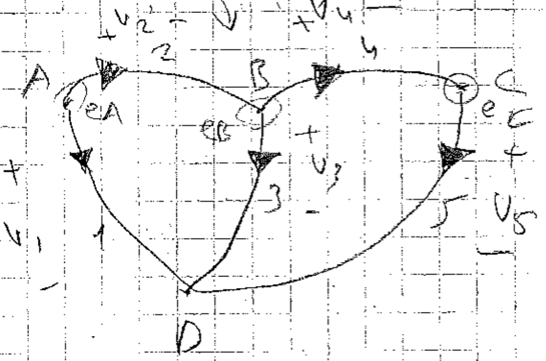




Graph of the circuit

Directed graph of the circuit





Incidence matrix

$$\underset{=}{A_a} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline A & 1 & 1 & 0 & 0 & 0 \\ B & 0 & -1 & 1 & 1 & 0 \\ C & 0 & 0 & 0 & -1 & 1 \\ D & -1 & 0 & -1 & 0 & -1 \end{array}$$

→ Branch #5 leaves node C.

→ 〃 〃 enters into node D.

Observation:

① About KCL: KCL at every node can be written:

$$1^{st}\text{ Row}\rightarrow \overbrace{\begin{bmatrix}1&1&0&0&0\\ &&A_a&&\\ &&&&\end{bmatrix}}^{A_a}\overbrace{\begin{bmatrix}J_1\\J_2\\J_3\\J_4\\J_5\end{bmatrix}}^{J}=\overbrace{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}^{0}\rightarrow 1^{st}\text{ Row}\Rightarrow J_1+J_2=0$$

Question: Is the equation system $A_a J = 0$

(4 equations with 5 unknowns for this example) redundant?

That is, are the equations independent?

$x+y=2$

$-x-y=-2$

$2x+3y=5$

(or $3x+4y=7$)

we don't need this equation.

$$\begin{bmatrix}1&1\\3&4\\2&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\7\\5\end{bmatrix}$$

Rank of this matrix = 2

(smallest square matrix inside it)

Answer: $A_a \cdot J = 0$ equation system contains a single redundant equation.

$$\underset{\substack{\text{Reduced}\\\text{Incidence}\\\text{Matrix}}}{A} = \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ A & 1 & 1 & 0 & 0 & 0 \\ B & 0 & -1 & 1 & 1 & 0 \\ C & 0 & 0 & 0 & -1 & 1 \end{array} \longrightarrow \underset{\substack{\uparrow\\\text{No redundant equation.}}}{A \cdot J = 0}$$

Comment: In general we select a node (Node D in this example) as a reference node / Datum node / Ground node and do not write KCL equations for this node, since it would be a redundant equation.

② About Branch Voltages

$$\underbrace{v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_5 \end{bmatrix}}_{\substack{\text{Branch}\\\text{volt.}\\\text{vector}}} \qquad \underbrace{e = \begin{bmatrix} e_A \\ e_B \\ e_C \end{bmatrix}}_{\substack{\text{Node}\\\text{Voltage}\\\text{Vector.}}} \longleftarrow \text{voltage between a node and datum/ground/reference node.}$$

$$\underset{\sim}{v} = \underset{=}{A}^T \cdot \underset{\sim}{e} \longrightarrow v = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_A \\ e_B \\ e_C \end{bmatrix}$$

③ Conservation of Power (Tellegen's Theorem):

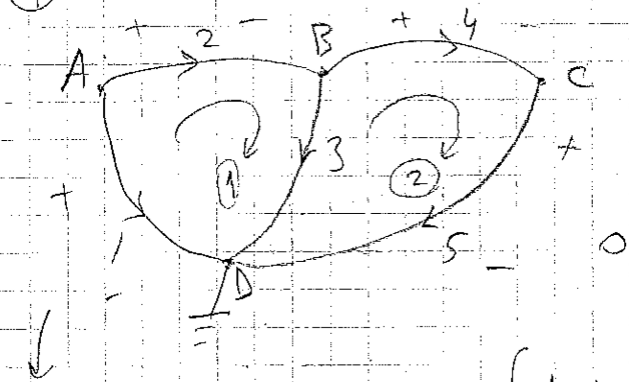
$$\sum_{k=1}^{\#\text{Branches}} \underbrace{v_k j_k}_{P_k(t)} = 0 \longleftarrow \text{Conservation of power statement.}$$

$P_k(t)$ ← Instanteneous power absorbed by k^{th} branch.

Proof:
$$\sum_{k=1}^{\#\text{branches}} P_k(t) = \sum_{k}^{\#\text{branches}} v_k(t) j_k(t) = [v_1 \ v_2 \ldots v_k] \begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_k \end{bmatrix} = v^T \cdot j = (A^T e)^T \cdot j$$
$$= (e^T \cdot A) \cdot j$$
$$= e^T \underbrace{(AJ)}_{0}$$
$$= e^T \cdot 0$$
$$= 0$$

So, conservation of power is verified.

④ Mesh Matrix and Mesh Currents



Passive sign convention.

About KVL:

Mesh$_1$: $+v_2+v_3-v_1=0$

Mesh$_2$: $v_4+v_5-v_3=0$

Outer mesh : $v_2+v_4+v_5-v_1=0$

$$\underbrace{\begin{bmatrix}-1 & 1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 1\\ -1 & 1 & 0 & 1 & 1\end{bmatrix}}_{M_a}\underbrace{\begin{bmatrix}v_1\\v_2\\v_3\\v_4\\v_5\end{bmatrix}}_{v}=\underbrace{\begin{bmatrix}0\\0\\0\end{bmatrix}}_{0}$$

Outermesh KVL equation is redundant, so we focus on individual meshes

$$\boxed{M}\cdot v = 0 \longrightarrow \text{mesh matrix}$$

$$\begin{bmatrix}-1 & 1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 1\end{bmatrix}$$

Mesh Currents:

$$\boxed{j = M^T \cdot i}$$

branch current vector

mesh current vector

$$\begin{bmatrix}i_1\\i_2\\ \vdots\\ i_m\end{bmatrix}=i \leftarrow \text{mesh current vector}$$

$j_1 = -i_1$

$j_2 = i_1$

$j_3 = i_1 - i_2$

$j_4 = i_2$

$j_5 = i_2$

this i_1 and i_2 are imaginary values, we can't measure them.

Dual Circuits/Variables/Graphs etc.

Previously we have expressed a circuit in terms of node voltages or mesh currents. To do that we have used KCL and KVL equations and we have observed that the theory is very similar for both cases.

Node Analysis

$$A \cdot j = 0$$

$$v = A^T \cdot e$$

branch current vector

branch voltages

node voltage

Mesh Analysis

$$M \cdot v = 0$$

$$j = M^T \cdot i$$

branch voltage vector

branch current

mesh current

Dual component

A dual component is a component whose math. description is achieved by the interchange of current (i) and voltage of the original component.

Ex: $\vartheta = 2(i)^2 \underset{Dual}{\longleftrightarrow} i = 2(V)^2 \leftarrow$ interchange $i \leftrightarrow \vartheta$

$\vartheta_{(\Omega)} = k\frac{di(t)}{dt} \underset{Dual}{\longleftrightarrow} i(t) = k\frac{dv(t)}{dt}$

(k Henry Inductor) (k Farad capacitor)

$V = R \cdot i \underset{Dual}{\longleftrightarrow} i = R \cdot \vartheta$

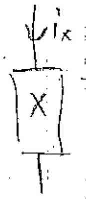
(R Ω resistor) (R $\mho$ = $\frac{1}{R}$ Ω resistor)

Dual Variables	Duals Components
flux ⟷ charge	R Ω ⟷ $\frac{1}{R}$ Ω
Volt ⟷ current	k F (cap.) ⟷ k H (inductor)
	Vol. source ⟷ Current source

Dual Equations

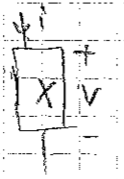
KVL ⟷ KCL

Dual circuit
↓
on Wednesday.



$V_x = func_x(ix) \xrightarrow{Dual}$

$V_x = 3(i_x)^2$



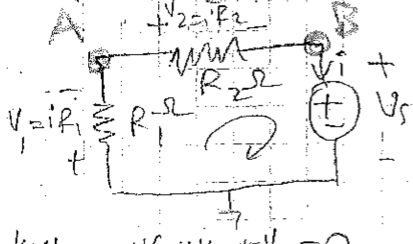
$i = func(V)$

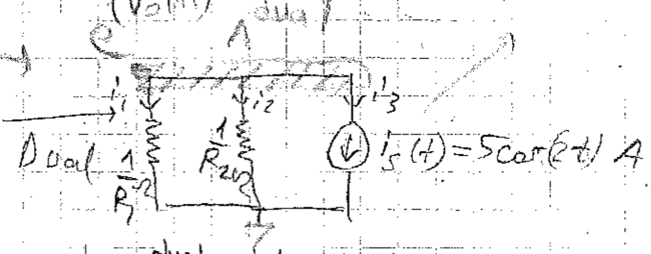
$i = 3V^2$

Akım aşağı yönlü

Ex: C: $i(t) = k\frac{dv(t)}{dt}$ ↪ L: $v(t) = k\frac{di(t)}{dt}$ $\underset{Dual}{C \longleftrightarrow L}$

Dual Circuit $\xrightarrow{Dual}$





KVL: $+V_1 + V_2 + V_3 = 0$

$iR_1 + iR_2 + 5cos2t = 0$

$i = -\frac{5cos2t}{R_1 + R_2}$ A

KCL at A^{dual}, $i_1 + i_2 + i_3 = 0$

$\frac{e}{\frac{1}{R_1}} + \frac{e}{\frac{1}{R_2}} + 5cos(2t) = 0$

$e = -\frac{5cos(2t)}{R_1 + R_2}$ Volts

Circuits A and B are duals of each other if branch voltages of say circuit A is identical to the branch currents of circuit B.

How to find dual circuit?

① Map components to their dual components.

② Map circuit configuration to the dual configuration.
(Parallel ⟷ Series)

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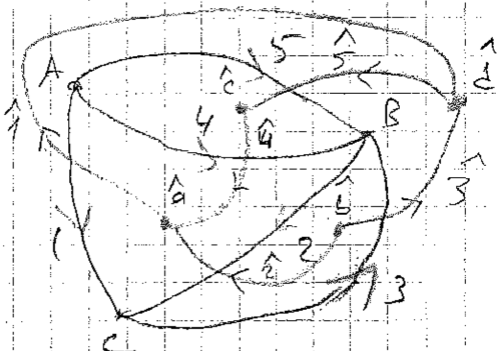


→ Circuit graph

outer mesh → 1-3-5 loop

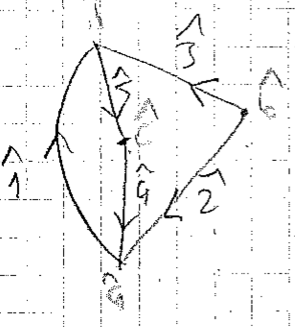
→ $\hat{a}$, $\hat{b}$, $\hat{c}$ — the names are the nodes aren't important.

How to find dual graph?

① Place a node at the center of each mesh including outer mesh.

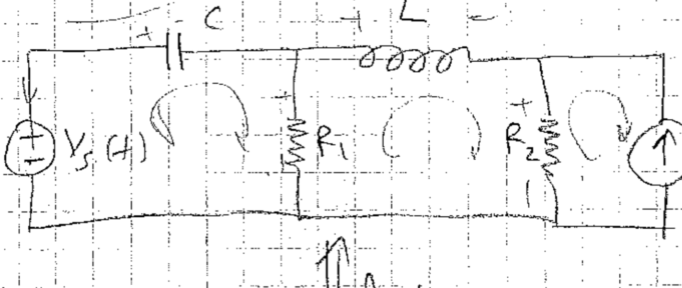
② Every branch is now in between two nodes that we put.

③ Connect the nodes we put and rotate the direction of the link in the circuit graph by 90°.
→ Clockwise direction



$-V_s + V_c + i R_1 = 0$

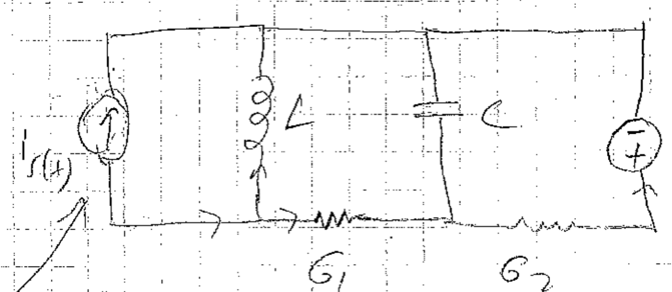
Ex:



→ Work at this example.

passive sign convention

⇕ Dual

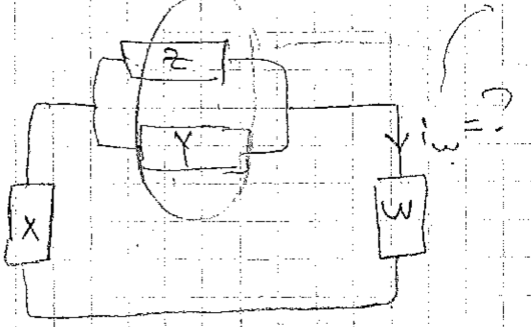


wrong direction!

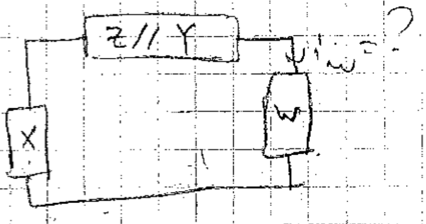
Series and Parallel Combinations

Circuit Simplifications

Our only focus

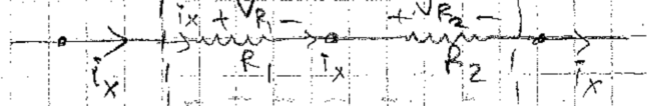


Simplify two components into an equivalent single component.



Series combination of LTI resistors

LTI (Linear Time-Invariant) Resistor → $V = i \cdot R$



$V_{R_1} = R_1 \cdot i_x$

$V_{R_2} = R_2 \cdot i_x$

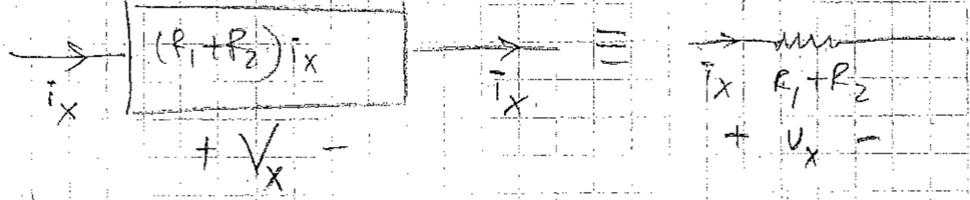
$V_x = V_{R_1} + V_{R_2}$

$= R_1 \cdot i_x + R_2 \cdot i_x$

$\boxed{V_x = (R_1 + R_2) i_x}$

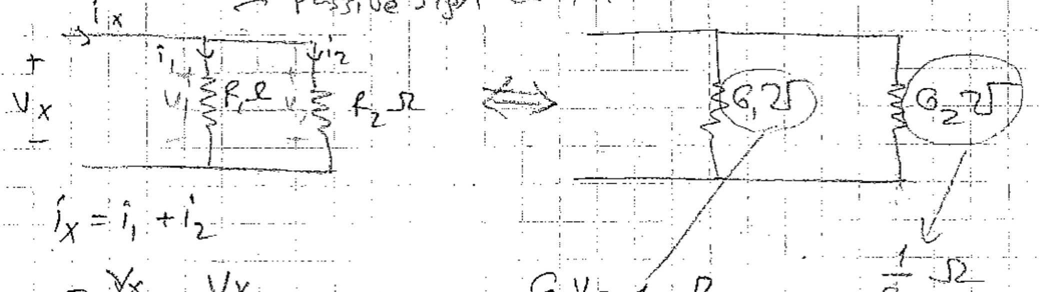
$+ \quad V_x \quad -$

|||



Parallel combination

→ Passive Sign convention



$i_x = i_1 + i_2$

$= \frac{V_x}{R_1} + \frac{V_x}{R_2}$

$G_1 \mho = \frac{1}{R_1} \Omega$ $\qquad \frac{1}{R_2} \Omega$

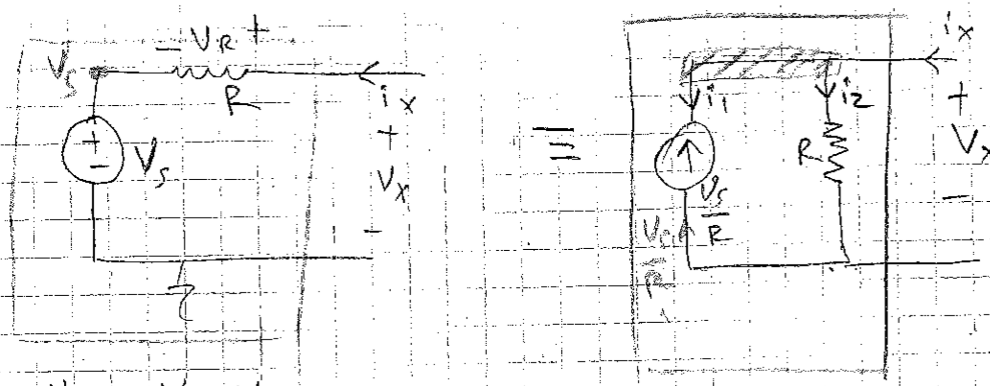
$i_x = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) V_x \longrightarrow V_x = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} i_x$

The box is equivalent to resistance $\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} \Omega$ or $(G_1 + G_2) \mho$

$G_1 = \frac{1}{R_1}$

Conductance of a resistor.

Source Transformation



$V_R = V_x - V_s$

$i_R = \frac{V_R}{R} = \frac{V_x - V_s}{R}$

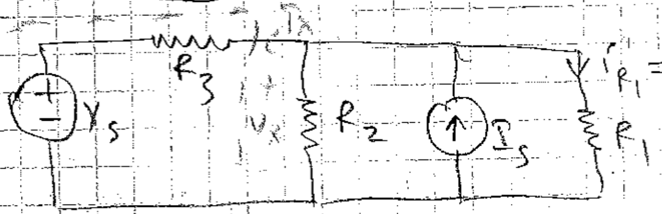
$$\boxed{i_x = \frac{V_x}{R} - \frac{V_s}{R}}$$

$i_x = i_1 + i_2$

$$\boxed{i_x = \frac{-V_s}{R} + \frac{V_x}{R}}$$

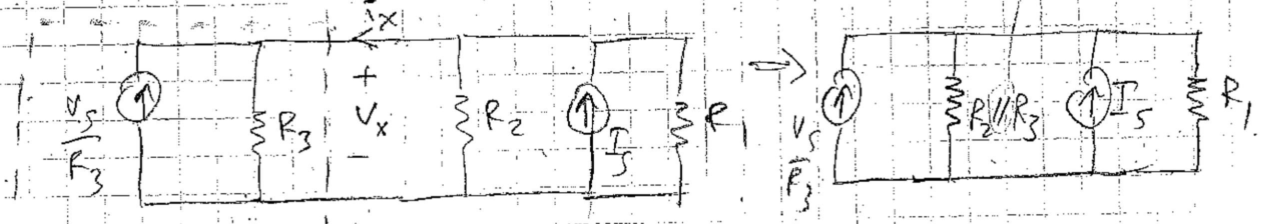
The same

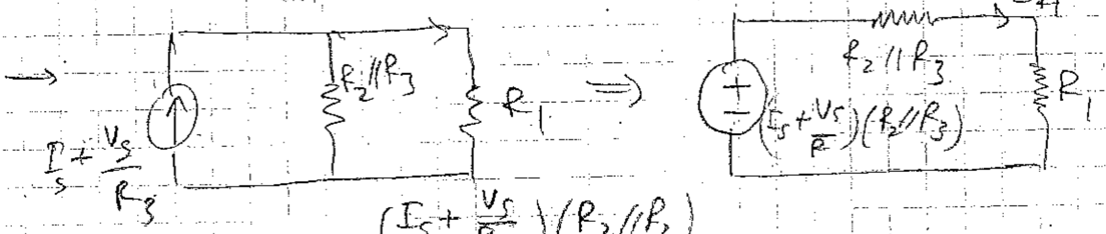
Ex:



Find i_{R_1}

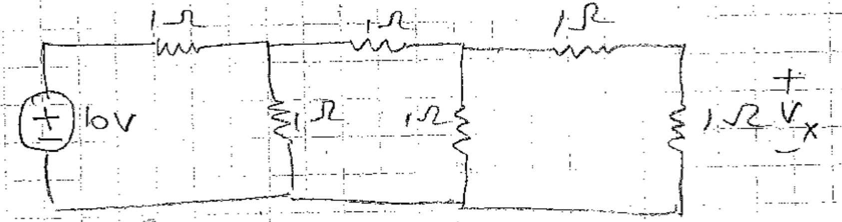
$R_2 // R_3 = \left(\frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$





$$\rightarrow I_{R_1} = \frac{\left(I_s + \frac{V_s}{R_3}\right)(R_2//R_3)}{R_2//R_3 + R_1}$$

Ex:

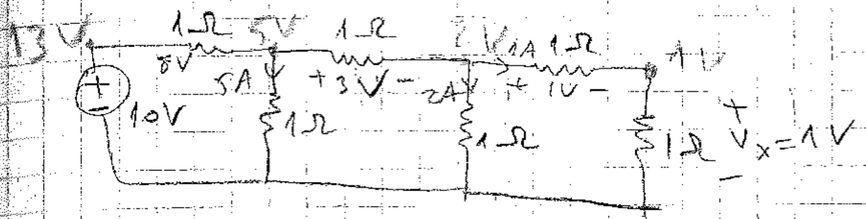


Find V_x.

Ladder Network

I will use linearity of the circuit, that is all components in the circuit is linear, hence the relation betwee input (10 V source) and output V_x.

Let's make a guess on V_x. Let $V_x = 1V$



Then If V_s were 13V
→ V_x would be 1V.
But I know $V_s = 10V$
and circuit is linear.

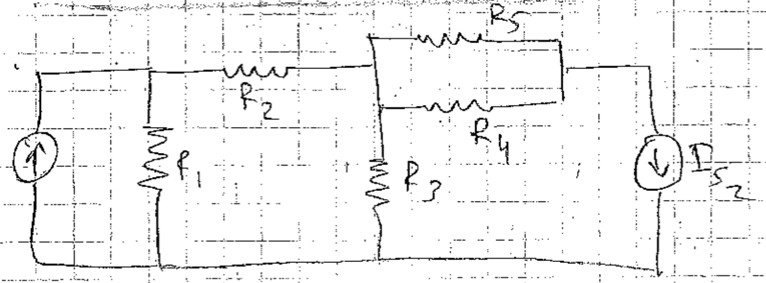
13	1
10	V_x

$$V_x = \frac{10}{13} V$$

Solution is achieved by making use of linearity of the circuit.

Node Analysis Method:

Ex:



Find branch voltages and branch currents.

Circuit Analysis: Goal: finding branch currents and voltages such that;

① All component equations (terminal equations) should be satisfied,

② KCL equations should be satisfied, ← # of independent KCL equations = # of nodes − 1

③ KVL " " "

↳ # of independent KVL equations = # of meshes (Not counting outer mesh)

(only valid for planar circuits)

From ① 7 branch currents/voltages should be set such that each branch component's i-v relation is satisfied.

② 3 independent KCL equations (constraints) ↳ $i_1 + i_2 - i_3 = 0$ ↙ constraint

③ 4 " KVL constraints.

Apply Node Analysis (to solution)

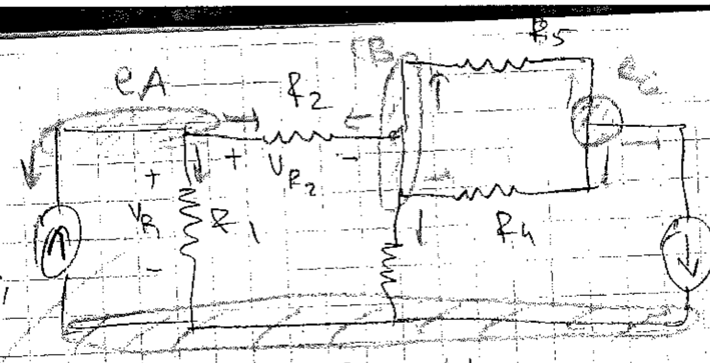
Procedure:

① Select a datum node

② Assign node voltages to the remaining nodes ($e_1, e_2, \ldots$), ($e_A, e_B \ldots$)

③ Write KCL at every node except reference/datum node using node voltages.

④ Solve KCL equations → find node voltages → find branch voltages → find branch currents.



KCL at e_A: $-I_{s_1} + \frac{e_A}{R_1} + \frac{e_A - e_B}{R_2} = 0$

$V_{R_1} = e_A - 0$
$= e_A$

KCL at e_B: $\frac{e_B}{R_3} + \frac{e_B - e_C}{R_4} + \frac{e_B - e_C}{R_5} + \frac{e_B - e_A}{R_2} = 0$

KCL at e_C: $\frac{e_C - e_B}{R_4} + \frac{e_C - e_B}{R_5} + I_{S_2} = 0$

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} & 0 \\ -1/R_2 & \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} + 1/R_2 & -\frac{1}{R_4} - \frac{1}{R_5} \\ 0 & -\frac{1}{R_4} - \frac{1}{R_5} & \frac{1}{R_4} + \frac{1}{R_5} \end{bmatrix} \begin{bmatrix} e_A \\ e_B \\ e_C \end{bmatrix} = \begin{bmatrix} I_{S_1} \\ 0 \\ -I_{S_2} \end{bmatrix}$$

↳ Solve for e_A, e_B, e_C

Then, since any branch is between two nodes the branch voltages can be found immediately from node voltages.

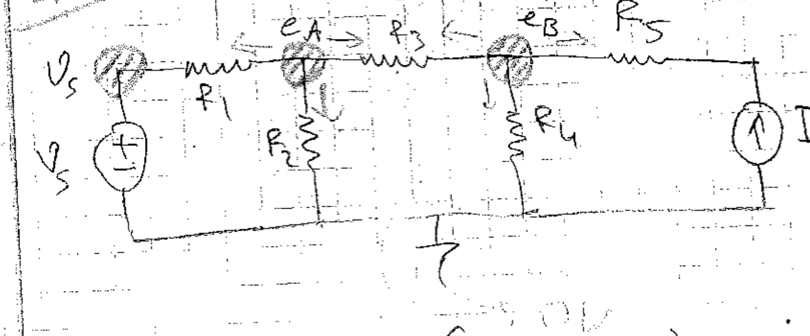
$V_{R_1} = e_A - 0 = e_A$, $V_{R_4} = e_B - e_C$, ...

Then, every branch voltage is known, from component equations branch currents can be immediately calculated.

$i_{R_4} = V_{R_4} / R_4$, ...

Node analysis aims to find node voltages $(e_A, e_B, ...)$, and unknowns of node analysis should be only node voltages.

Ex - 2 (Node analysis with voltage sources)



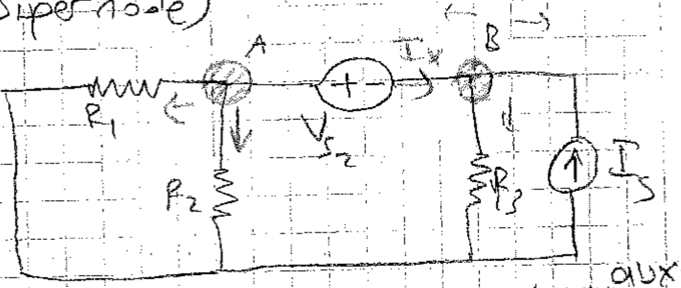
((Important!))

KCL at e_A: $\frac{e_A - V_S}{R_1} + \frac{e_A}{R_2} + \frac{\text{[illegible]}}{R_3} = 0$

KCL at e_B: $\frac{e_B}{R_4} + \frac{e_B - e_A}{R_3} - I_S = 0$

} 2 equations and 2 unknowns solve for e_A and e_B

Ex: (Supernode)



Apply node analysis.

auxilary unknown

KCL at e_A: $\frac{e_A}{R_1} + \frac{e_A}{R_2} + I_x = 0$ (I)

KCL at e_B: $\frac{e_B}{R_3} - I_S - I_x = 0$ (II)

] 2 equation 3 unknown

Auxilary Equation: $e_A - e_B = V_{S_2}$ (III) ⟵ 3rd equation

Let's solve for e_A and e_B

$$\text{(I)} + \text{(II)} \rightarrow \frac{e_A}{R_1} + \frac{e_A}{R_2} + \frac{\overset{(e_A - V_{S_2})}{e_B}}{R_3} - I_S = 0$$

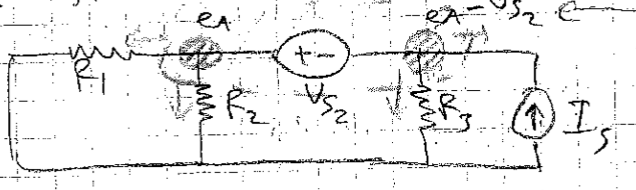
$$\text{(III)} \longrightarrow e_A - e_B = V_{S_2}$$

$$\text{(I)} \quad e_A = \frac{I_S + V_{S_2}/R_3}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \text{ Volts}$$

$$\text{(II)} \quad e_B = e_A - V_{S_2} \text{ Volts}$$

(III)

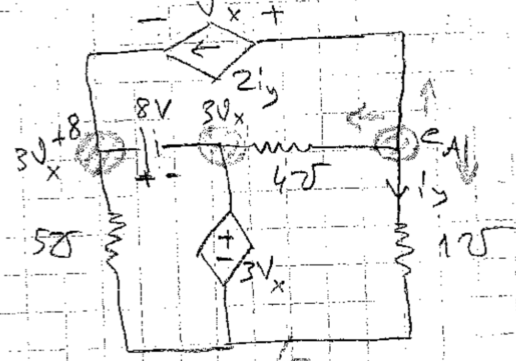
Instead of Introducing auxiliary variable I_x, we may use supernode approach to shorten the solution.



⟵ supernode

KCL at supernode: $\frac{e_A}{R_1} + \frac{e_A}{R_2} + \frac{e_A - V_{S_2}}{R_3} - I_S = 0$

Ex: ZPS-2
Pr, 3f.



Apply node analysis and find branch currents voltages.

KCL at e_A: $\frac{e_A}{1} + \frac{e_A - 3(V_x)}{1/4} + 2I_y = 0$

i_y in terms of node voltages

$I_y = \frac{e_A}{1}$ ✓

$V_x = e_A - (3V_x + 8)$

$\boxed{V_x = \frac{e_A - 8}{4}}$ ✓

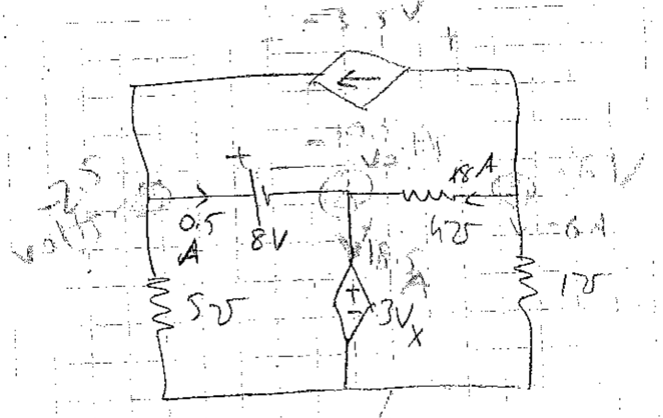
$$e_A + \frac{e_A - \frac{3}{4}(e_A - 8)}{1/4} + 2\frac{e_A}{1} = 0$$

$e_A + 4e_A - 3e_A + 24 + 2e_A = 0$

$4e_A = -24$

$e_A = -6A$

$V_x = -\frac{7}{2}$ Volts



Power of 8V source = ?

$P_{8V} = v(t) \cdot i(t)$
$= 8 \times \frac{1}{2} = 4$ Watts

4 Watts absorbed

$P_{3V_x} = (18.5) \times (-10.5) = \boxed{-194.25 \text{ watts}}$ absorbed

$= 194.25$ watts delivered

$P_{2i_y} = (-3.5)(-12) = 42$ Watts absorbed

$P_{1\Omega} = \frac{V_{1\Omega}^2}{R} = 36$ watts

$P_{5\Omega} = I^2 R = (12.5)^2 \cdot \frac{1}{5} = (2.5)(12.5) = 31.25$ watts

$P_{4\Omega} = I^2 R = 18^2 \cdot \frac{1}{4} = 81$ watts

$P_x = V_x i_x$ — Gives me absorbed power in watts

Let's check whether power is conserved.

$$P_{3v_x} + P_{2i_y} + P_{5V} + P_{5\Omega} + P_{4\Omega} + P_{1\Omega} \overset{?}{=} 0$$

$$-194.25 + 42 + 4 + 31.25 + 81 + 36 = 0$$ ✓

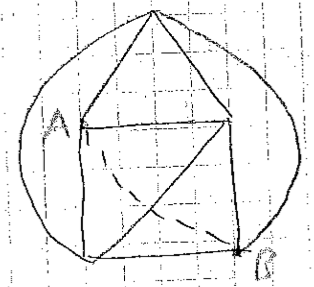
HW#2 is posted ← Also check Alexander-Sadiku for Node and Mesh analysis

Mesh Analysis

For a planar circuit, the possible analysis is the mesh analysis.

Planar circuit: A circuit whose graph can be drawn on two dimensional surface, i.e. drawn on paper.

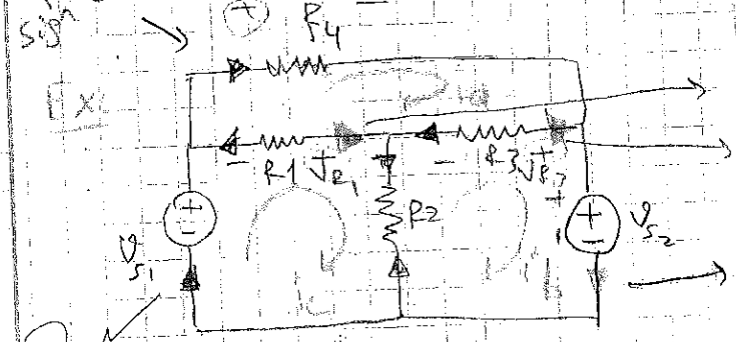
Non-planar graph: ← Non-planar graph



← A↔B connection cannot be made on a 2-D plane.

passive sign convention

Ex



→ Branch current of R_1: $i_{R_1} = i_c - i_a$

→ $i_{R_3} = i_b - i_a$

→ $i_{S_2} = i_b$

Since circuit is planar, mesh analysis is applicable. To do mesh analysis;

① Introduce mesh currents in each mesh,

② Write KVL equations using mesh currents as unknowns.

③ Solve for mesh currents → get branch currents → get branch voltages from component equations

KVL around i_a: $V_{R_4} + V_{R_3} + V_{R_1} = 0$

$$R_4(i_a) + R_3(i_a - i_b) + R_1(i_a - i_c) = 0$$

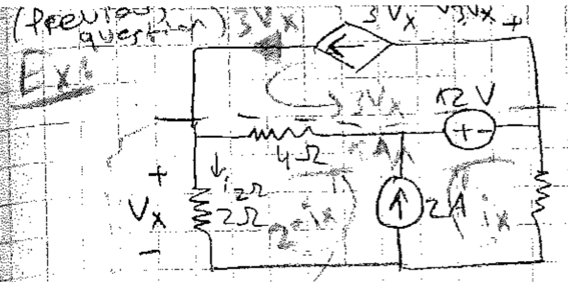
KVL around i_b: $R_2(i_b - i_c) + R_3(i_b - i_a) + V_{S_2} = 0$

KVL around i_c mesh: $-V_{S_1} + R_1(i_c - i_a) + R_2(i_c - i_b) = 0$

3 equ. 3 unknown

Every branch voltage can be found $V_{R_3} = i_{R_3} R_3$ ← every branch current can be found $i_{R_3} = i_b - i_a$ ← solve for i_a, i_b, i_c

Ex:



$? + i_x = 2$

$? = 2 - i_x$

Supermesh

$V_x = 2 \cdot i_{2\Omega} = 2(2 - i_x)$

KVL: $4(\underbrace{3V_x}_{6(2-i_x)} + i_x - 2) + 12 + 6i_x + 2(i_x - 2) = 0$

$$i_x = \frac{-40 - 12 + 4}{-20 + 6 + 2}$$

$$i_x = \frac{-48}{-12} = \boxed{4A}$$

KVL for V_{3V_x}: $-12 + 4(2 - i_x - 3V_x) - V_{3V_x} = 0$

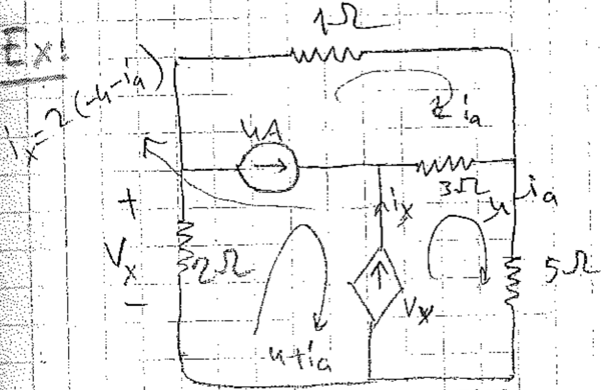
$V_{3V_x} = -12 + 4 \cdot 10$

$V_{3V_x} = 28$ Volts

? ?

1 KVL equat is enough on # of meshes − # of CS = 1

Ex!



Analyze the circuit.

Mesh Analysis

Only one unknown : i_a

Outer mesh is supermesh

Node Analysis :

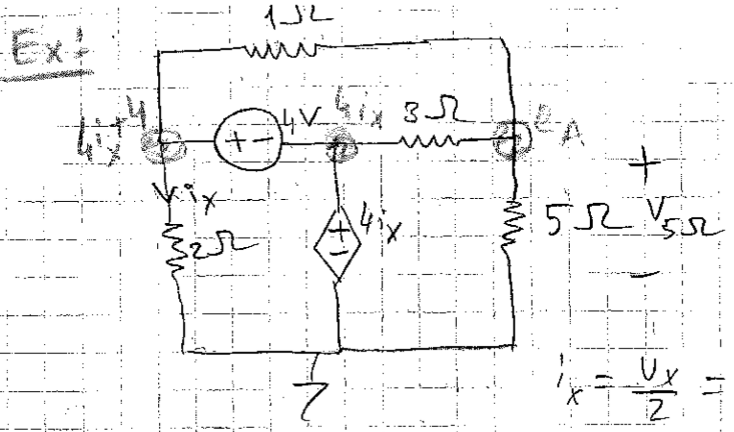
e_A, e_B, e_C 3 unknowns to solv

☆ General comment on # of equations for Node/Mesh analysis.

① Mesh analysis : # unknowns = # meshes − # of current sources (dependent or independent)

② Node analysis : # of unknowns = # of node voltages − # of voltage sources (dependent or Independent)

Ex1:



Analyze the circuit and find $V_{5Ω}$.

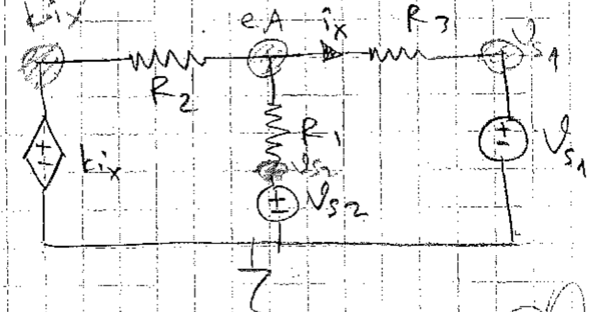
$$i_x = \frac{V_x}{2} = \frac{4i_x+4}{2} \rightarrow i_x = -2A$$

KCL at e_A:

$$\frac{e_A}{5} + \frac{e_A-(-8)}{3} + \frac{e_A-(-4)}{1} = 0$$

$$\longrightarrow e_A = \frac{-40-60}{3+5+15} = \frac{-100}{23} \text{ Volts}$$

Circuits with multiple inputs (independent sources) ← sources



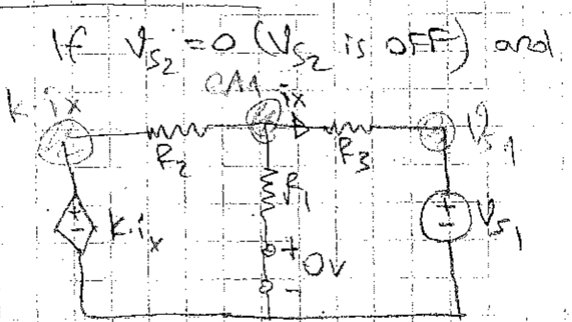
Analyze using node analysis.

KCL at e_A: $$\frac{e_A - v_{s2}}{R_1} + \frac{e_A - k\cdot i_x}{R_2} + \frac{e_A - v_{s1}}{R_3} = 0$$

$$i_x = \frac{e_A - v_{s1}}{R_3}$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{k}{R_2R_3} + \frac{1}{R_3}\right)e_A = \frac{1}{R_1}v_{s2} - \frac{k}{R_2R_3}V_{s1} + \frac{1}{R_3}V_{s1} \quad (I)$$

If $v_{s2}=0$ (v_{s2} is OFF) and $v_{s1} \neq 0$ (v_{s1} is ON)



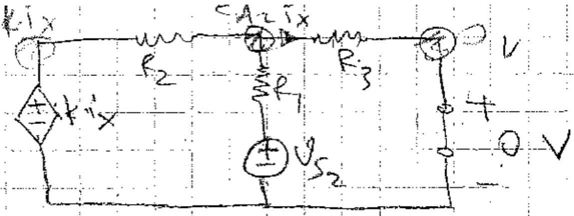
KCL at e_{A1}:

$$\frac{e_{A1} - v_{s1}}{R_3}$$

$$\frac{e_{A1}}{R_1} + \frac{e_{A1} - k\cdot i_x}{R_2} + \frac{e_{A1} - v_{s1}}{R_3} = 0$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{k}{R_2\cdot R_3} + \frac{1}{R_3}\right)e_{A1} = \frac{-k\, v_{s1}}{R_2R_3} + \frac{v_{s1}}{R_3} \quad (\bigstar_1)$$

($\bigstar_2$) Take $\vartheta_{S_1} = 0$ (OFF)
ϑ_{S_2} : ON



KCL at e_{A_2}:

$$\left(\frac{e_{A_2} - \vartheta_{S2}}{R_1}\right) + \frac{e_{A_2} - k \cdot i_x}{R_2} + \frac{e_{A2}}{R_3} = 0$$

$$\hookrightarrow i_x = \frac{e_{A2}}{R_3}$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{k}{R_2 R_3} + \frac{1}{R_3}\right) e_{A_2} = \frac{1}{R_1} \vartheta_{S2} \quad (\bigstar_2)$$

If V_{S_1}: ON, ϑ_{S_2}: ON, Q! Can I use ($\bigstar_1$) and ($\bigstar_2$) solutions to find the solution (general solution for $\vartheta_{S_1} \neq 0$, $V_{S_2} \neq 0$)?

A: $e_A = e_{A_1} + e_{A_2}$

Solution of $\bigstar_1$ — Solution of $\bigstar_2$

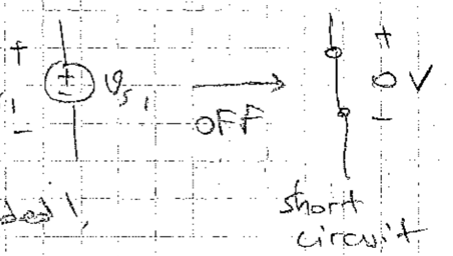
$$A \cdot e = b_1 \cdot \vartheta_{S_1} + b_2 \cdot \vartheta_{S_2}$$

$$e = \begin{bmatrix} e_A \\ e_B \end{bmatrix}$$

$$e = A^{-1}(b\vartheta_{S_1} + b_2 \cdot \vartheta_{S_2})$$
$$= (A^{-1} b_1 \vartheta_{S_1}) + (A^{-1} b_2 \vartheta_{S_2})$$

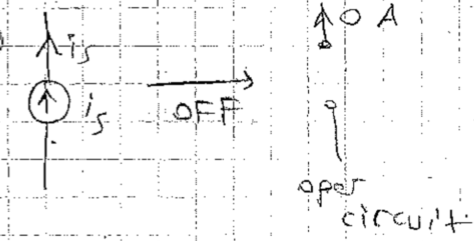
Superposition Principle: For linear circuits, the output can be calculated by individually turning off/on the independent sources and summing the response to each ON/OFF configuration.

NOTES: ①



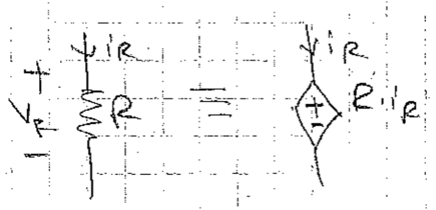
Not recommended!

②

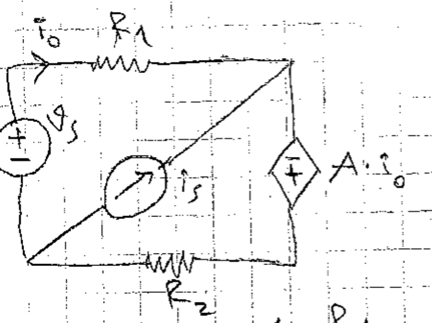


③ Never try to turn ON/OFF dependent sources (dependent components).

A dependent source is just another component; it is not an input to the circuit; hence like all components it affects LHS of node/mesh equation. Always remember that a simple resistor can also be described as a dependent source.



EX:

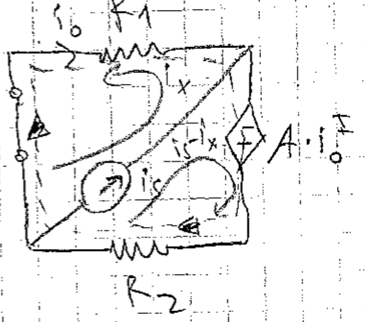


Find i_0 (Apply superposition)

$$I_0 = (\ ?\ )\, v_s + (\ ?\ )\, i_s$$

a scalar   a scalar

(I) V_s: OFF
i_s: ON



$$R_1(-i_x) - A i_0 + R_2(i_s - i_x) = 0$$

$-i_x$

$$i_x = \frac{-R_2}{-R_1 + A - R_2} i_s$$

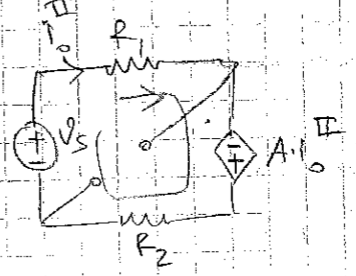
$$\boxed{i_0^{I} = \frac{R_2}{A - R_1 - R_2} \cdot i_s}$$

$$i_x(A - R_1 - R_2) + R_2 i_s = 0$$

Node analysis: # of unknowns = # nodes (other than ground) − # voltage sources

Mesh analysis: # of unknowns = # meshes − # current sources

(II) V_s: ON
i_s: OFF



KVL:

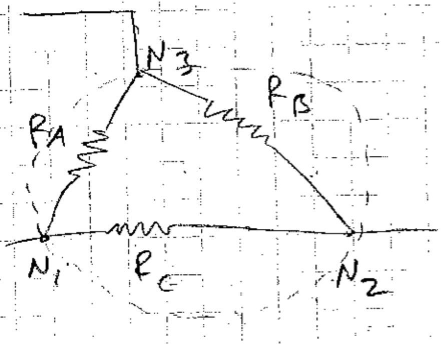
$$-v_s + R_1 i_0^{II} - A i_0^{II} + R_2 i_0^{II} = 0$$

$$i_0^{II} = \frac{1}{R_1 + R_2 - A} v_s$$

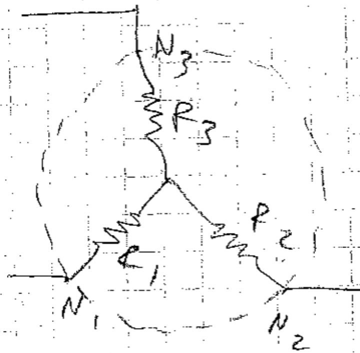
when V_s: ON
i_s: ON → $i_0 = i_0^{I} + i_0^{II}$

$$\boxed{i_0(t) = \left(\frac{R_2}{A - R_1 - R_2}\right) i_{s(t)} + \left(\frac{-1}{A - R_1 - R_2}\right) v_{s(t)}}$$

Δ-Y Transformation



Δ-connection



Y-connection

Y → Δ

$$R_A = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_2}$$

$$R_B = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1}$$

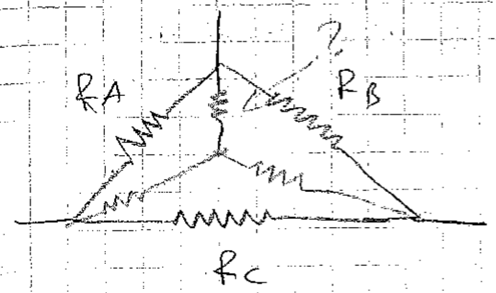
$$R_C = \frac{(R_1R_2 + R_1R_3 + R_2R_3)}{R_3}$$

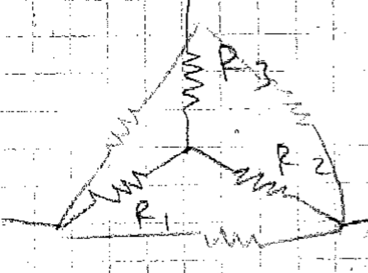
Δ → Y

$$R_1 = \frac{R_A R_C}{R_A + R_B + R_C}$$

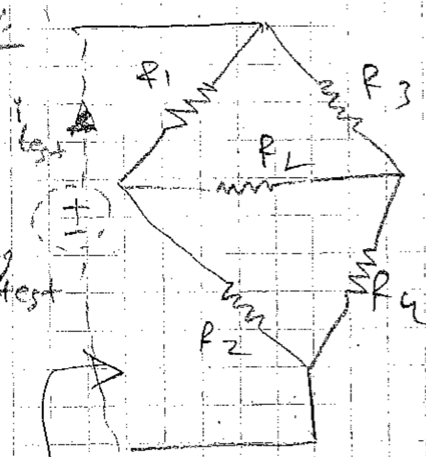
$$R_2 = \frac{R_B R_C}{R_A + R_B + R_C}$$

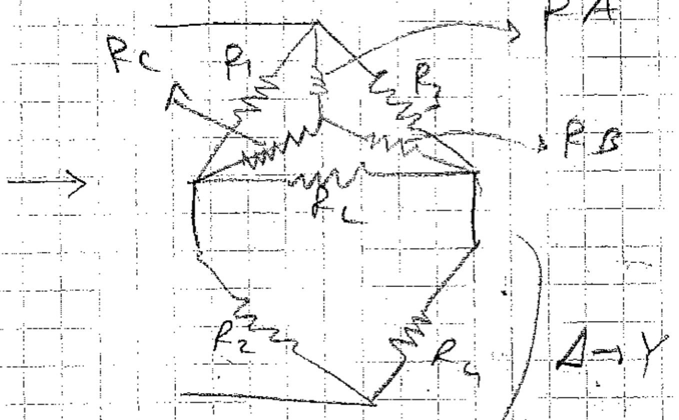
$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C}$$





Ex:





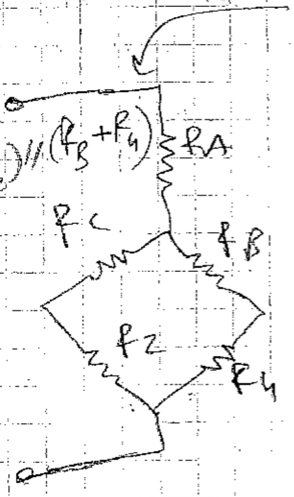
$R_{in} = ?$

$i_{test} = \frac{V_{test}}{R_{in}}$

$$R_A = \frac{R_1 R_3}{R_1 + R_L + R_3}$$

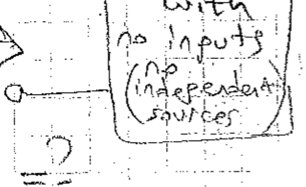
$$R_B = \frac{R_3 R_L}{R_1 + R_C + R_3}$$

$$R_C = \frac{R_1 R_L}{R_1 + R_L + R_3}$$



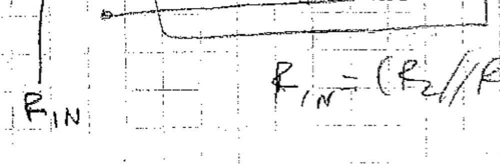
$R_{IN} = R_A + (R_C + R_2) // (R_B + R_4)$

Input Resistence Calculation



$R_{IN} = ?$

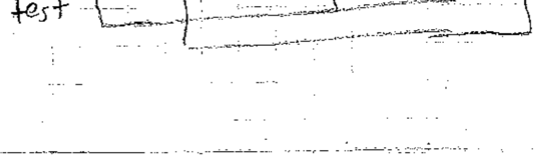
①



R_{IN}

$R_{IN} = (R_2 // R_3) + R_1$

②

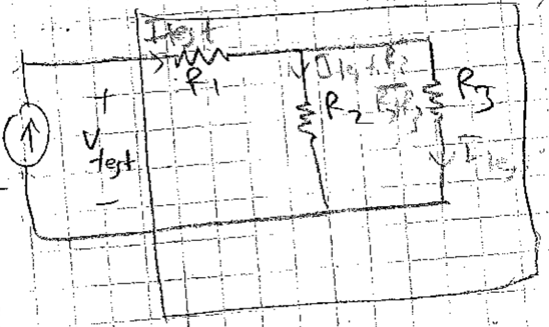


$R_{IN} = \frac{V_{test}}{i_{test}}$ ← Voltage applied; ← Current measured (calculated)

So to find R_{IN} I can apply V_{test} to the terminals of the box, and then "check" I_{test} R_{IN} is equal to $\frac{V_{test}}{I_{test}}$

(or i_{test})

③



$$V_{test} = I_{test} R_1 + I_{test} \frac{R_2 R_3}{R_2+R_3}$$

$$R_{IN} = \frac{V_{test}}{i_{test}} = R_1 + \frac{R_2 R_3}{R_2+R_3}$$

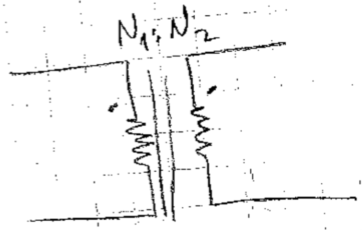
Transformers:

Transformer is a 2-port component. N_1, N_2 are called turns-ratio.

Transformer operates only the principle of rate of change of flux is proportional to the voltage (Faraday's Law).

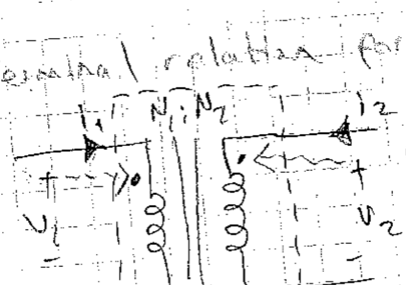
Hence, transformers operate only with A.C. signals, but in EE 201 we introduce (assume) a hypothetical transformer which also operates with DC voltages.

This is the reason in problem sets of EE201, the transformers are also shown as:



- plus is alligned with the dot.
- current is entering the dot.

Terminal relation for transformers:



$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

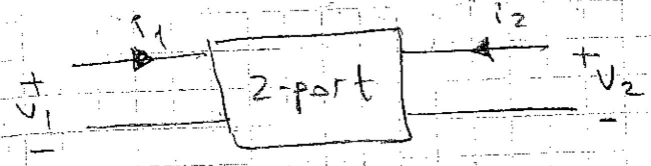
(Voltages are proportional to turns ratio)

$$\frac{i_1}{i_2} = -\frac{N_2}{N_1}$$

(Currents are inversly proportional to turns ratio)

Those equations are correct for this dot configuration

Power of a 2-port



$$P_{2\text{-port}} = v_1 i_1 + v_2 i_2$$

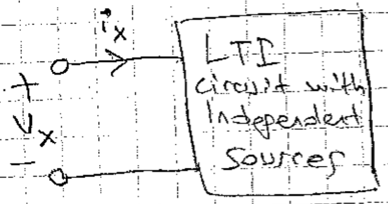
For transformer: $P_{\text{Transformer}} = v_1 \cdot i_1 + \underbrace{v_2}_{\frac{N_2 \cdot v_1}{N_1}} \overset{-\frac{N_1 i_1}{N_2}}{i_2} = 0$

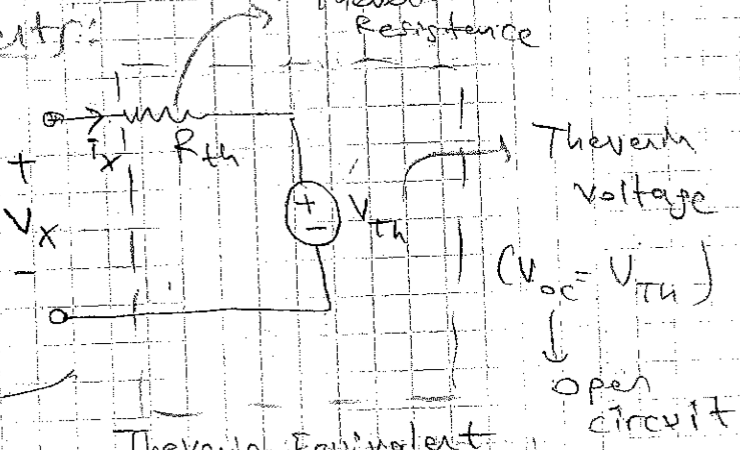
minus sign

Transformer doesn't consume energy

Please note the dots in the transformer symbol and how they define polarity/direction of V_1, V_2 and i_1, i_2.

Thevenin - Norton Equivalents:

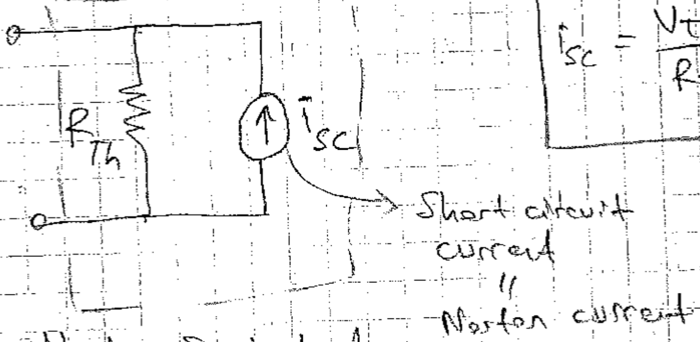




Thevenin Equivalent

$$\boxed{i_{sc} = \frac{V_{th}}{R_{Th}}}$$

$\equiv$



Norton Equivalent

Dependent source olsaydı?

Procedure for finding Thevenin - Norton Equivalent

Procedure ①: Ⓐ Find V_{oc}
Ⓑ Find R_{Th}

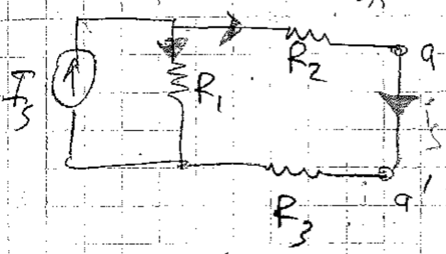
To find R_{Th}:

① Turn-off all independent sources and find equivalent resistance seen from terminals of the box.

② Turn-off all independent source and apply V_{Test} and calculate I_{test}. $R_{th} = \dfrac{V_{Test}}{I_{test}}$

Procedure ② : * V_{oc} (as before)

* i_{sc}

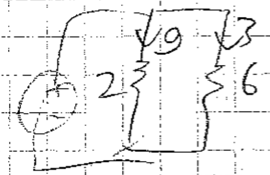


$$i_{sc} = I_s \cdot \frac{R_1}{R_1+R_2+R_3}$$

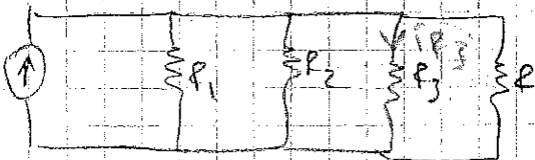
$$(V_{oc} = R_1 \cdot I_s)$$

$$R_{Th} = \frac{V_{oc}}{i_{sc}} = R_1+R_2+R_3$$

Note: Pay attention to V_{oc} and i_{sc} signs and directions

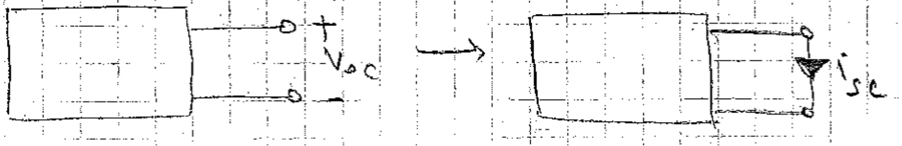


Current Division:

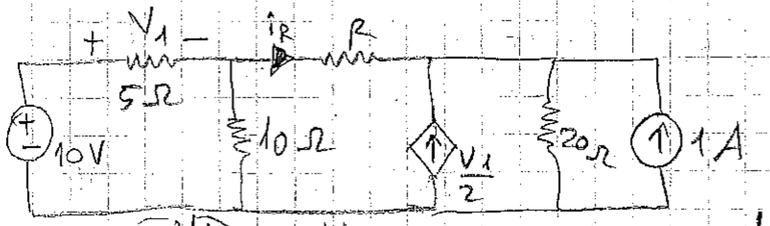


$$i_{R_3} = \frac{\overset{G_3 \text{ (conductance)}}{\boxed{1/R_3}}}{1/R_1 + \frac{1}{R_2} + 1/R_3 + \frac{1}{R_4}} I_s$$

V_{oc} polarity and its relation with i_{sc} current:

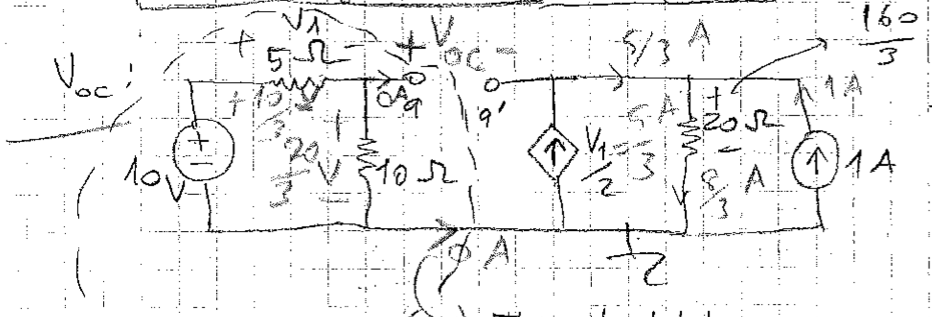


Ex:



Find i_R by finding Thevenin equivalent seen by R.

V_{oc}:



$\frac{160}{3}$ Volts

$\rightarrow V_1 = \frac{10}{3}$ $\quad \frac{160}{17}$

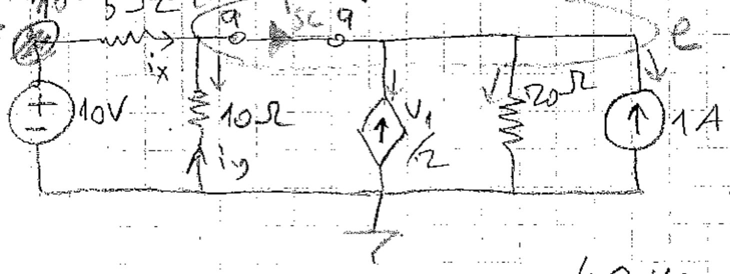
$V_{a'} = \frac{160}{3}$ Volts

$V_a = \frac{20}{3}$ Volts

$V_a - V_{a'} = V_{oc} = -\frac{140}{3}$ Volts

This should be zero as well by KCL.

i_{sc}:



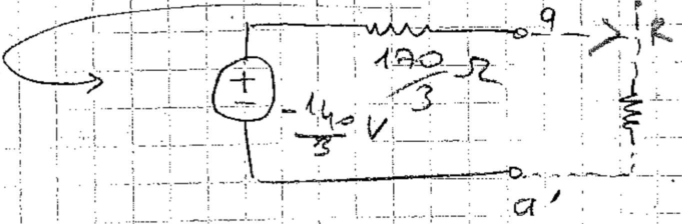
KCL at e:

$$\frac{e-10}{5} + \frac{e}{10} - \frac{\overbrace{V_1}^{}}{2} + \frac{e}{20} - 1 = 0 \qquad V_1 = 10-e$$

$$e = \frac{40+100+20}{4+2+10+1} = \frac{160}{17} \text{ Volts}$$

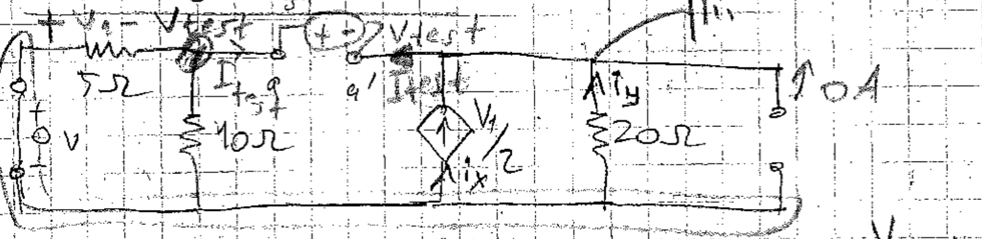
$$I_{sc} = i_x + i_y = \frac{10-e}{5} - \frac{e}{10} = 2 - \frac{3e}{10} = 2 - \frac{48}{17} = \boxed{\frac{-14}{17}\,A}$$

$$R_{Th} = \frac{V_{oc}}{I_{sc}} = \frac{-140/3}{-14/17} = \frac{170}{3}\,\Omega$$



Procedure ①:

R_{Th} using V_{Test} method



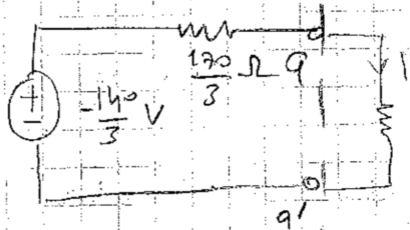
$$\underline{\text{KCL at e:}}\quad \frac{e-V_{Test}}{5} + \frac{e-V_{Test}}{10} + \frac{\overset{e-V_{Test}}{V_1}}{2} + \frac{e}{20} = 0$$

$$\xrightarrow{\times 20}\; e = \frac{4+2+10}{4+2+10+1}\,V_{Test} = \frac{16}{17}\,V_{Test}$$

$$i_{test} = i_x + i_y = \frac{V_1}{2} + \frac{e}{20} = \frac{e-V_{Test}}{2} + \frac{e}{20} = \frac{11e}{20} - \frac{V_{test}}{2}$$

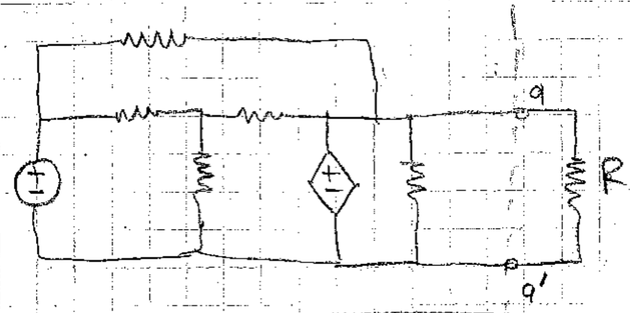
$$= \frac{11}{\cancel{20}_{10}}\,\frac{\cancel{16}^{8}}{17} - \frac{1}{2} = \frac{88}{170} - \frac{85}{170} = \frac{3}{170}\,V_{Test}$$

$$R_{Th} = \frac{V_{Test}}{i_{Test}} = \frac{V_{test}}{\frac{3}{170}V_{test}} = \frac{170}{3}\,\Omega$$



$$i_R = \frac{-140/3}{\frac{170}{3}+R}\,A$$

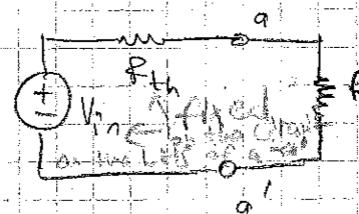
Maximum Power Transfer



Q: What should be R_L s.t. power transfered to R_L is maximum?

$$P_{R_L} = (i_{R_L})^2 \cdot R_L = \frac{v_{R_L}^2}{R_L}$$

→ Find the Thevenin equivalent of Left hand side (LHS) of a-a' line:



$$P_{R_L} = \Big(\underbrace{\frac{V_{Th}}{R_{Th}+R_L}}_{(i_{R_L})^2}\Big)^2 \cdot R_L$$

fixed

$$\frac{d}{dR_L} P(R_L) = \left[(V_{Th})^2 \cdot \left(\frac{1}{R_{Th}+R_L}\right)^2 \cdot R_L\right]' = V_{Th}^2 \cdot \left[\frac{1\cdot(R_{Th}+R_L)^2 - R_L \cdot 2 \cdot (R_{Th}+R_L)}{(R_{Th}+R_L)^4}\right] = 0$$

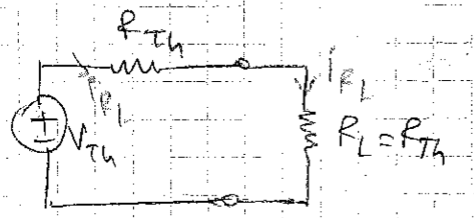
derivative = 0 (max-min problem)

$$R_{Th} + R_L = 2R_L$$

$$\boxed{R_{Th} = R_L}$$

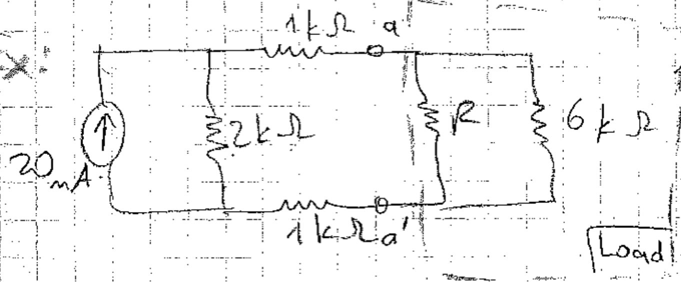
$\cancel{R_{Th} = -R_L}$ can't be possible

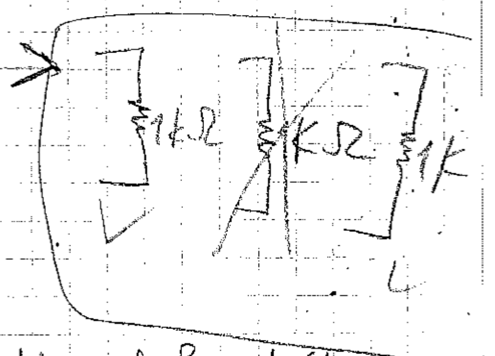
A:



$R_L = R_{Th}$ So efficiency is 50%!

Ex:

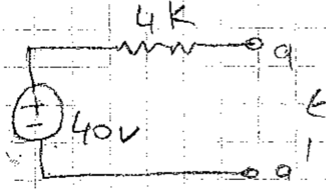




Q: Find R such that P_{Load} (Load is combination of R and 6kΩ) is maximized.

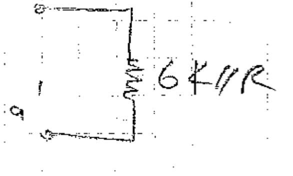
A: The resistance seen by the load should be identical to the load resistance for max. power transfer.

Thevenin equivalent of LHS of a-a':



$R_{Th} = 4K$

Thevenin Eq. of RHS of a-a':



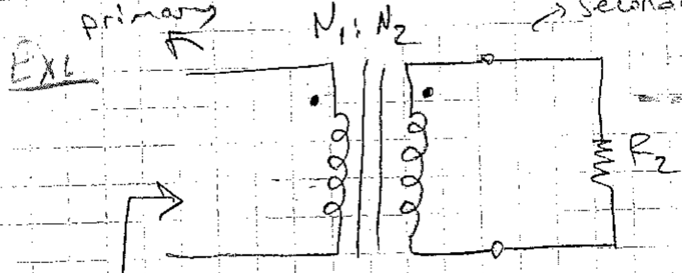
Then for max. power

$$4K = 6K // R$$

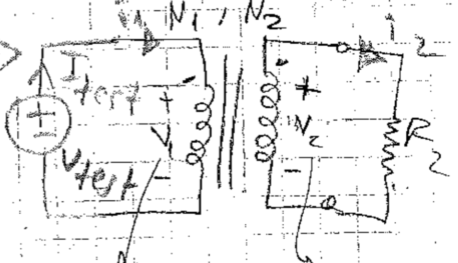
$$\overset{2}{\cancel{4}} = \frac{\overset{3}{\cancel{6}}R}{6+R} \qquad 3R = 12 + 2R$$

$$\boxed{R = 12K}$$

Ex1



Let's apply V_{test}



$R_1 = ?$ Resistance seen from primary side of transformer.

$V_1 = V_{test}$

$V_2 = \frac{N_2}{N_1} \cdot V_{Test}$

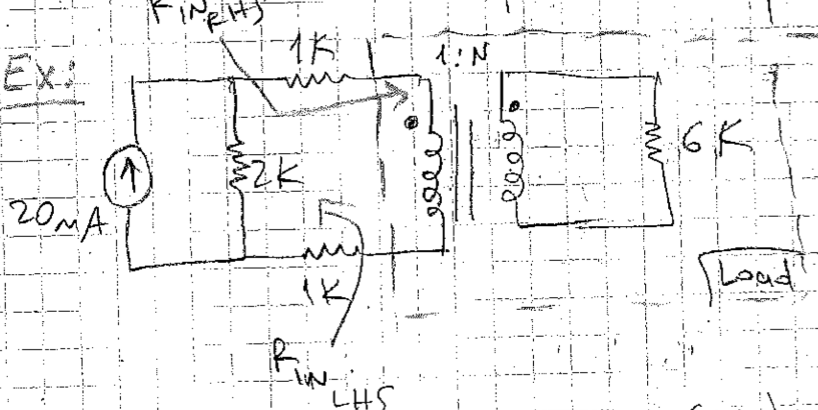
$$i_1 = \frac{N_2}{N_1} \cdot i_2 \quad \rightarrow i_2 = \frac{V_2}{R_2} = \frac{N_2}{N_1} \cdot \frac{V_{test}}{R_2}$$

$$= \frac{N_2^2}{N_1^2} \cdot \frac{1}{R_2} \cdot V_{Test}$$

$$R_{in} = \frac{V_{test}}{i_{test}} = \frac{\cancel{V_{test}}}{\frac{N_2^2}{N_1^2} \cdot \frac{1}{R_2} \cdot \cancel{V_{test}}} = \boxed{R_2 \cdot \left(\frac{N_1}{N_2}\right)^2}$$

(Resistance Reflection Rule)

Ex:



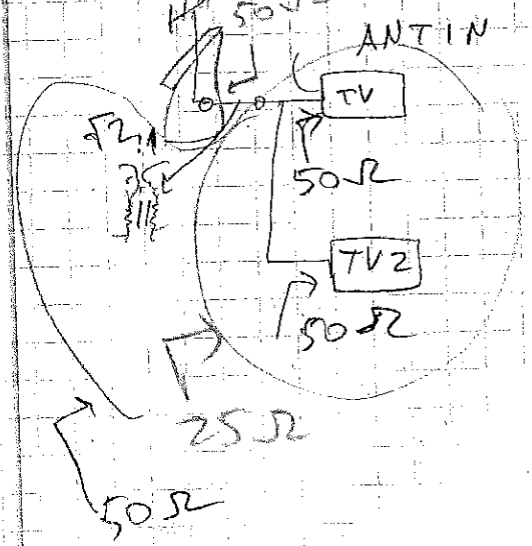
Find N (turns ratio) such that P_{Load} is maximum.

$R_{IN_{LHS}} = 4$ K

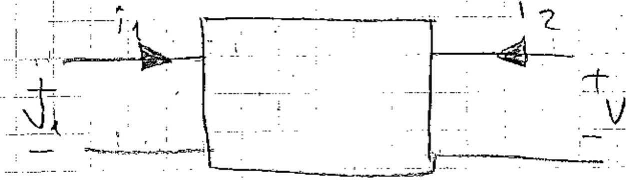
$R_{IN_{RHS}} = (6K) \cdot \frac{1}{N^2}$

$4K = 6K \cdot \frac{1}{N^2}$

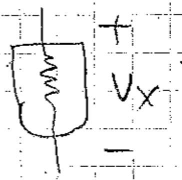
$N = \sqrt{\frac{3}{2}}$



Two-Ports



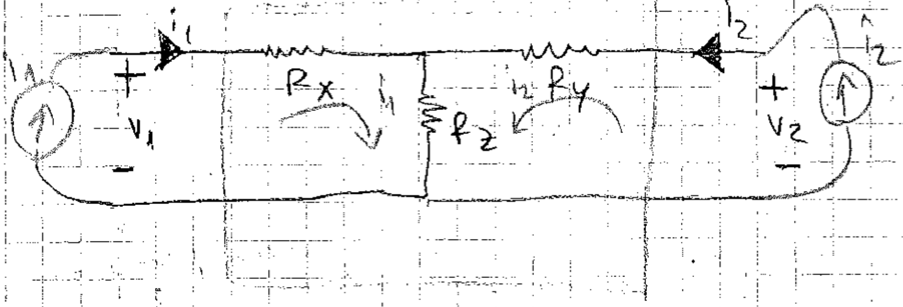
1-port



$\rightarrow V_X = R \cdot i_X$

$i_X = \frac{1}{R} \cdot V_X$

Simple 2-ports:



Assume that $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ as the input and then $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ as the output.

So after the connection of i_1, i_2 sources; we need to solve the circuit and find V_1 and V_2.

$$V_1 = R_X \cdot i_1 + R_Z (i_1 + i_2)$$

$$V_2 = V_{R_Y} + R_{R_Z} = R_Y \cdot i_2 + R_Z (i_1 + i_2)$$

$$\downarrow$$

$$\underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{\text{Calculated output}} = \underbrace{\begin{bmatrix} R_X + R_Z & R_Z \\ R_Z & R_Y + R_Z \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}}_{\text{input}}$$

$$\boxed{V = R \cdot i}$$

Resistance Parameters → Good at series connection

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

r_{ij} has the unit of Ω

$$V = R \cdot i$$

Conductance Parameters

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$\boxed{\text{Note: } G = R^{-1}}$

calculated outputs $\boxed{i = G \cdot V}$ inputs

Hybrid-I parameter set:

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}}_{H} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

$$v_1 = r_{11} i_1 + r_{12} i_2$$
$$v_2 = r_{21} i_1 + r_{22} i_2$$
$$\downarrow\; i_2 = \frac{v_2 - r_{11} i_1}{r_{22}}$$

Hybrid-II parameter set:

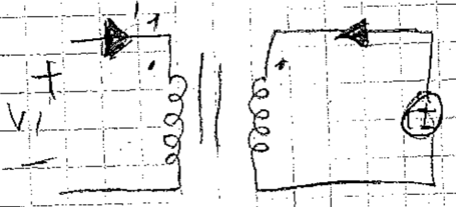
$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = H^{-1} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

ABCD parameters (Transfer parameters) → Like a transformer...

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$$

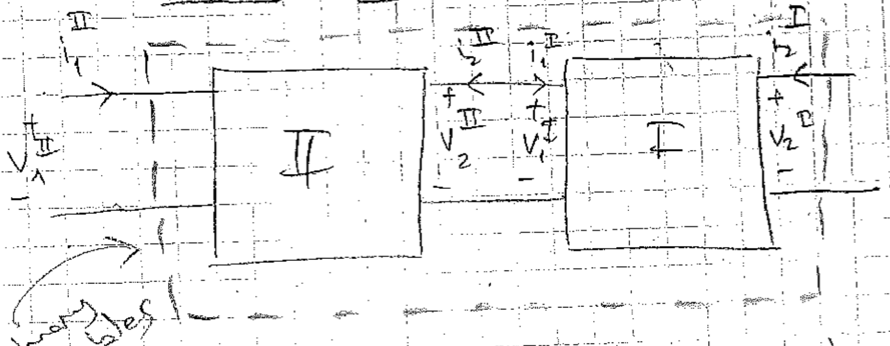
→ Please note the sign.

Advantageous in cascade two-ports. (multiple)



input

secondary variables



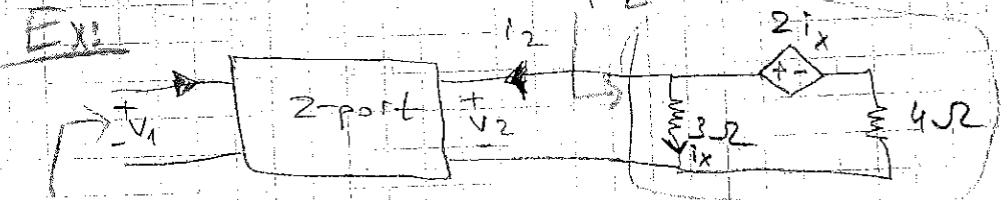
primary variables

$$(ABCD)_{cascade} = (ABCD_{II}) \times (ABCD)_{I}$$

ABCD representation of the 2-ports I and II in cascade.

$$-i_1^{I} = i_2^{II}$$
$$v_1^{I} = v_2^{II}$$

Ex:



R_{IN}?

R_{IN}

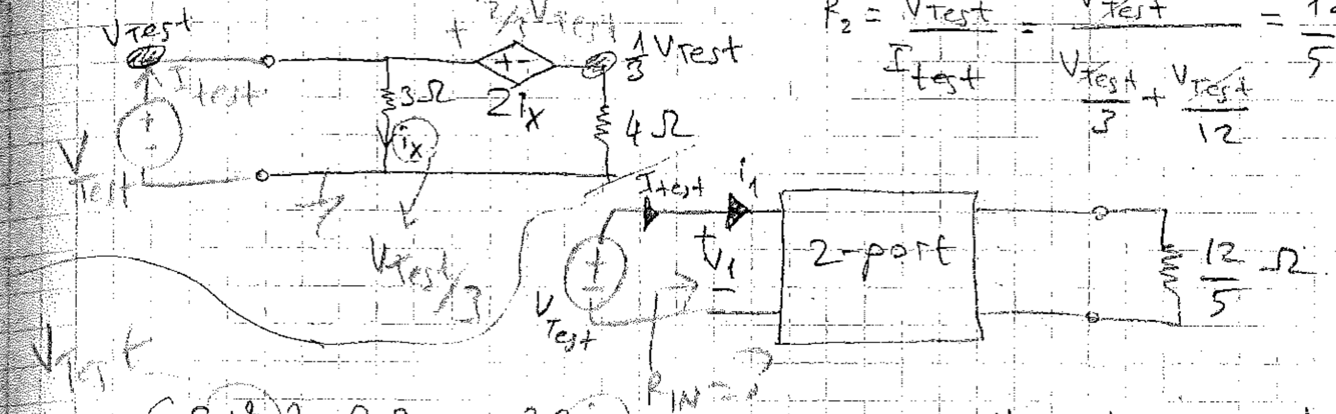
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Z-parameters

terminated 2-port
one side of 2-port is connected with load.

Let's find R_2 first

$$R_2 = \frac{V_{Test}}{I_{test}} = \frac{\cancel{V_{Test}}}{\frac{\cancel{V_{Test}}}{3} + \frac{\cancel{V_{Test}}}{12}} = \frac{12}{5}\,\Omega$$



2-port → $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ → I_{test} → Need to express i_2 in terms of V_{Test}, I_{test}

due to termination $V_2 = \frac{-12}{5} i_2$

$V_2 = 2I_{Test} + 4i_2$

$\longrightarrow V_2 = \frac{-12}{5} i_2 \longrightarrow \frac{-12}{5} i_2 = 2I_{test} + 4i_2$

$\overset{2.4}{}$

$i_2 = \frac{-2}{6.4} I_{test} = \frac{-1}{3.2} I_{test}$

From the first Row of the 2-nd port equation:

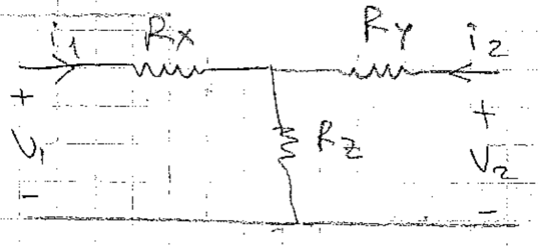
$$V_{Test} = 3I_{Test} + 2i_2$$

$$= 3I_{test} - \frac{2}{3.2} I_{test}$$

$$= \left(\frac{4.8 - 1}{1.6}\right) I_{Test}$$

$$R_{IN} = \frac{V_{Test}}{I_{Test}} = \frac{38}{16} = \frac{19}{8}\,\Omega$$

T - Network:

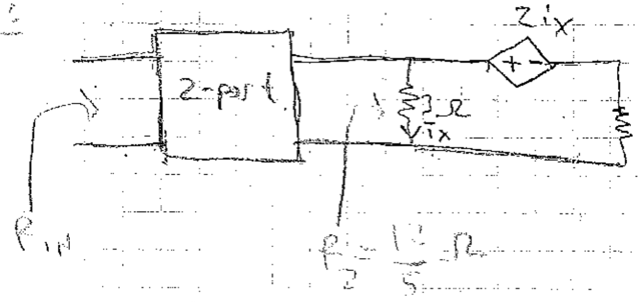


$\longrightarrow R = \begin{bmatrix} R_X + R_Z & R_Z \\ R_Z & R_Y + R_Z \end{bmatrix}$

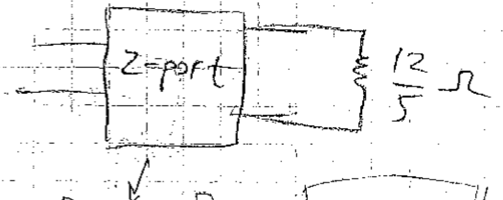
$V = R \cdot i$

Let's go back to the original problem:

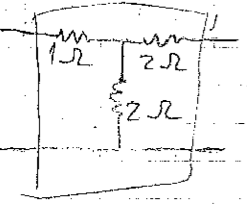
Ex:



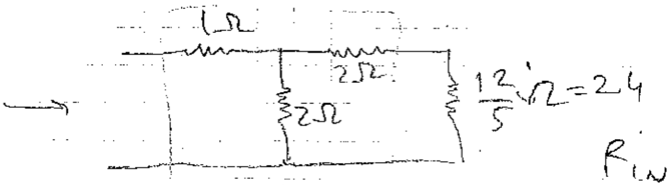
$\longrightarrow$



$R = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \longrightarrow$



$\longrightarrow$

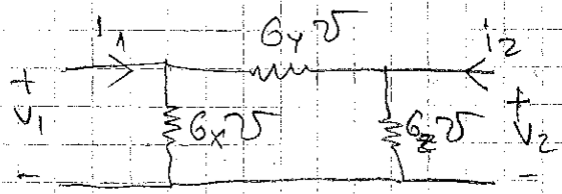


$R_{IN} = 1 + 2 // 4.4 = \frac{19}{8}\,\Omega$

The replacement with T network is applicable when $\underline{\underline{R}}$ is symmetric.

$$A = A^T$$

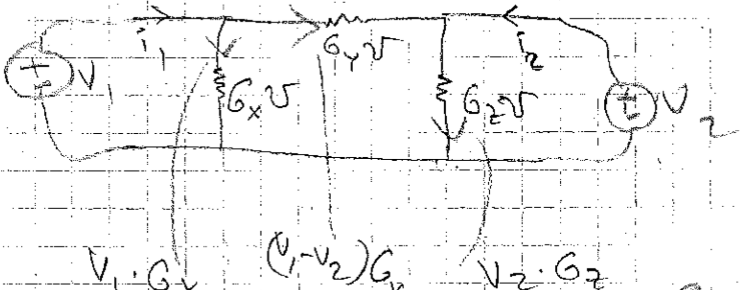
π - Network



$\underline{\underline{G}} = ?$ ⊖ → vector (?)

↖ conductance matrix.

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \underline{\underline{G}} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$



$i_1 = V_1 \cdot G_x + (V_1 - V_2) G_y$
$i_2 = V_2 \cdot G_z - (V_1 - V_2) G_y$

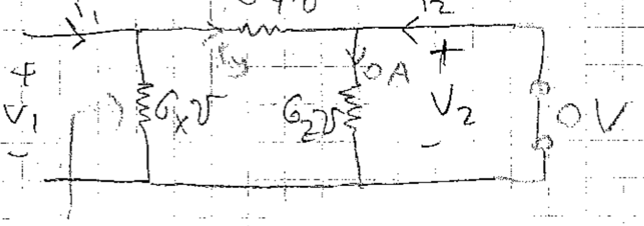
$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \overset{g_{11}}{G_x + G_y} & \overset{g_{12}}{-G_y} \\ \underset{g_{21}}{-G_y} & \underset{g_{22}}{G_y + G_z} \end{bmatrix}}_{\underline{\underline{G}}} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

(Note: $\underline{\underline{G}} = \underline{\underline{G}}^T$ fo π network i.e. symmetric matrix)

- a 2^{nd} method to find parameters:

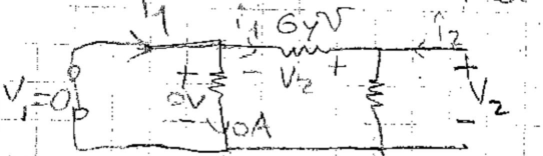
$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$ Let's find only g_{11}:

→ Set $V_2 = 0$ and find $\frac{i_1}{V_1} = g_{11}$, Let's apply this method to π - network.



$$\frac{i_1}{V_1} = G_x + G_y$$

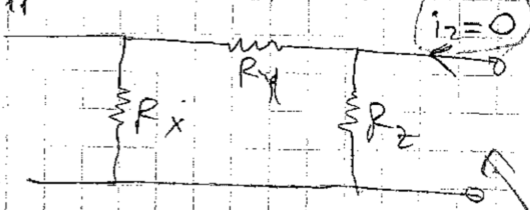
$g_{12}=?$ Set $V_1=0 \xrightarrow{then} \frac{i_1}{V_2}=?$



$V_2 = \frac{1}{G_Y}(-i_1)$

$$\boxed{\frac{i_1}{i_2} = -\frac{1}{G_Y}}$$

- Find r_{11} of π-network



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

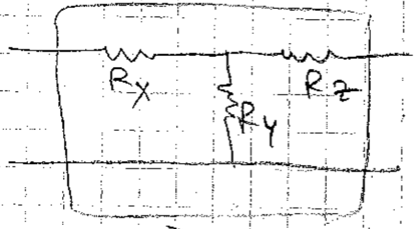
$R_X = \frac{1}{G_X}$, $R_Y = \frac{1}{G_Y}$, $R_Z = \frac{1}{G_Z}$

open circuit

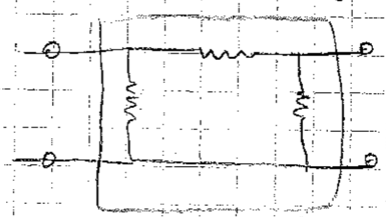
$r_{11} = R_X // (R_Y+R_Z) = \frac{R_XR_Y+R_XR_Z}{R_X+R_Y+R_Z}$

Rember Δ↔Y conversion

The proof of Δ↔Y conversion formulas is simply:



Y-connection

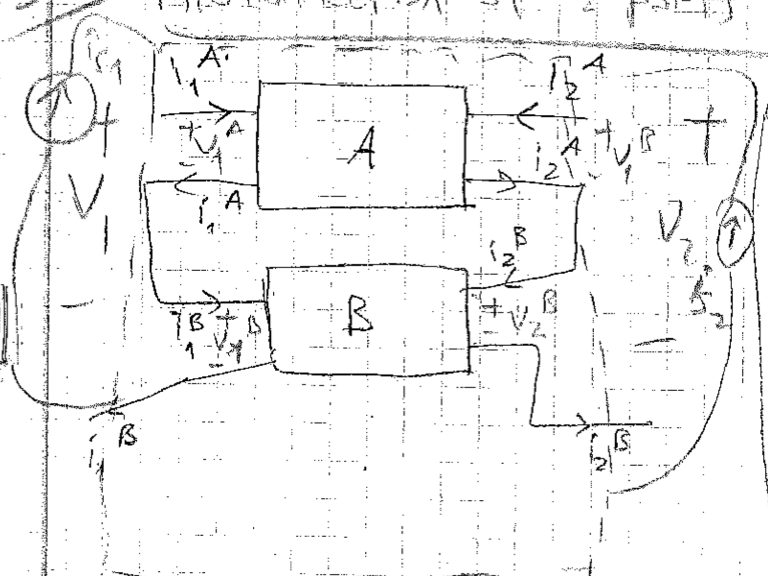


Δ-connection

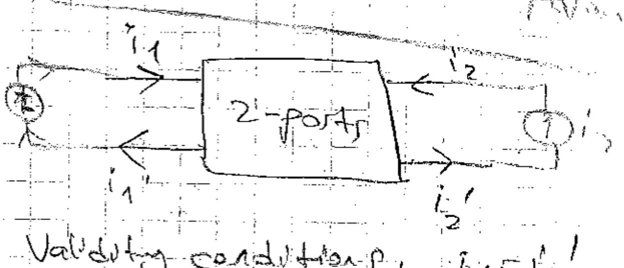
Y→Δ: Find R parameters of T network → $G = R^{-1}$ → Draw π network from G(conductance) parameter

Δ→Y: Find $\underline{\underline{G}}$ from π-network → $R = G^{-1}$ → Draw T network by inspection from $\underline{R}$ matrix

Series Interconnection of 2-ports



Validity condition for 2-port formulas



Validity conditions: $i_1=i_1'$
$i_2=i_2'$

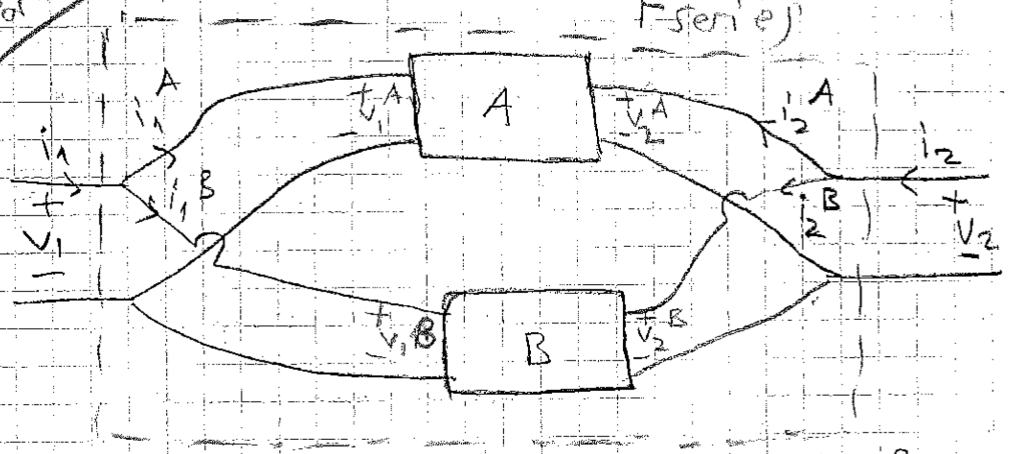
$\underline{\underline{R}}$ of series combination?

(Matrix)

$$\begin{bmatrix} V_1^{series} \\ V_2^{series} \end{bmatrix} = \begin{bmatrix} V_1^A+U_1^B \\ V_2^A+V_2^B \end{bmatrix} = \begin{bmatrix} V_1^A \\ V_2^A \end{bmatrix} + \begin{bmatrix} V_1^B \\ V_2^B \end{bmatrix} = R_A \begin{bmatrix} i_{s_1} \\ i_{s_2} \end{bmatrix} + R_B \begin{bmatrix} i_{s_1} \\ i_{s_2} \end{bmatrix}$$

$$\underline{\underline{R_{series}}} = \underbrace{\left(\underline{\underline{R_A}}+\underline{\underline{R_B}}\right)}_{R_{series}} \begin{bmatrix} i_{s_1} \\ i_{s_2} \end{bmatrix}$$

Parallel



Due to parallel comb. $V_1 = V_1^A = V_1^B$

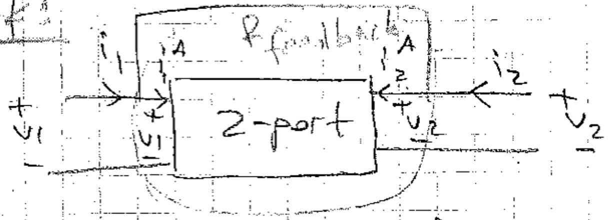
$V_2 = V_2^A = V_2^B$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} i_1^A \\ i_2^A \end{bmatrix} + \begin{bmatrix} i_1^B \\ i_2^B \end{bmatrix}$$

$$= \left(\underline{\underline{G_A}}+\underline{\underline{G_B}}\right) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

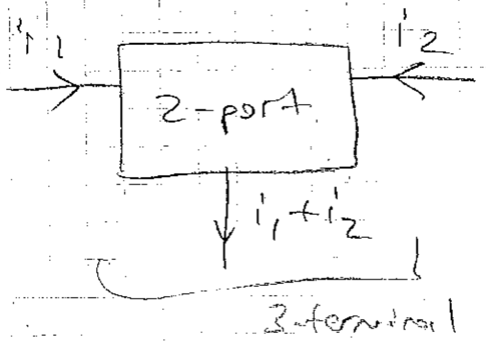
$\underline{\underline{G}}$ matrix of the 2-port

Remark:



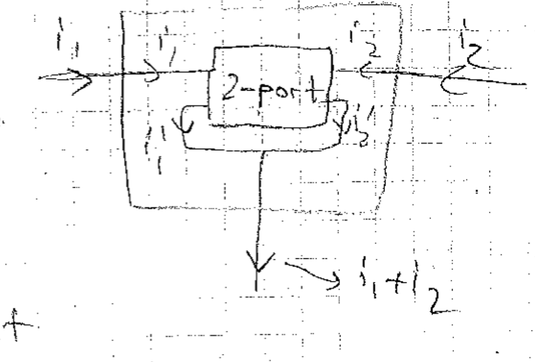
Two-port parameters, if given for the above configuration, can be utilized for finding a relation between $i_1^A, i_2^A, V_1^A, V_2^A$; but not i_1, i_2, V_1, V_2.

3-Terminal 2-port

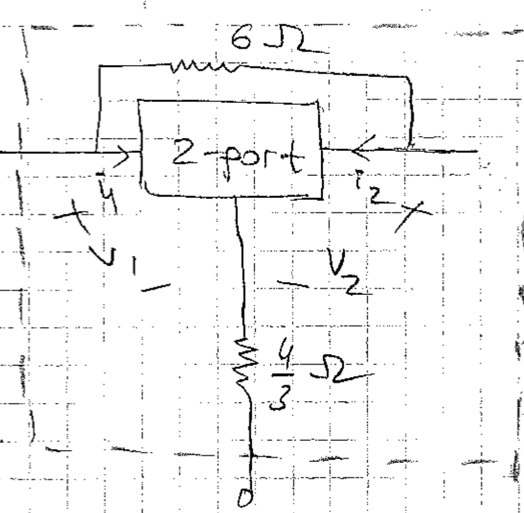


3-terminal 2-port

$\equiv$



Ex: 2PS-II
P.10

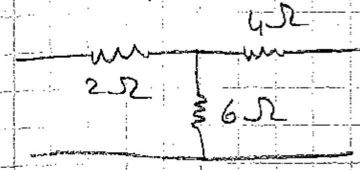


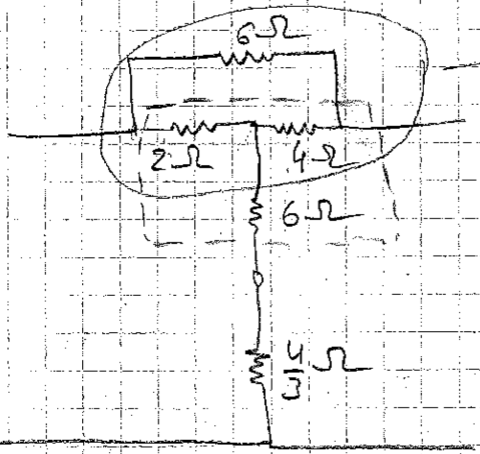
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Find 2-parameters of the combination.

$$\begin{Bmatrix} 8 & 6 \\ 6 & 10 \end{Bmatrix}$$

↓

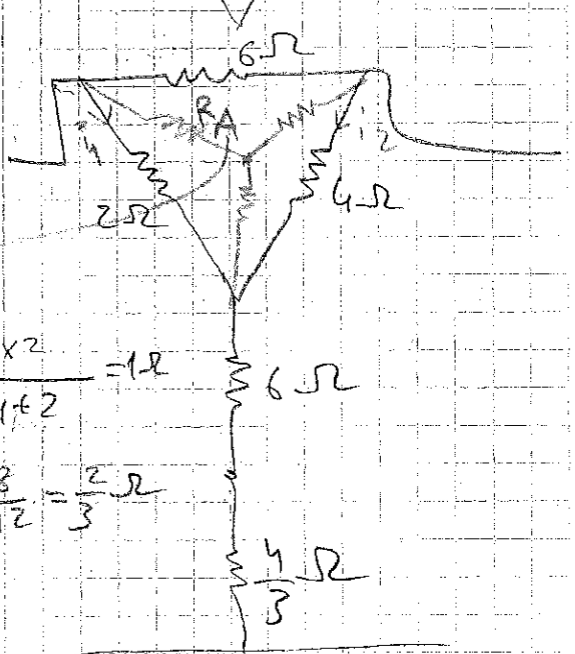




∇ (π) network

Π T

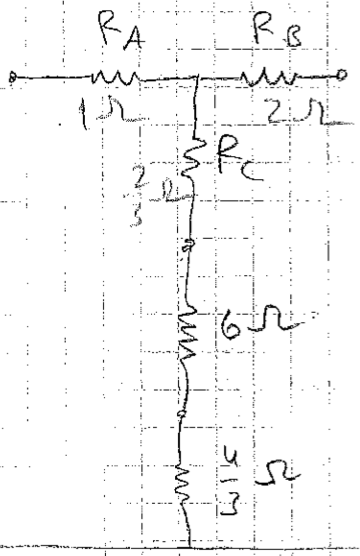
∇



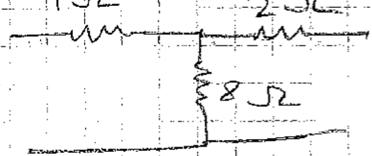
$$R_A = \frac{6 \times 2}{6+4+2} = 1 \Omega$$

$$R_B = \frac{24}{12} = 2 \Omega$$

$$R_C = \frac{8}{12} = \frac{2}{3} \Omega$$

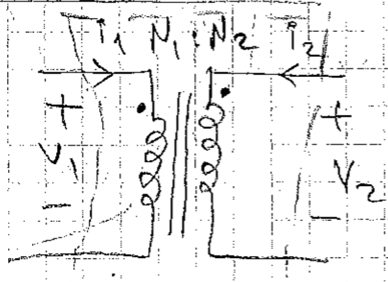


→



$$\rightarrow Z = \begin{bmatrix} 9 & 8 \\ 8 & 10 \end{bmatrix}$$

Ex:



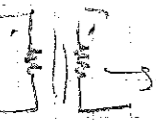
$$\frac{V_1}{V_2} = \frac{N_1}{N_2} \quad , \quad \frac{I_1}{I_2} = -\frac{N_2}{N_1}$$

ABCD parameters of transformers:

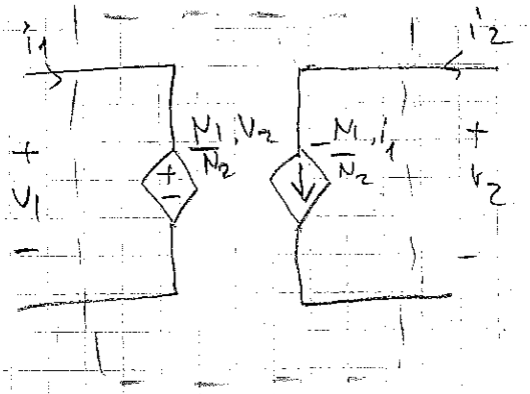
$$\begin{bmatrix} V_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -i_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} \frac{N_1}{N_2} & 0 \\ 0 & \frac{N_2}{N_1} \end{bmatrix} \begin{bmatrix} V_2 \\ -i_2 \end{bmatrix}$$

★ Sometimes it is drawn like this



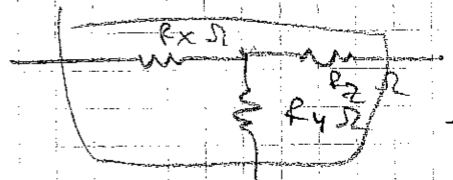
→ Not a resistor !!



2-ports (continued)

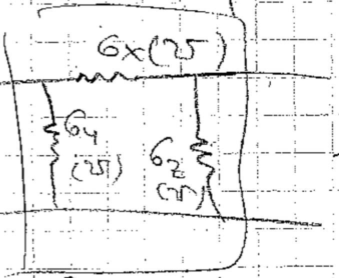
Today: Reciprocity and Symmetric 2-ports / [illegible]

T Networks



$$\rightarrow R = \begin{bmatrix} R_x+R_y & R_y \\ R_y & R_z+R_y \end{bmatrix}$$

π networks



$$\longrightarrow G = \begin{bmatrix} G_x+G_y & -G_x \\ -G_x & G_x+G_z \end{bmatrix}$$

Pls note that R and G matrices for T and π network are symmetric.

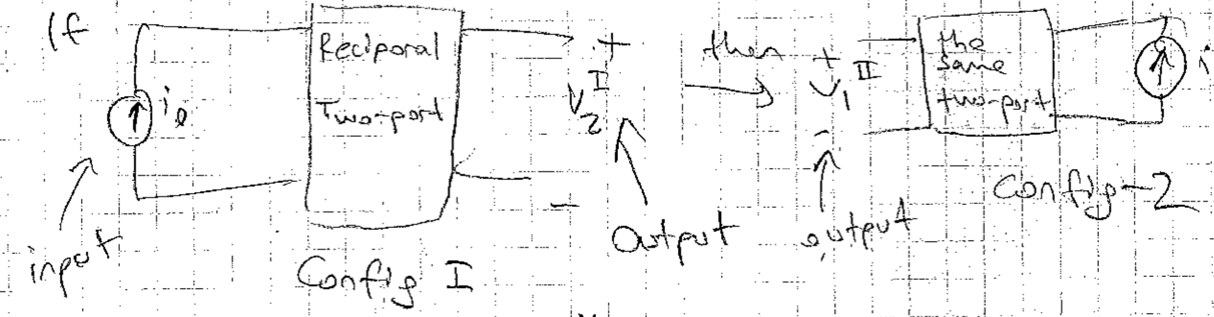
Reciprocity: If a 2-port doesn't contain any dependent sources that is it is composed of only resistances; then it's R matrix or G matrix is guaranteed to be symmetric, that is

$r_{12} = r_{21}$ or $g_{12} = g_{21}$ (Such networks are called reciporal networks)

For such networks, some special operations involving exchange of input and output results in the same mapping.

① Since $r_{12} = r_{21}$,

If



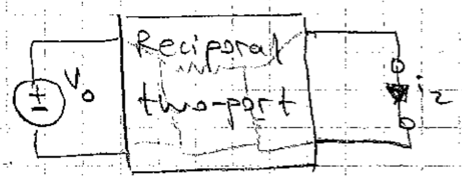
Config I

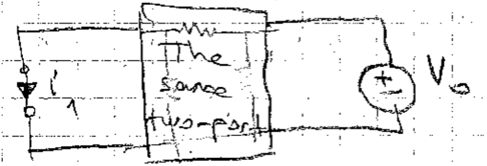
$\longrightarrow$ Since the 2-port is reciporal:

$$\boxed{V_1^{II} = V_2^{I}}$$

② Since $g_{12}=g_{21}$,

Config-2



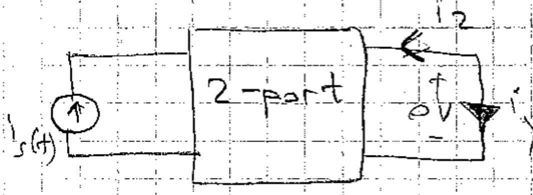


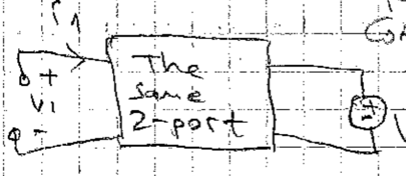
Config-1 Due to reciprocity ($g_{12}=g_{21}$) $\boxed{i_1=i_2}$

$$\begin{bmatrix}i_1\\i_2\end{bmatrix}=\begin{bmatrix}g_{11}&g_{12}\\g_{21}&g_{22}\end{bmatrix}\begin{bmatrix}V_1\\V_2\end{bmatrix}$$

$\begin{bmatrix}V_o\\0\end{bmatrix}$ for Config-1 $\begin{bmatrix}0\\V_o\end{bmatrix}$ for Config-2

③ $h_{12}=-h_{21}$ (for reciporal 2-ports)





$V_o=f(t)$ Volts

$i_s(t)=f(t)$ A

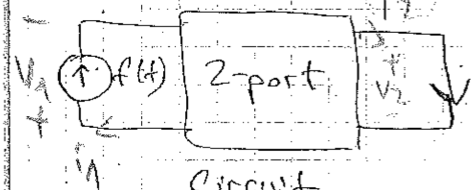
a function

for reciporal 2-ports: $\boxed{V_1=i_y}$

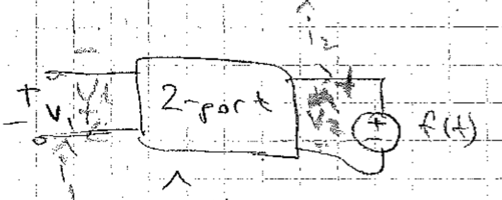
$$\begin{bmatrix}V_1\\i_2\end{bmatrix}=\begin{bmatrix}h_{11}&h_{12}\\h_{21}&h_{22}\end{bmatrix}\begin{bmatrix}i_1\\V_2\end{bmatrix}$$

negatives of each other

Let's give a proof for the 3rd case involving h parameters.



Circuit



$\widehat{\text{Circuit}}$

We will use Telegen's theorem to show the reciprocity case.

$$\sum_{k=1}^{N}v_k\hat{i}_k=\sum_{k=1}^{N}\hat{v}_k i_k=0$$

Then since 2-port is reciporal I can assume that it contains only LTI resistors.

$$v_1\hat{i}_1+v_2\hat{i}_2+\underbrace{\cancel{\sum_{k=3}^{N}v_k\hat{i}_k}}_{\text{all branches of 2-port ??}}=\hat{v}_1 i_1+\hat{v}_2 i_2+\cancel{\sum_{k=3}^{N}\hat{v}_k i_k}$$

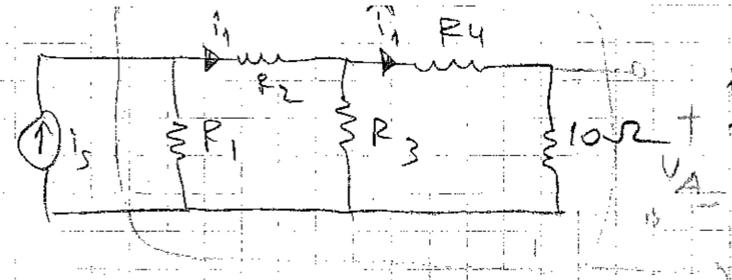
$v_k=R_k i_k$ $\hat{v}_k=R_k\hat{i}_k$

$$v_1\hat{i}_1+v_2\hat{i}_2=\hat{v}_1 i_1+\hat{v}_2 i_2$$

$$V_1\cdot 0+0\cdot\hat{i}_2=(-V_1)\cancel{f(t)}+\cancel{f(t)}\,i_y \qquad \boxed{V_1=i_y}$$

Ex 1

C_1:



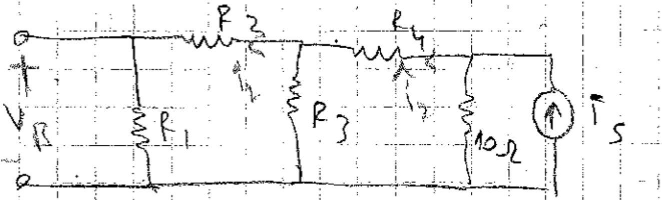
(circuit 1)

$i_1 = 0.6\, i_s$

$\hat{i}_1 = 0.3\, i_s$

→ Our two port

C_2:



$i_2 = 0.2\, i_s$

$\hat{i}_2 = 0.5\, i_s$

Find R_1 from measurements from circuits C_1 and C_2.

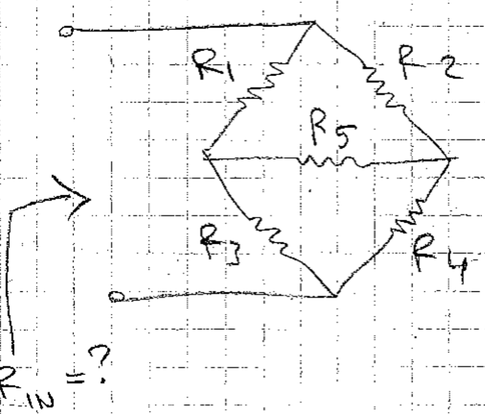
By reciprocity $(r_{12} = r_{21}) \rightarrow V_A = V_B$

$$10 \cdot \hat{i}_1 = R_1 \cdot i_2$$

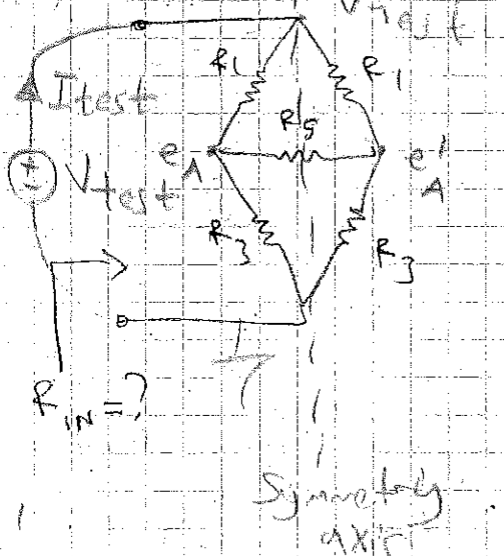
$$10\,(0.3\, i_s) = R_1 (0.2)\, i_s$$

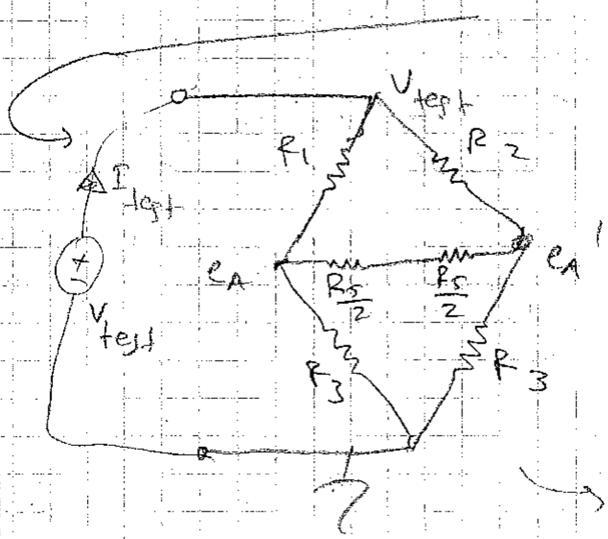
$$\hookrightarrow \boxed{R_1 = 15\ \Omega}$$

Symmetric Circuits

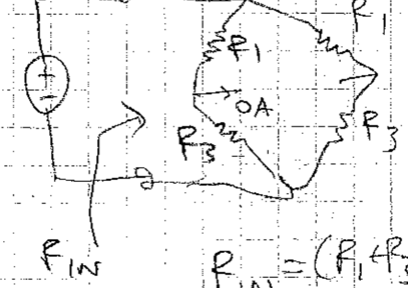


Case 1: $R_1 = R_2$
$R_3 = R_4$





By symmetry $e_A = e_A'$



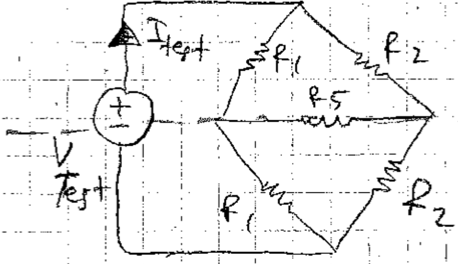
$$R_{IN} = (R_1 + R_3) // (R_1 + R_3)$$

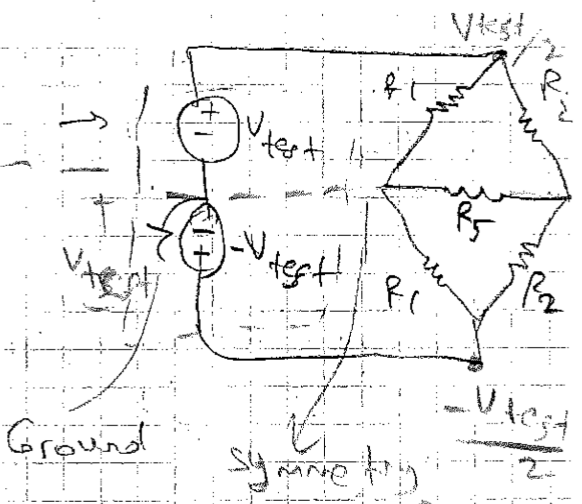
$$= \frac{R_1 + R_3}{2}$$

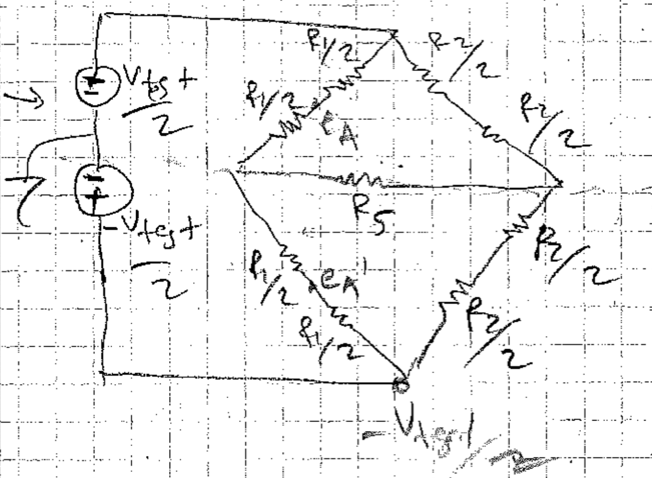
Case 2:

$R_1 = R_3$

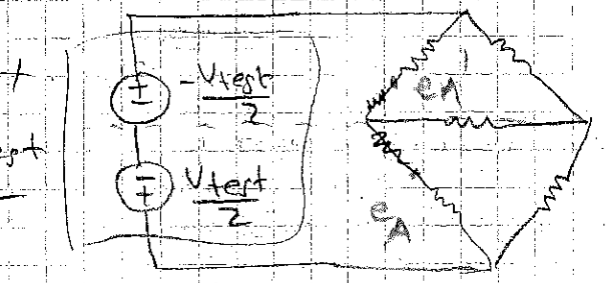
$R_2 = R_4$







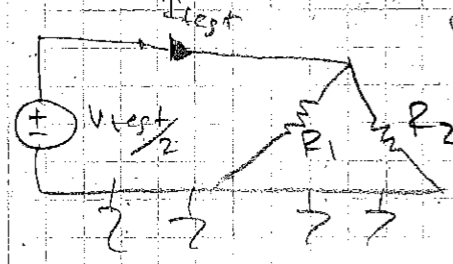
By folding upper-part to bottom and lower part to top we get



$e_x = e_x'$ but

Node X is on Symmetry axis $\quad e_x = -e_x'$

$\rightarrow e_x = 0$



$e_A = -e_A'$

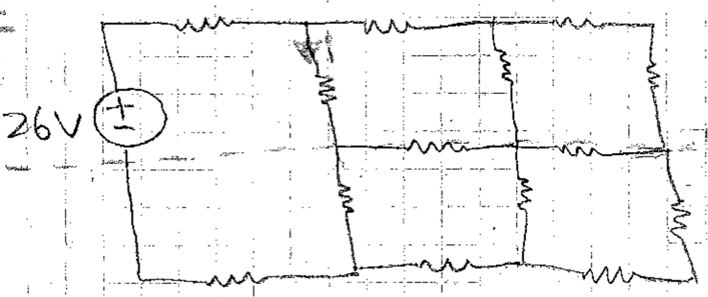
$$I_{test} = \frac{V_{test}/2}{R_1 // R_2}$$

$$\frac{V_{test}}{I_{test}} = 2(R_1 // R_2)$$

??

Nasıl Ground oldu?

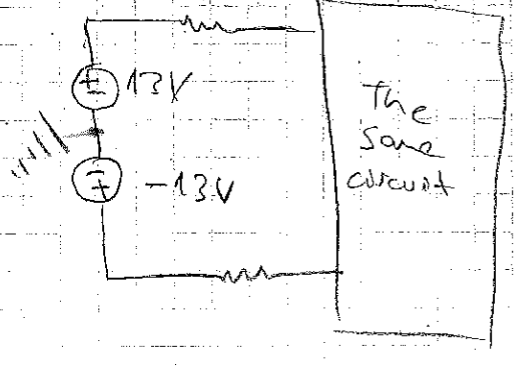
Ex:

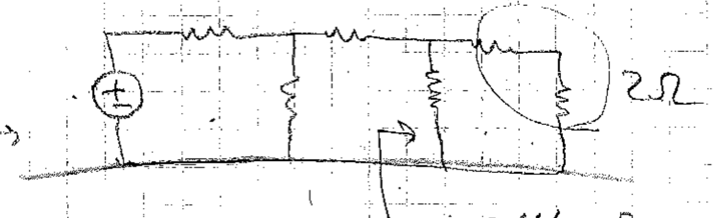


All resistors are 1Ω.

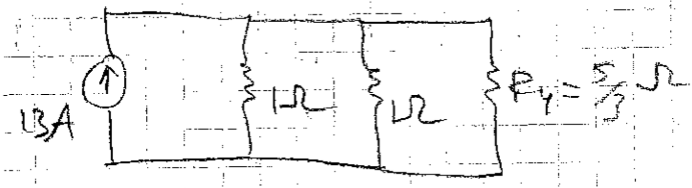
Find i.

Symmetry axis





$R_x = 2 // 1 = \frac{2}{3}\,\Omega$



$$i = 13 \cdot \frac{1/1}{1/1 + \frac{1}{1} + \frac{1}{5/3}}$$

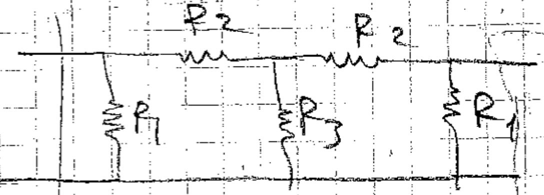
$$= 13 \cdot \frac{5}{13}$$

$$= 5A$$

Also check ? ...notes p. 43 for another application of symmetry

ZPS !!

Remark:



Find R parameters.

$$R = \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix}$$

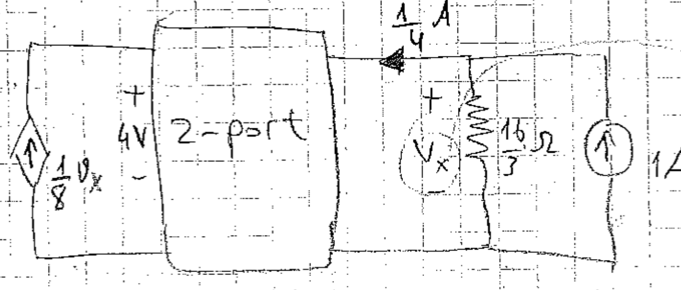
Comment (1): 2-port is reciprocal then $r_{12} = r_{21}$

Comment (2): 2-port has symmetry across the axis. The labeling of primary and secondary ports for 2-port is arbitrary.

$\rightarrow r_{11} = r_{22}$

Ex:

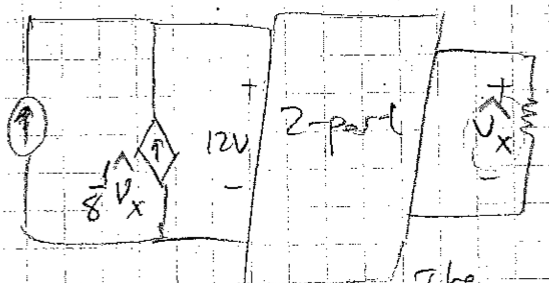
Config 1:



Dummy Variable

They are Not the same!

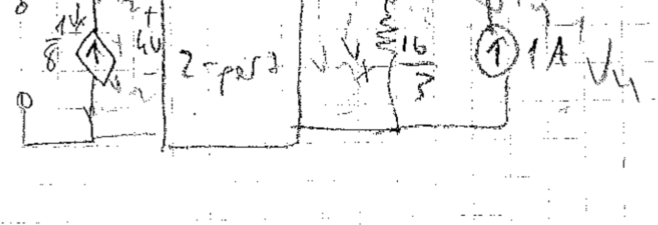
Config 2:
Hat (^) Configuration

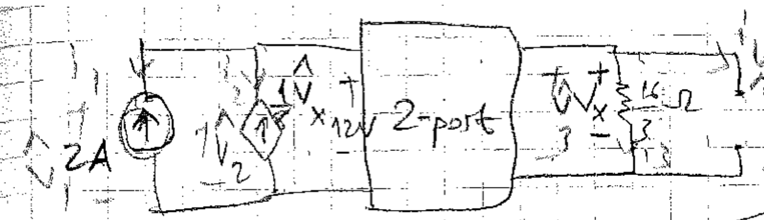


let's say it is $\hat{V}_b$

Find $\hat{V}_x$ of circuit 2. (2-port The consist of LTI (reciprocal))

Apply Telegen's Theorem.





$$\sum_{k=1}^{\#\text{of branches}} v_k \cdot \hat{i}_k = \sum_{k=1}^{\#\text{of branches}} \hat{v}_k \cdot i_k = 0$$

$$v_1 \cdot \hat{i}_1 + v_2 \hat{i}_2 + v_3 \cdot \hat{i}_3 + v_4 \cdot \hat{i}_4 + \sum_{k=5}^{\#\text{of branches}} (R i_k) \hat{i}_k = \hat{v}_1 i_1 + \hat{v}_2 i_2 + \hat{v}_3 i_3 + \hat{v}_4 \cdot i_4 + \sum_{k=5}^{\#\text{of branches}} (R \cdot \hat{i}_k) i_k$$

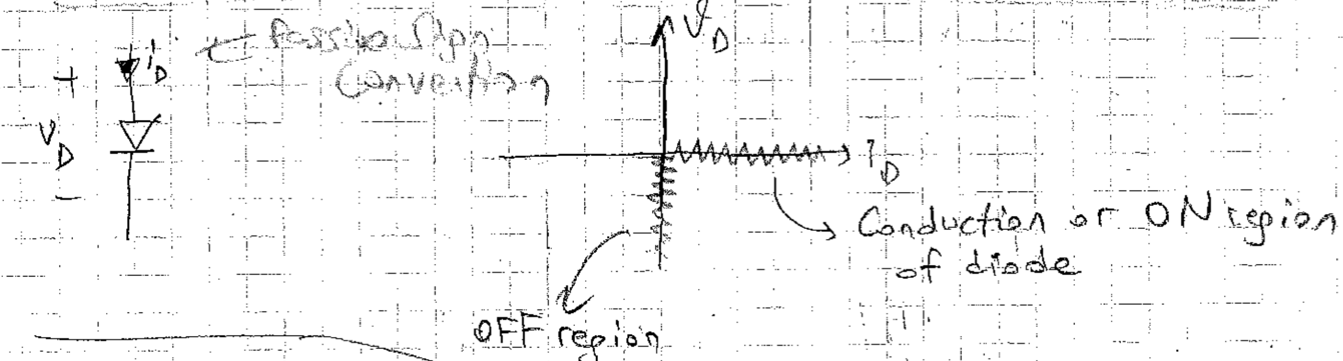
$$4(-2) + 4 \cdot \left(-\frac{1}{8}\hat{v}_x\right) + 4\left(\frac{\hat{v}_x}{16/3}\right) + 4 \cdot 0 = 12 \cdot 0 + 12\left(\frac{-1}{2}\right) + \hat{v}_x \cdot \frac{3}{4} + \hat{v}_4 (-1)$$

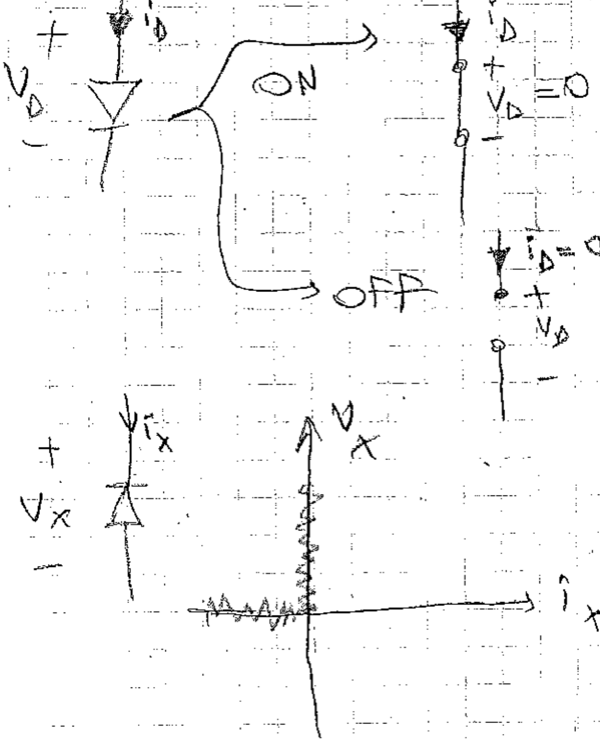
$$-2 - \frac{1}{8}\hat{v}_x + \frac{3}{16}\hat{v}_x = \frac{-3}{2} + \hat{v}_x \cdot \frac{3}{16} - \frac{\hat{v}_x}{4}$$

$$\boxed{\hat{v}_x = 4V}$$

Note: Don't confuse the branch numbers of the 2-port.

Ideal Diodes and Introduction to Non-Linear Components



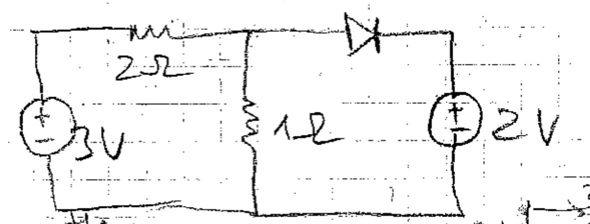


provided that $i_D \geq 0$

provided that $v_d \leq 0$

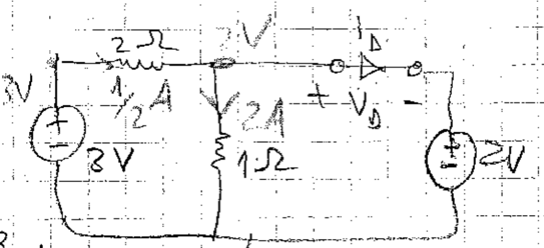
Remark: Ideal diodes have i-v characteristics whose one leg is in (+) positive part of the axis and other one in the (−) part of the axis.

Ex1: Analyze the circuit



Assume the diode is ON!

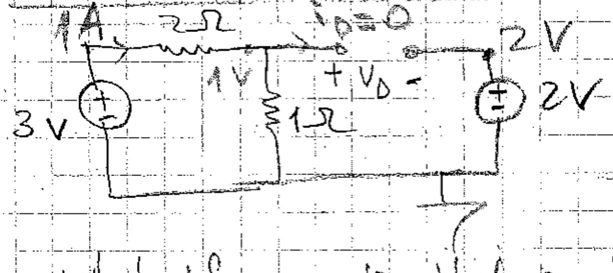
(Requires $i_D \geq 0$)



$i_D = \frac{-3}{2} A$

The assumption is not correct since the ON condition is NOT satisfied.

Assume the diode is OFF:



Condition: $V_D < 0$

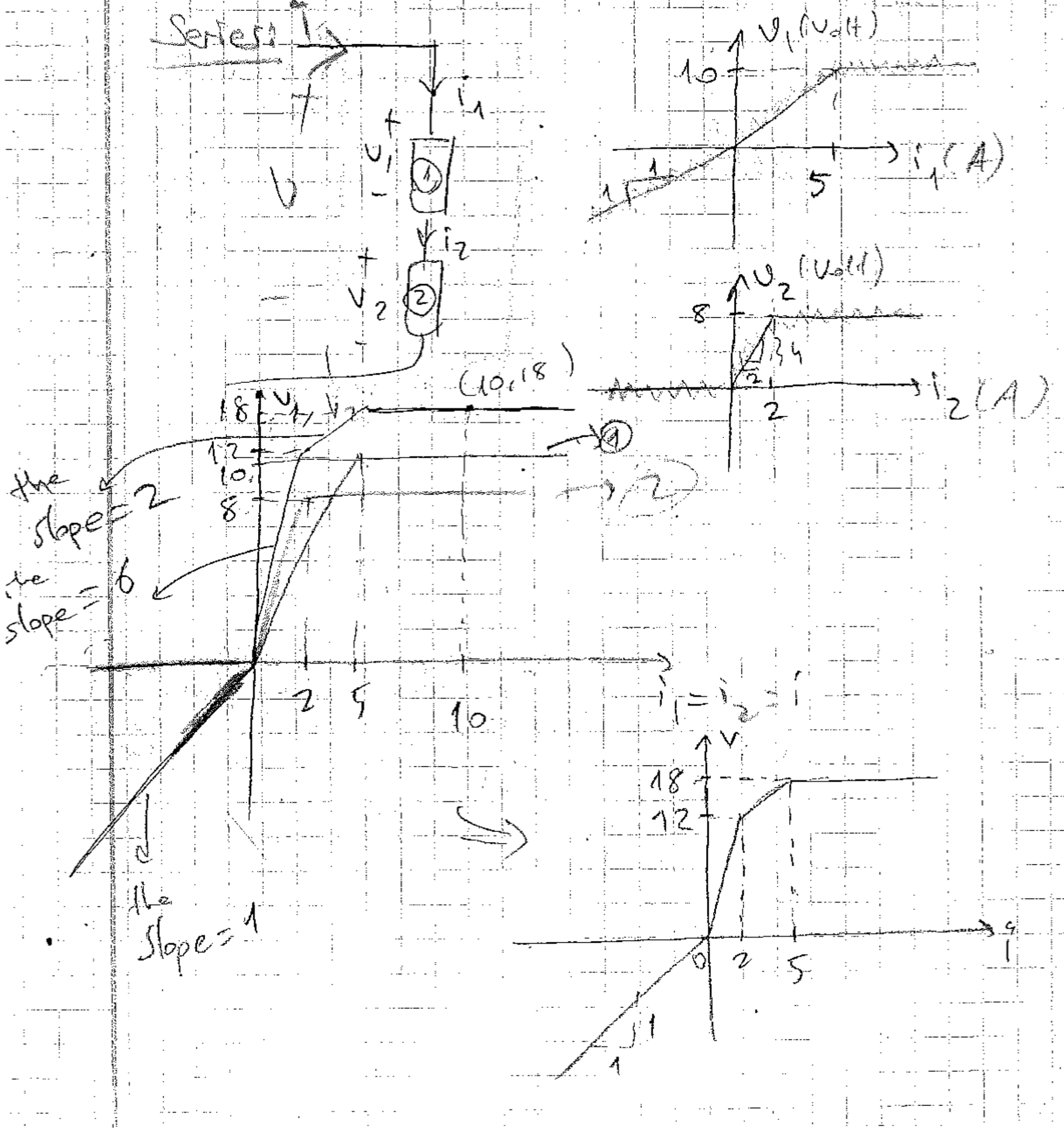
$V_D = 1 - 2 = -1$ Volts ✓

Assumption is satisfied. ✓

What if a circuit has 50 diodes? 2^{50} possibilities.

Series and Parallel Combination through Graphical Analysis

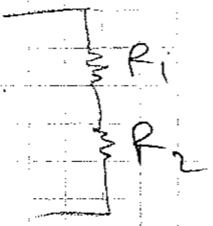
Series:

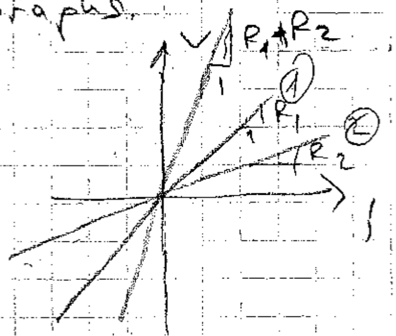


Series combination: For the same current value, we add the voltages.

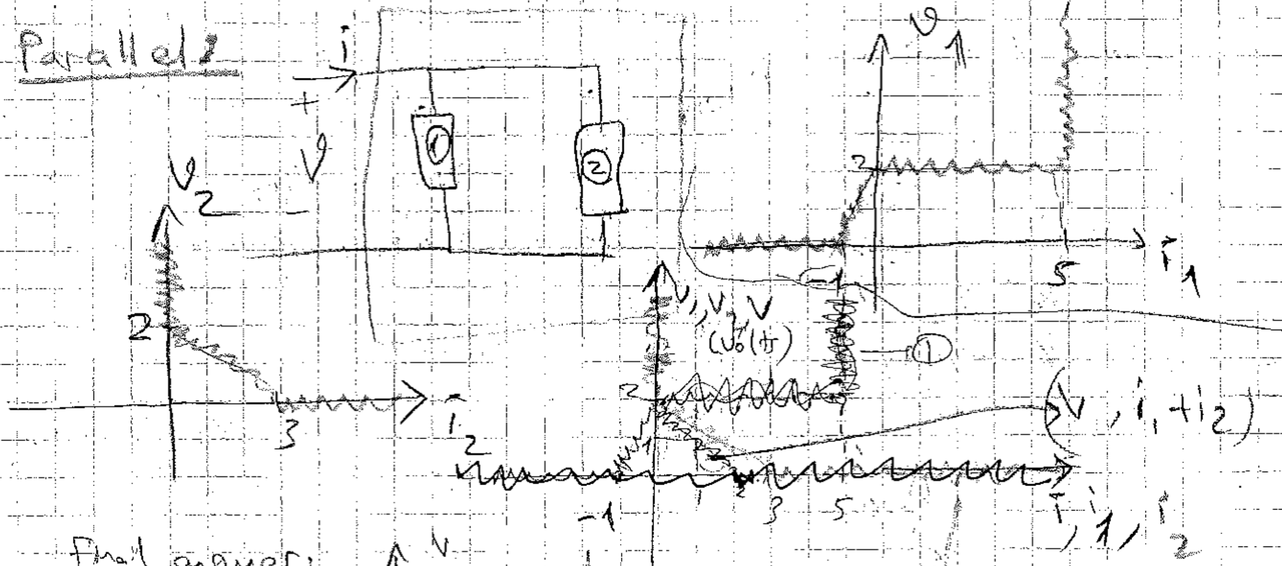
In this example, the voltage addition is the vertical addition of two graphs.

Note:

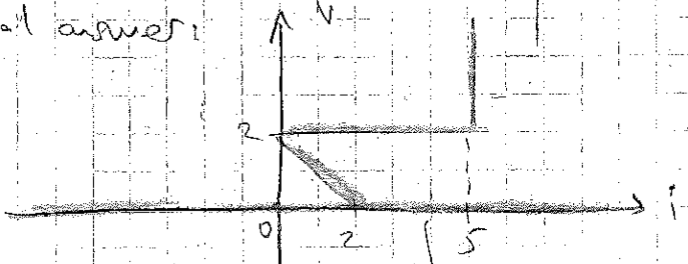




Parallel:



Final answer:



(The whole axis)

i_1 can be any negative number < -1

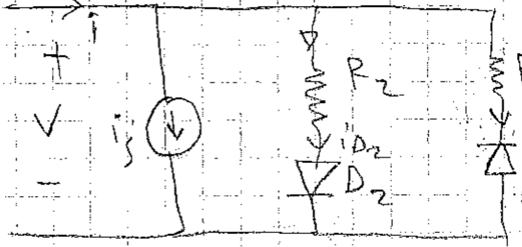
i_2 can be any positive number > 3

Ex $-50 + (+50) = 0$ A.

Parallel combination: We add current values for the same voltage, $i = i_1 + i_2$ for the fixed $V = V_1 = V_2$.

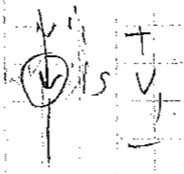
For this example, adding currents is the horizontal addition of graphs.

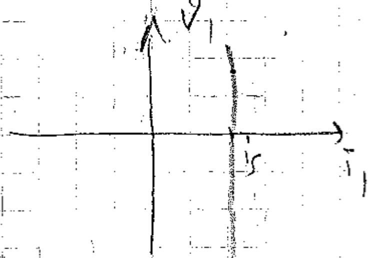
Ex:



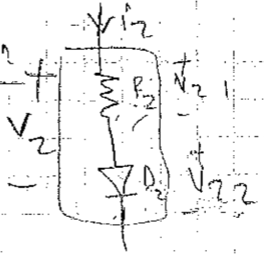
Find (i, V) characteristics.

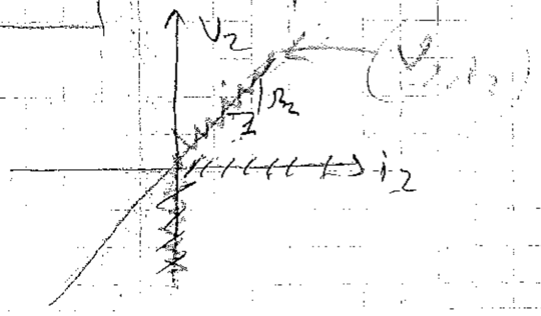
1st branch:



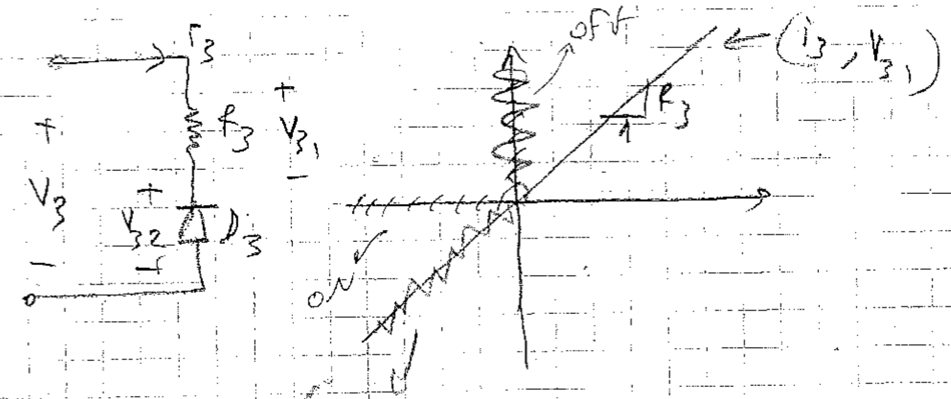


2nd branch:

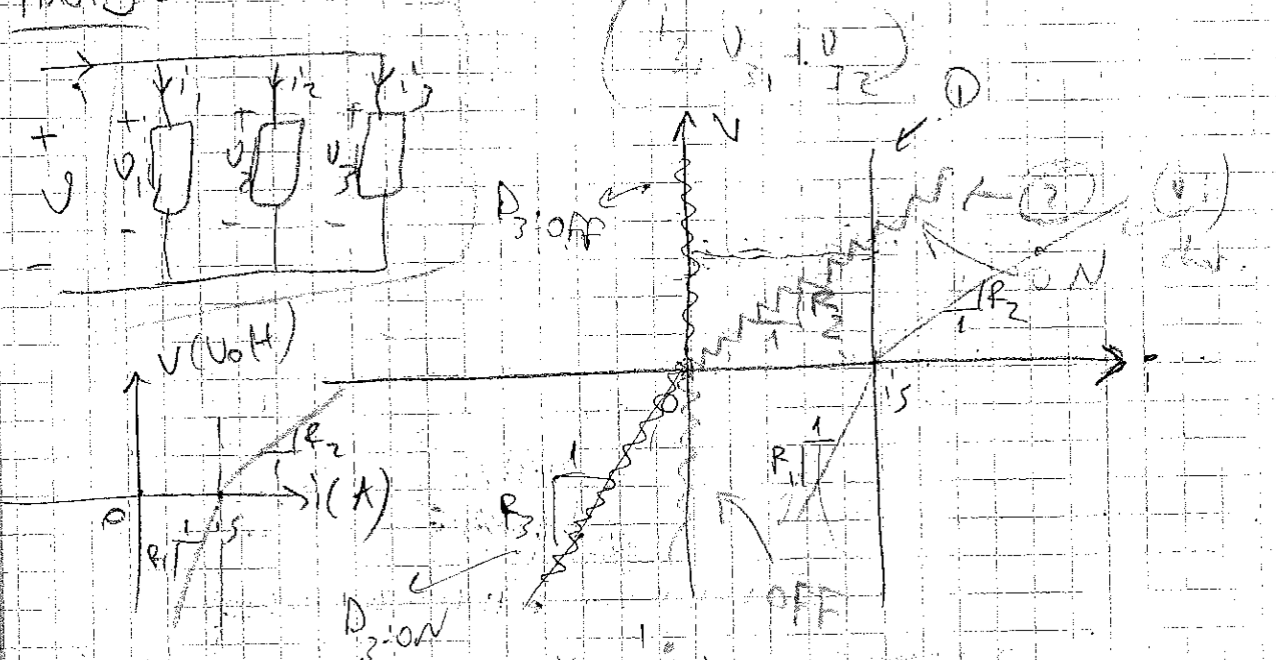




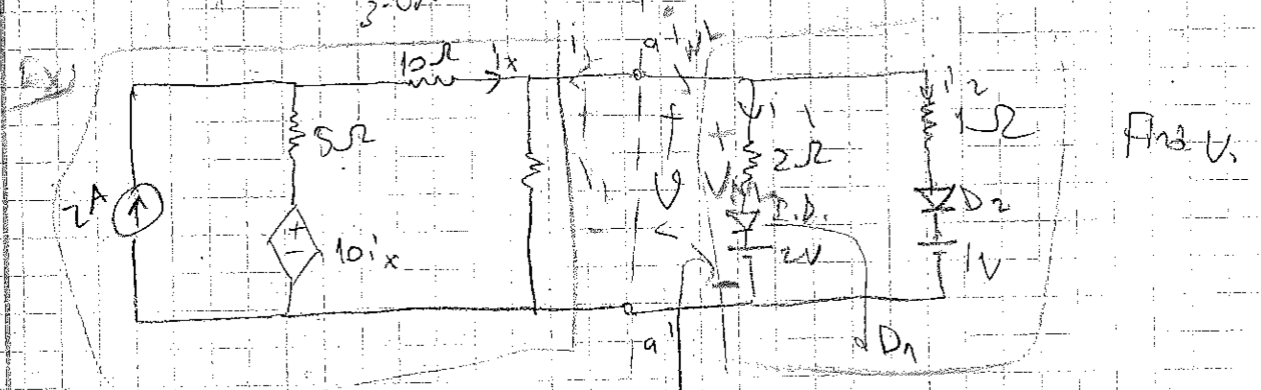
3rd Branch:



Finally:



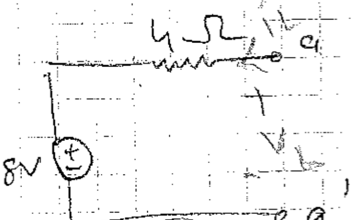
Ex:

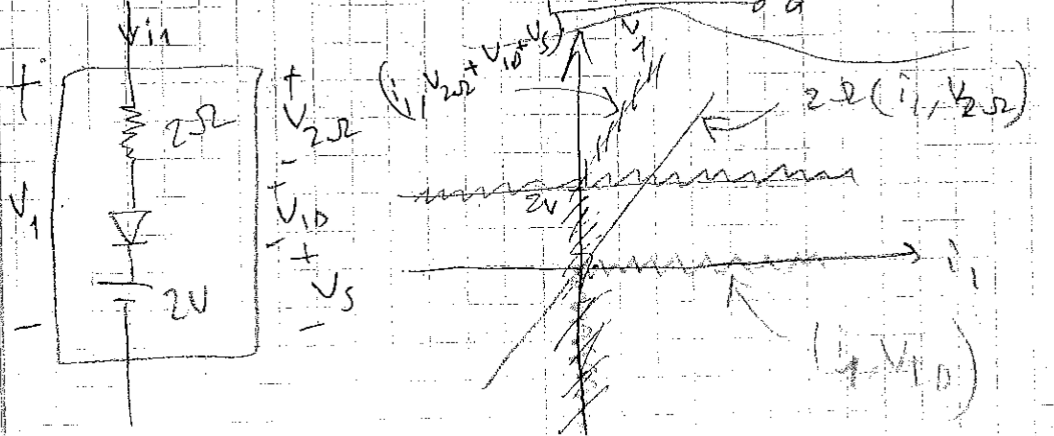


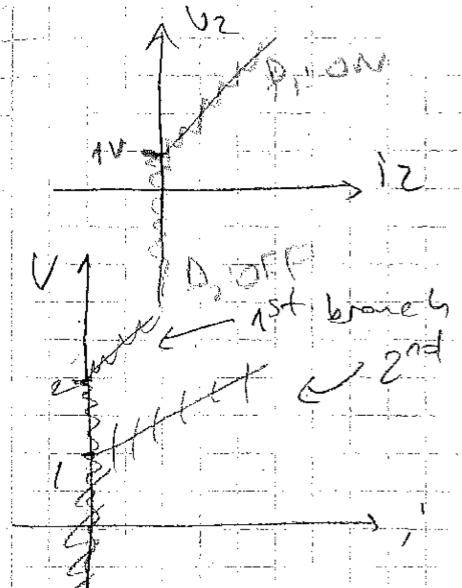
Find V_1

To simplify the solution let's first find Thevenin equivalent of LHS of a-a' line.

Thevenin Equiv

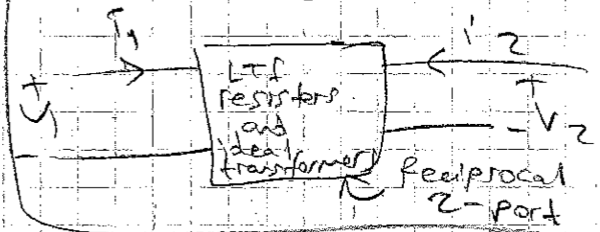




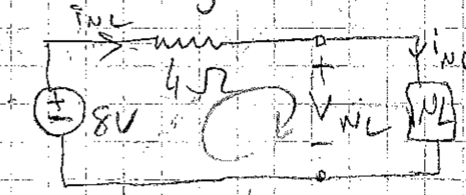


Notes: Ideal transformer is a reciprocal component, since its h-parameter representation is:

$h_{12} = -h_{21}$, so



Combining LHS and RHS of a-a' lines:

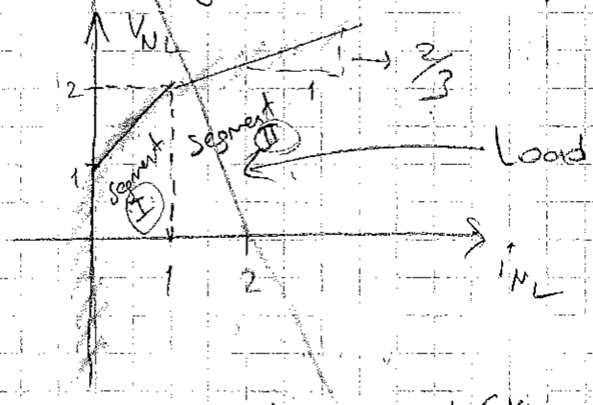


KVL: $-8 + 4i_{NL} + V_{NL} = 0$

$V_{NL} = 8 - 4i_{NL}$ ← load line

8

Solution by load line (graphical method)



Load line

Operating point is the intersection of the load line and the NL components' characteristic.

Assume Operating point (the intersection) is in Segment Ⓘ:

$$\left.\begin{aligned} &\text{Ⓘ}: V_{NL} = i_{NL} + 1 \\ &\text{Load Line}: V_N = 8 - 4i_{NL} \end{aligned}\right\} i_{NL} = \frac{7}{5}\,A \quad \times \quad \Big|\; V_{NL} = \frac{12}{5} \text{ Volts} \quad \times$$

This assumption is wrong since segment Ⓘ is valid for $0 \leq i_{NL} < 1$.

Assume that op. point is in Segment Ⓘⓘ.

$$\left.\begin{aligned} &\text{Ⓘⓘ}: V_{NL} = \frac{2}{3} i_{NL} + \frac{4}{3} \\ &\text{Load Line}: V_N = 8 - 4i_{NL} \end{aligned}\right\} i_{NL} = \frac{10}{7}\,A \quad V_{NL} = \frac{16}{7}\,V$$

So $(i_{NL}, V_{NL}) = \left(\frac{10}{7}\,A, \frac{16}{7}\,V\right)$ is the operating point.

//

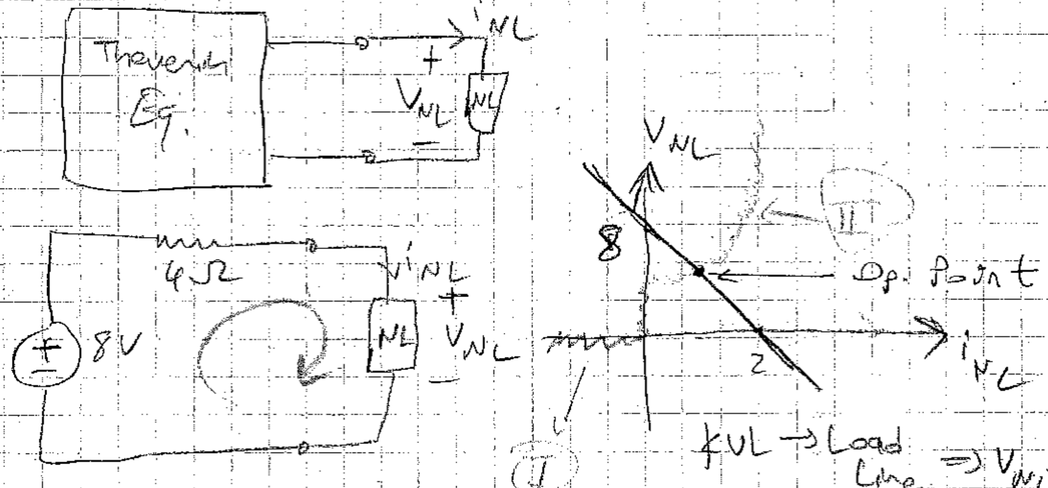
Q: Assume that the 2A source is replaced with a 3A source. What is the new solution?

Load line: $\boxed{V_{NL} = 12 - 4i_{NL}}$

Q_2: Assume the source is 2A but the NL component is expressed as

$$V_{NL} = \begin{cases} 4(i_{NL})^2+5, & i_{NL}>0, \\ 0, & i_{NL}<0 \end{cases}$$ Find the solution.

Answer of Q_2:



KUL → Load line ⇒ $V_{NL}=8-4i_{NL}$

Assume the solution is in segment ⓘⓘ:

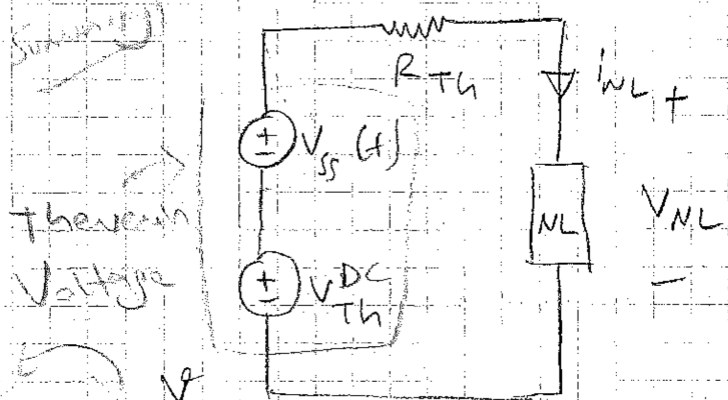
load line ⇒ $V_{NL}=8-4i_{NL}$

NL load ⇒ $V_{NL}=4(i_{NL})^2+5$

$\rightarrow 4(i_{NL})^2+4i_{NL}-3=0$

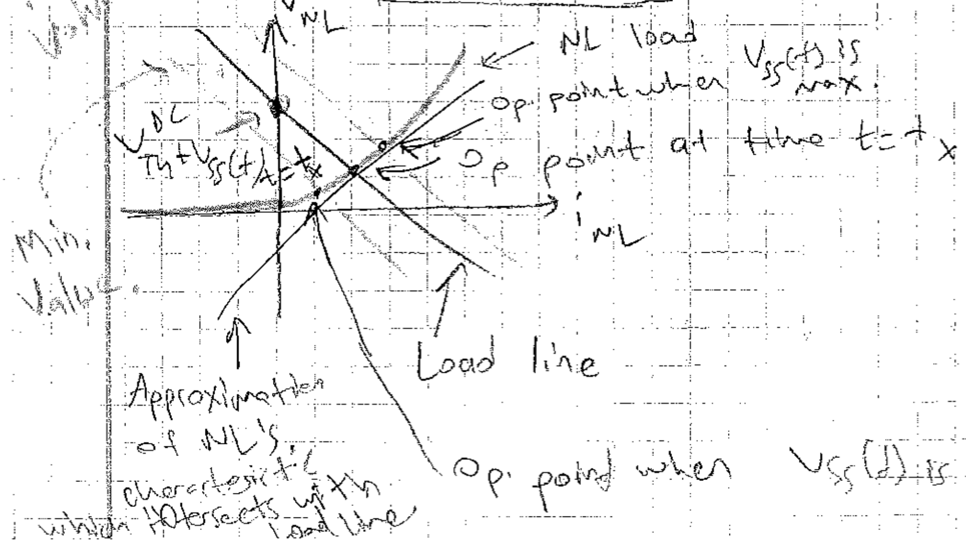
$i_{NL}=\{1/2, -3/2\}$

Small Signal Analysis



$V_{SS}(t)$ = Small signal voltage waveform (A.C.)

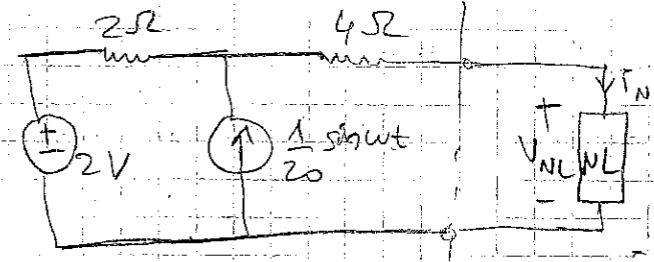
V_{Th}^{DC}: DC part of the thevenin voltage.



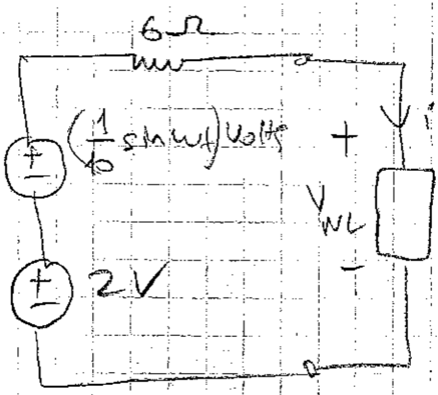
Small Signal assumption:

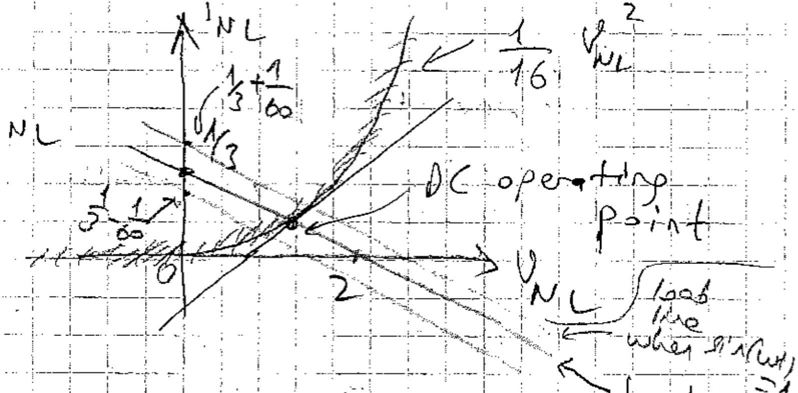
$|V_{SS}(t)| << V_{Th}$

Ex:



$$i_{NL} = \begin{cases} \frac{1}{16} v_{NL}^2 , & v_{NL} \geq 0 \\ 0 , & v_{NL} < 0 \end{cases}$$





Load line (KVL) : $v_{NL} = (2 + \frac{1}{10}\sin(\omega t)) - 6\, i_{NL}$

$$i_{NL} = \left[\frac{1}{3} + \frac{1}{60}\sin(\omega t)\right] - \frac{v_{NL}}{6}$$

Finally DC operating point:

Load Line : $i_{NL} = \frac{1}{3} ~~+ \frac{1}{60}\sin\omega t~~ - \frac{v_{NL}}{6}$

for DC operating point AC part is not taken into account.

NL Load $i_{NL} = \begin{cases} \frac{1}{16} v_{NL}^2 , & v_{NL} \geq 0 \\ 0 , & \text{otherwise} \end{cases}$

Assume $V_{NL} \geq 0 \rightarrow \frac{1}{16} v_{NL}^2 = \frac{1}{3} - \frac{v_{NL}}{6}$ $\quad v_{NL} = \{ \cancel{-4}, \frac{4}{3} \}$

$V_{NL} = \frac{4}{3}$ V is the DC operating point. $\rightarrow i_{NL} = \frac{1}{9}$ A.

$$\frac{1}{3} - \frac{2}{9} = \frac{3-2}{9} = \frac{1}{9}$$

$$\boxed{(v_{NL}^{DC}, i_{NL}^{DC}) = (\tfrac{4}{3}V, \tfrac{1}{9}A)}$$

Approximating NL with a straight line:

$$i_{NL} = \frac{1}{16} v_{NL}^2$$

① Taylor Series around DC op. point

$$f(v_{NL}) = f(v_{NL}^{DC}) + \frac{f'(v_{NL}^{DC}) \cdot (v_{NL} - v_{DC})}{1!} + \frac{f''(v_{NL}^{DC})(v_{NL} - v_{DC})^2}{2!} + \dots$$

$f(v_{NL}) = \frac{1}{16} v_{NL}^2$, $v_{NL}^{DC} = \frac{4}{3}$

$$f(v_{NL}) = \frac{1}{9} + \frac{f'(\frac{4}{3})(v_{NL} - \frac{4}{3})}{1!} + \dots$$

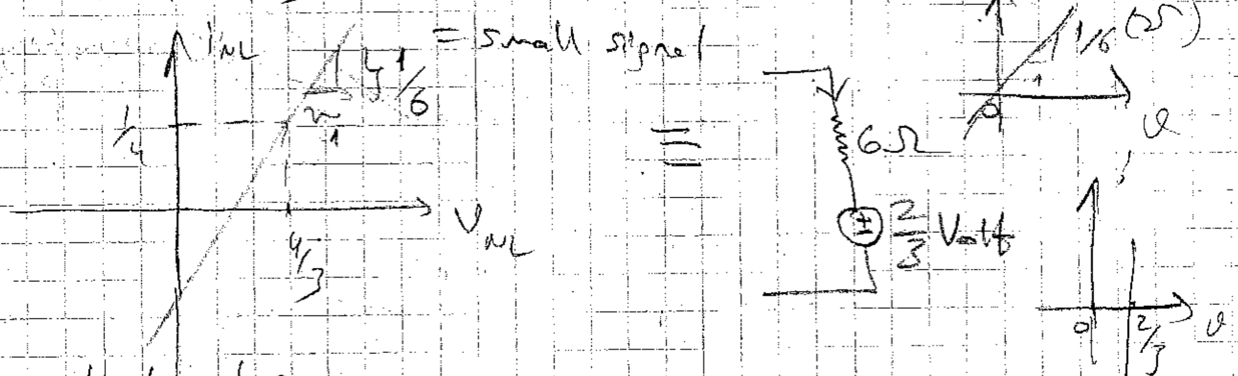
$$f'(v_{NL}^{DC}) = \frac{2}{16} v_{NL} \Big|_{v_{NL} = v_{NL}^{DC} = 4/3} = \frac{1}{6}$$

$$\boxed{f(\underset{\frac{1}{16}v_{NL}^2}{v_{NL}}) \cong \frac{1}{9} + \frac{1}{6}\left(v_{NL} - \frac{4}{3}\right)}$$

approximation around DC op. point.

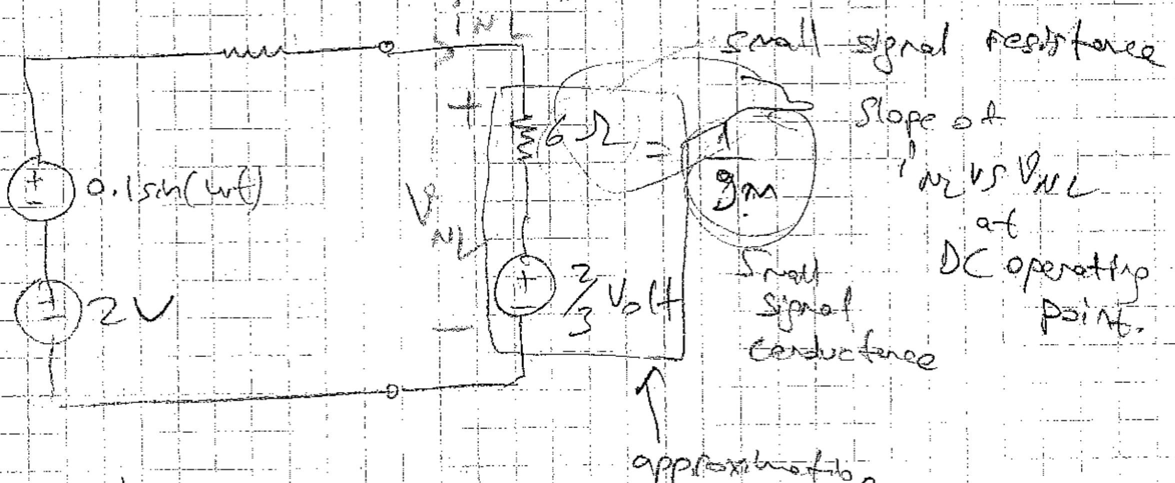
(a straight line)

② You can find the tangent line at DC operating point.



Last step

Replace the NL component with the approximation:



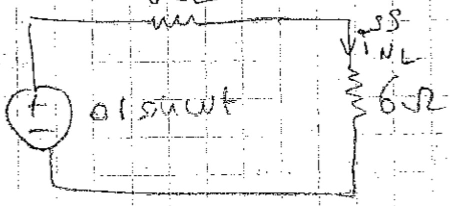
small signal resistance

Slope of i_{NL} vs V_{NL} at DC operating point.

Small signal conductance

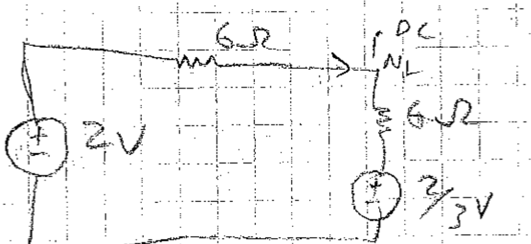
approximation

AC part: (small signal circuit)



$$i_{NL}^{SS}(t) = \frac{0.1\sin(\omega t)}{12}\ A$$

DC part:

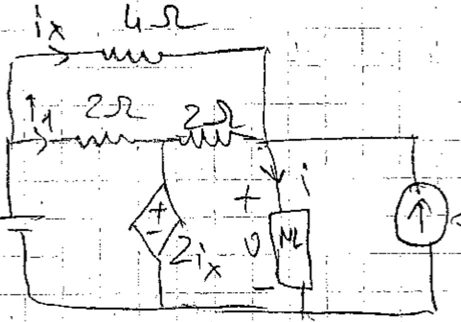


$$I_{NL}^{DC} = \frac{2 - 2/3}{12} = \frac{1}{9} A$$

$$i_{NL}(t) \cong I_{NL}^{DC} + i_{NL}^{SS}$$
$$= \frac{1}{9} + \frac{0.1}{12}\sin(\omega t)\ A$$

Ex: 2.8S-3
Pr. 46

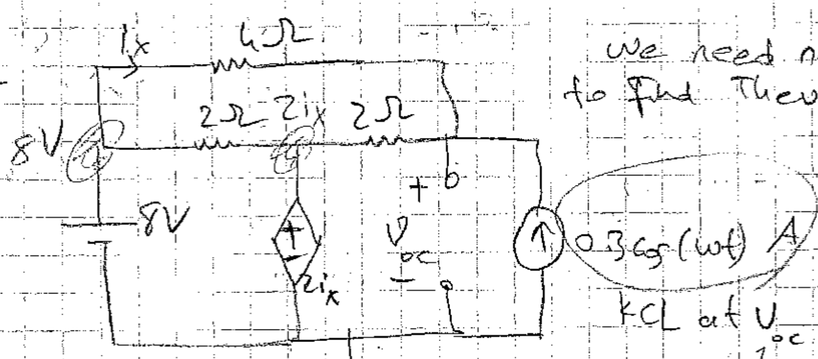
$i = \begin{cases} \frac{1}{2}v^2, & v \geq 0 \\ 0, & \text{otherwise} \end{cases}$



Find i_1 by small signal approximation

I need to find D.C. operating point. To do that, Thevenin equivalent will be utilized.

V_{oc}:



$= i_s(t)$

We need node (or mesh) analysis to find Thevenin eq.

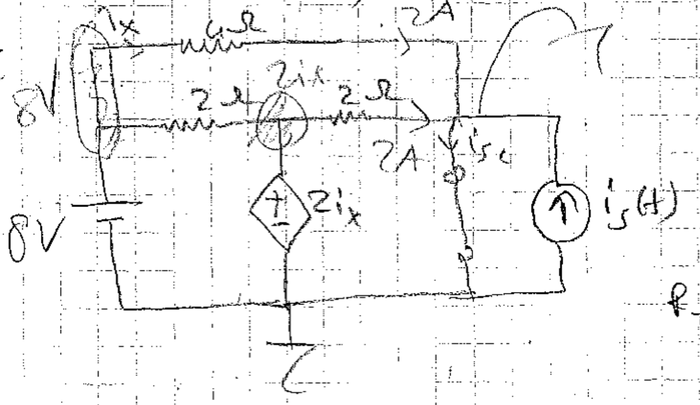
KCL at V_{oc}: $\frac{V_{oc} - 2i_x}{2} + \frac{V_{oc} - 8}{4} - i_s(t) = 0$

$i_x = \frac{8 - V_{oc}}{4}$

$V_{oc} = \frac{2 + 2 + i_s(t)}{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}}$

$V_{oc} = 4 + i_s(t)$

i_{sc}:

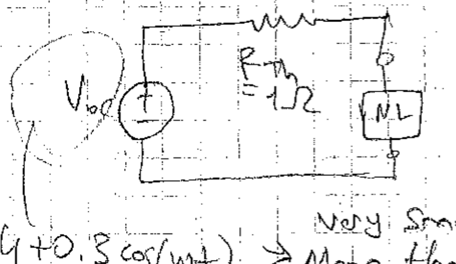


$i_x = 2A$

$i_{sc} = 2 + 2 + i_s(t)$

$i_{sc} = 4 + i_s(t)$

$R_{Th} = \frac{V_{oc}}{i_{sc}} = 1\Omega$



$4 + 0.3\cos(\omega t)$ → Very small: More than 10 times smaller.

Then DC operating point is $(v_{NL}^{DC}, i_{NL}^{DC})$
$= (2V, 2A)$

Let's find DC operating point:

KVL: $-4 + i_{NL} \times 1 + v_{NL} = 0$

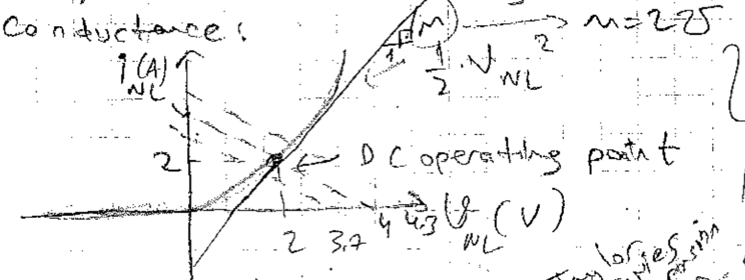
$\frac{1}{2}v_{NL}^2$ (provided that $v_{NL} \geq 0$)

$v_{NL}^2 + 2v_{NL} - 8 = 0$

$(v_{NL} + 4)(v_{NL} - 2) = 0$

$v_{NL} = \{\cancel{-4}, 2\}$

Then, Let's find small signal resistance/conductance:



$f(v_{NL}) = \frac{1}{2}v_{NL}^2$
then slope of $f(\cdot)$ is at DC op. point is

Taylor series expansion

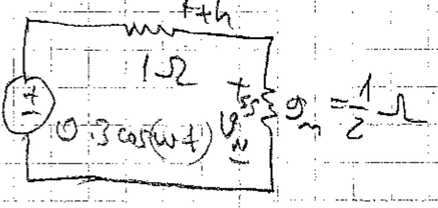
$\frac{d\, f(v_{NL})}{d v_{NL}} = \frac{2}{2} v_{NL} \Big|_{v_{NL} = v_{NL}^{DC} = 2} = 2$

$g_m = 2\mho \rightarrow r_m = \frac{1}{2}\Omega$

the small signal conductance

the small signal resistance

Then to find the AC part of the solution,



AC component of V_{oc}

$$v_N^{ss} = \frac{0.3\cos(wt)}{3}$$

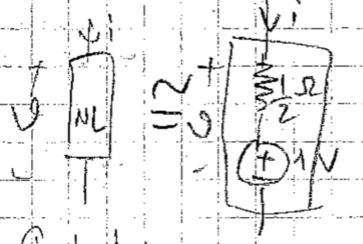
$$v_N^{ss} = 0.1\cos(wt)$$

$$i_{NL}^{ss}(t) = 0.2\cos(wt)$$

☆ Normally we can't use superposition in non-linear circuits but we make a linear approximation here.

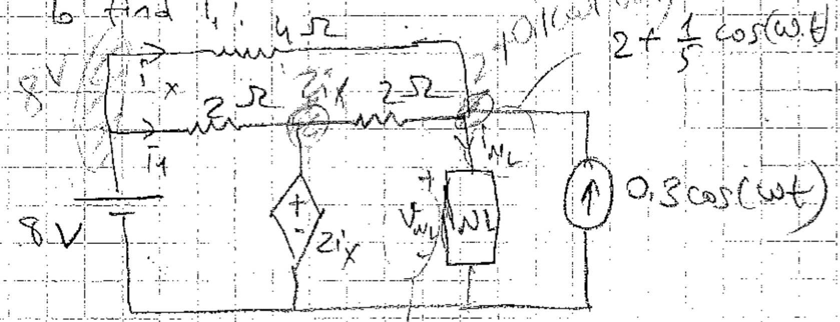
$$v_{NL}(t) \cong V_{NL}^{DC} + V_{NL}^{ss}(t) = 2 + 0.1\cos(wt)\ V$$

$$i_{NL}(t) \cong I_{NL}^{DC} + i_{NL}^{ss}(t) = 2 + 0.2\cos(wt)\ A$$



We approximated the non-linear component around D.C operating point.

To find i_1:



$2+0.1\cos(wt)$

$2 + \frac{1}{5}\cos(wt)\ A$

Again Node analysis:

$$i_1 = \frac{8 - 2i_x}{2} = 4 - i_x$$

$$i_x = \frac{8 - (2 + 0.1\cos(wt))}{4}$$

$$i_x = \frac{3}{2} - \frac{1}{40}\cos(wt)$$

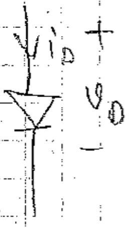
$$\boxed{i_1 = \frac{5}{2} + \frac{1}{40}\cos(wt)\ A}$$

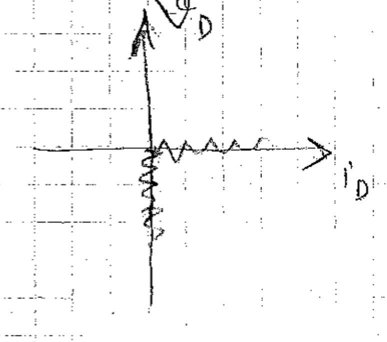
$2 + \frac{1}{10}\cos(wt)\ V$

Diodes: Analysis and Synthesis

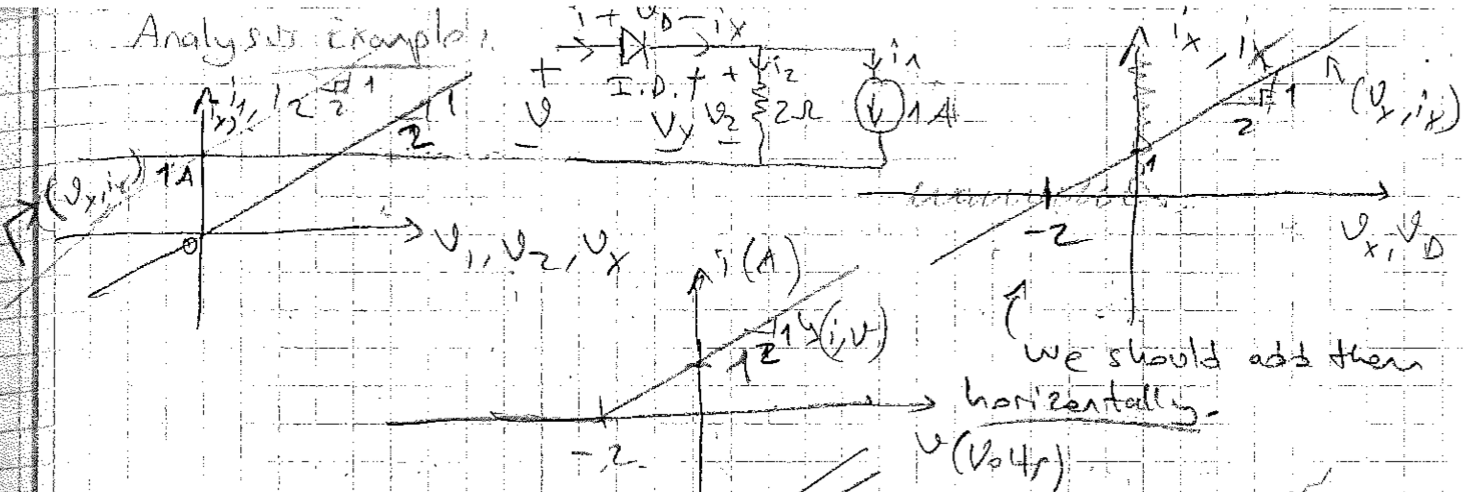
Analysis: Given the circuit → find (i, v) characteristic

Synthesis (Design): Given (i, v) char. → Design a circuit realizing (i, v) char.



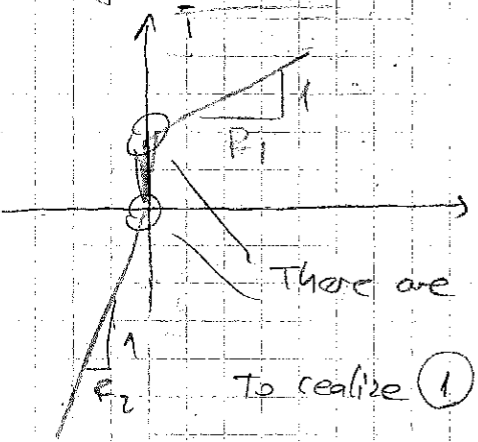


Analysis Example:

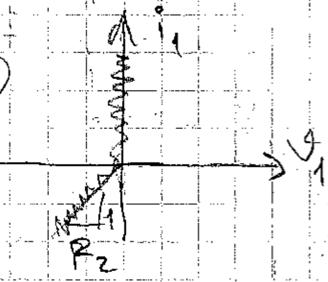


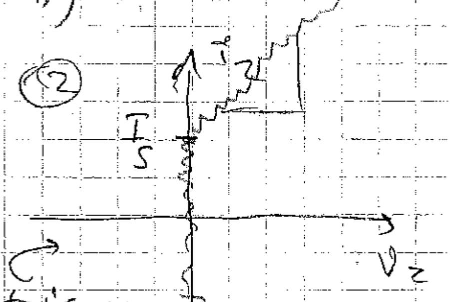
We should add them horizontally.

Design Example:



There are at least 2 diodes.



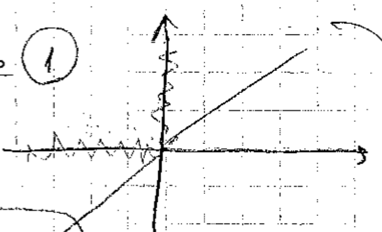


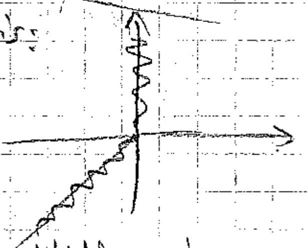
It's added horizontally.

① + ② ⇒ final answer!

in series to get this:

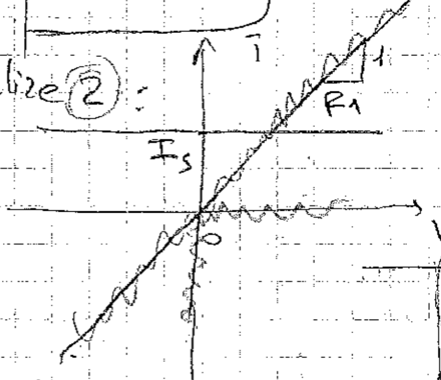
To realize ①



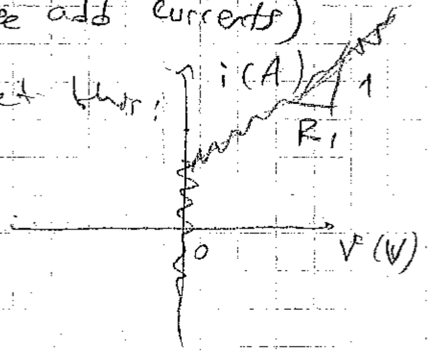


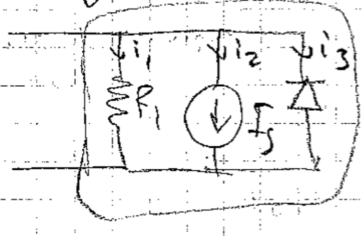
Vertical addition i.e parallel combination (for the same voltage add currents)

To realize ②:



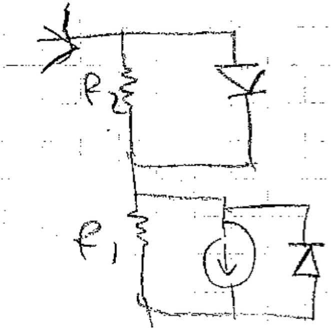
to get this!



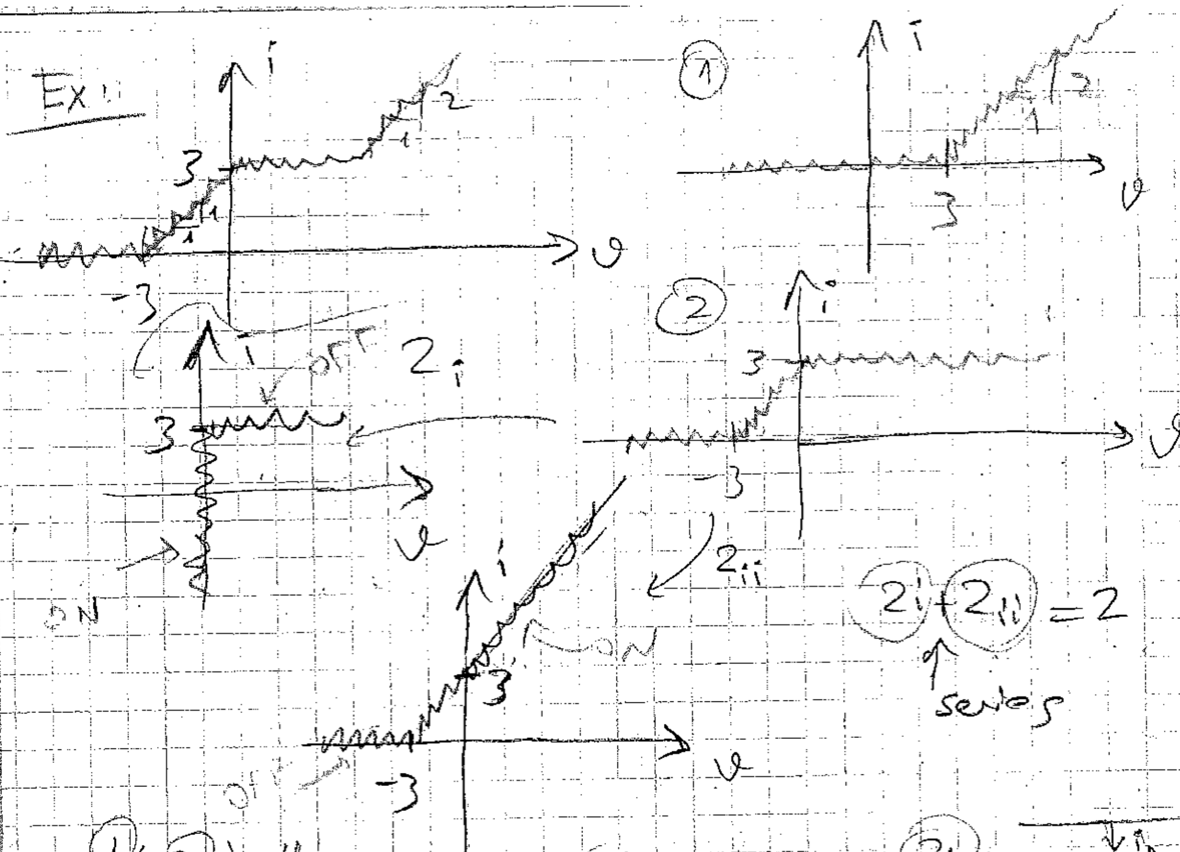


Vertical additional gives ②

Final design:



Ex.:

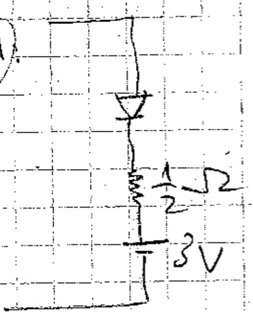


①//② is the final design

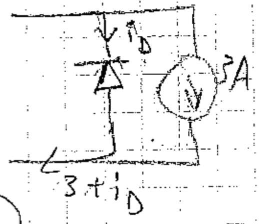
② = (2i) + (2ii)

for the same voltage values we add currents.

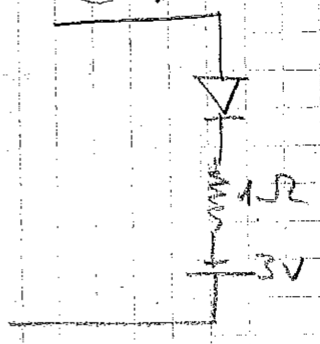
①:



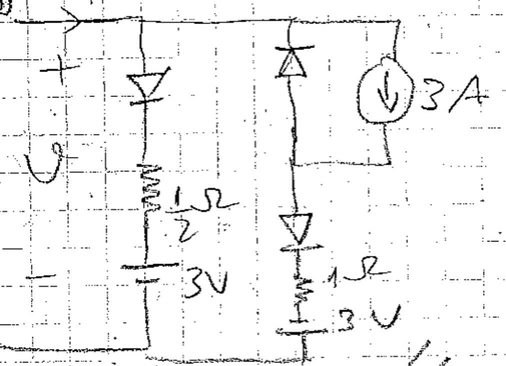
(2i):



(2ii):



Final Answer:

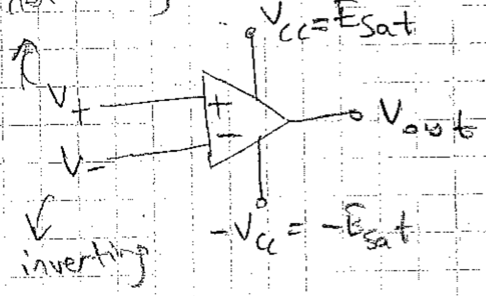


(around ~1968)

Operational Amplifiers

op-amp is an integrated chip

in EE311 we will be studying opamps further.

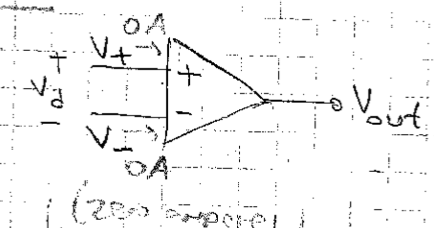


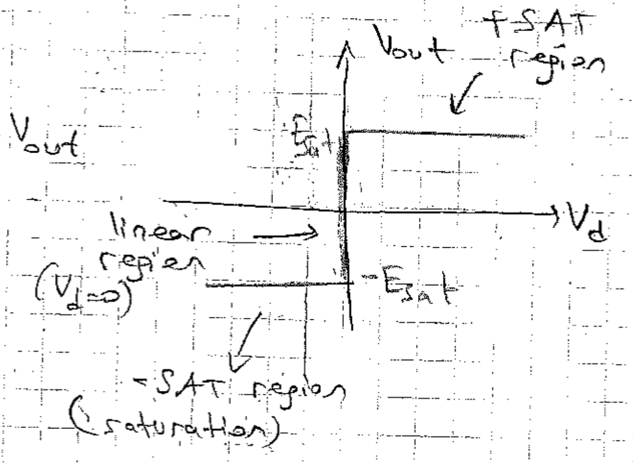
1 from

Ideal Op-Amp Model

$V_d = V_+ - V_-$

differential voltage



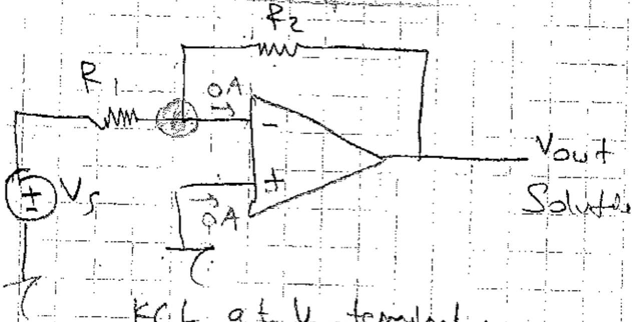


For Ideal op-amp:

$\begin{pmatrix} A=\infty \\ r_{out}=0 \\ R_{IN}=\infty \end{pmatrix}$

It is like a comperator.

Inverting Amplifier

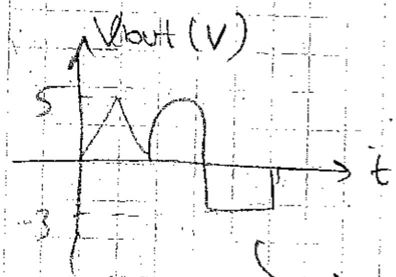


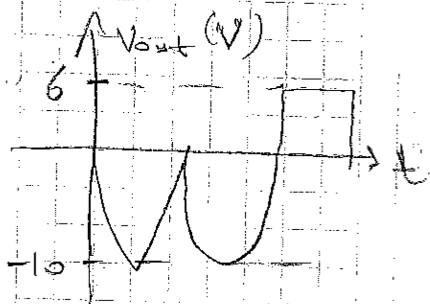
Assume op-amp is ideal and in linear region, find V_{out} in terms of V_s.

Solution: ① In linear region, $V_d = 0 \rightarrow V_+ = V_-$

② $I_+ = I_- = 0\,A$ Ideal op-amp in linear region

KCL at V_- terminal: $\frac{0-V_s}{R_1} + \frac{0-V_{out}}{R_2} = 0 \xrightarrow{(1),(2)} \boxed{V_{out} = \frac{-R_2}{R_1} V_s}$





↑ (output vs input)

Transfer characteristic

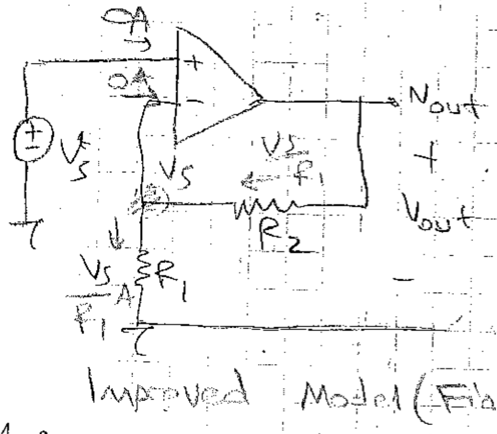
Ideal op-amp in linear region

$\boxed{I_{IN} = \frac{V_s}{R_1}\ A}$

Non-inverting Amplifier

Ideal op-amp linear region

$\boxed{V_{out} = V_s + \frac{V_s}{R_1} R_2 = V_s\left(\frac{R_1+R_2}{R_1}\right)}$

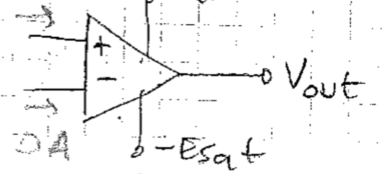


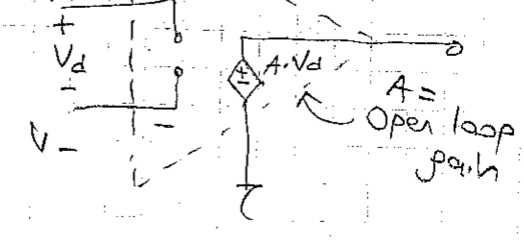
$\frac{R_1+R_2}{R_1}$ = amplification constant

$= 1+\frac{R_2}{R_1}$

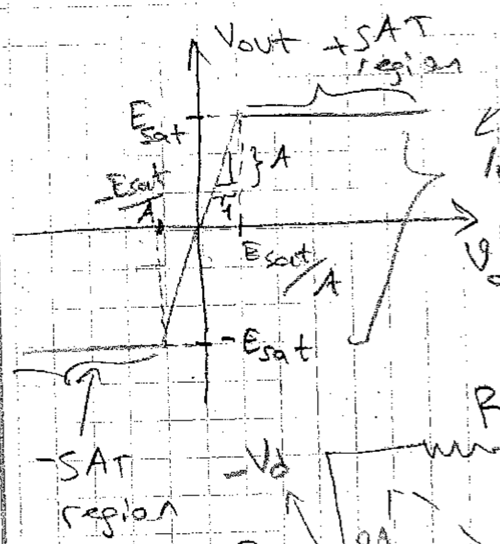
Improved Model (Finite Gain Model)

$\begin{pmatrix} A: \text{finite} \\ r_{out}=0 \\ R_{IN}=\infty \end{pmatrix}$

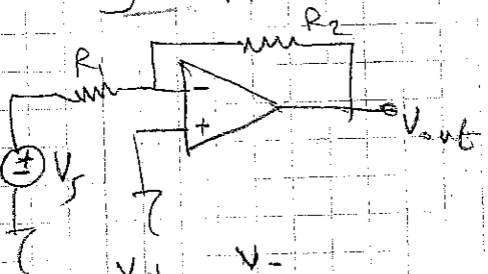




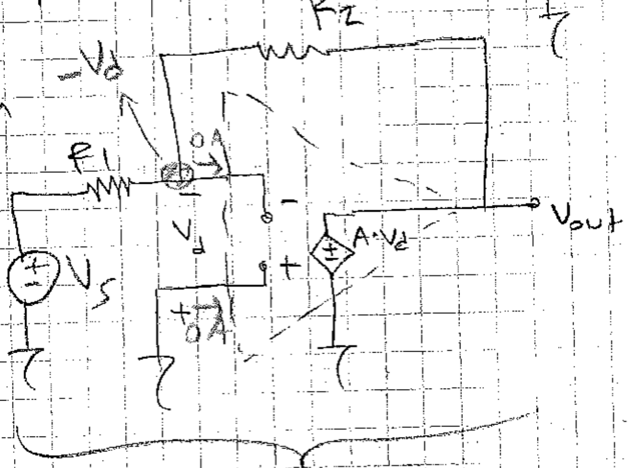
A = Open loop gain



Let's analyze inverting amplifier with finite gain model:



$$\underset{\sim}{\overset{V_+}{0}} - \overset{V_-}{(-V_d)} = V_d$$



Model for linear region

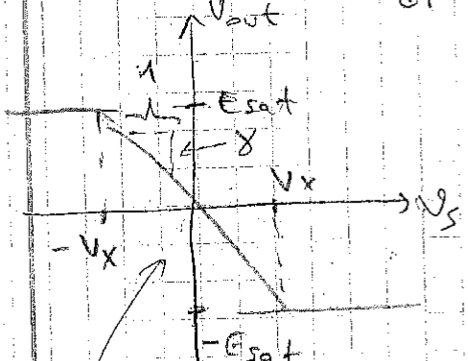
KCL at V_-:

$$\frac{-V_d - V_s}{R_1} + \frac{-V_d - \overset{A\cdot V_d}{V_{out}}}{R_2} = 0$$

$$V_d = \frac{-R_2}{R_1 A + R_1 + R_2} \cdot V_s$$

$$V_{out} = A\cdot V_d = \frac{-A\cdot R_2}{A\cdot R_1 + R_1 + R_2} \cdot V_s$$

Validity condition $|V_d| < \frac{E_{sat}}{A}$



$V_{out} = \gamma \cdot V_s$

$$|V_d| < \frac{E_{sat}}{A} \longrightarrow \boxed{|V_s| < \underbrace{\frac{E_{sat}}{A} \cdot \frac{R_1 A + R_1 + R_2}{R_2}}_{V_x}}$$

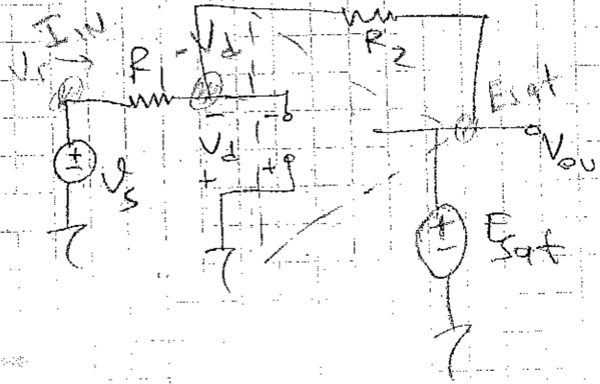
$$\uparrow \frac{-R_2}{R_1 A + R_1 + R_2} V_s$$

$$\boxed{\gamma = \frac{-A R_2}{R_1 A + R_1 + R_2}}$$

Note: As $A \to \infty$,
$V_{out} = \frac{-R_2}{R_1} V_s$, $V_x = E_{sat} \cdot \frac{R_1}{R_2}$

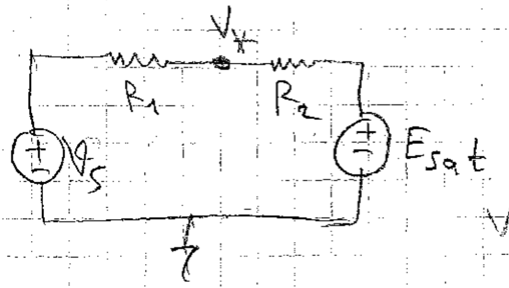
+SAT region

$V_{out} = E_{sat}$ provided that $V_d > \frac{E_{sat}}{A}$



$V_{out} = E_{sat}$

Let's find V_s interval for which op-amp is in +SAT region.



$$\boxed{V_x = \frac{R_2}{R_1+R_2} v_s + \frac{R_1}{R_2+R_1} \cdot E_{sat}}$$ ↙ from linearity

$V_x = -V_d$

$$V_d = -\left(\frac{R_2 v_s + R_1 \cdot E_{sat}}{R_1+R_2}\right)$$

$$V_d > \frac{E_{sat}}{A} \longrightarrow \frac{R_2 v_s + R_1 E_{sat}}{R_1+R_2} < \frac{-E_{sat}}{A}$$

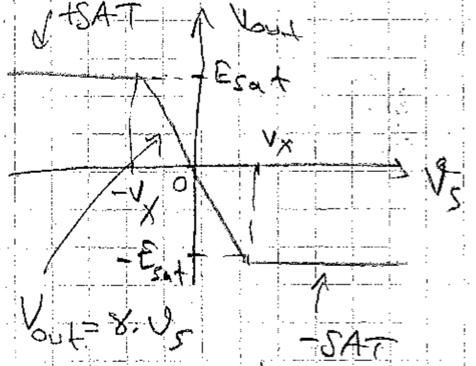
$$v_s < \frac{-E_{sat}\left(\frac{R_1+R_2}{A}\right) - R_1 E_{sat}}{R_2}$$ $-V_x$

Similliarly, when op-amp is in -SAT region;

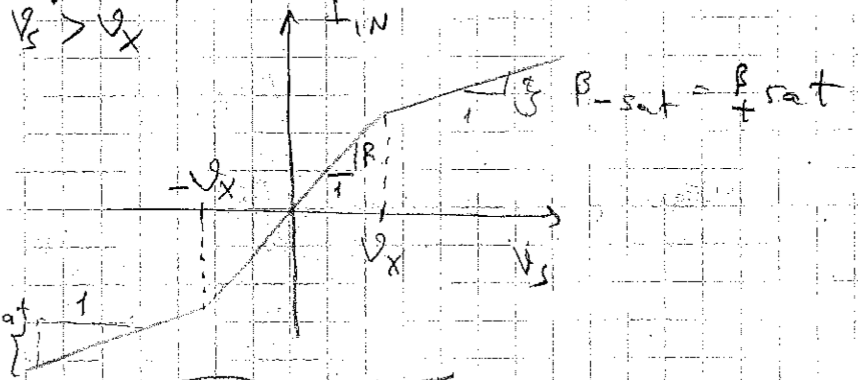
$V_{out} = -E_{sat}$ and $V_d < -\frac{E_{sat}}{A}$

$\Updownarrow$

$v_s > v_x$



transfer characteristic of the circuit



Input Characterization

$$I_{IN} = \frac{v_s - (-v_d)}{R_1}$$ $\longrightarrow$ Linear region

$$V_d = \frac{-R_2}{R_1 \cdot A + R_1 + R_2} \cdot v_s$$

$$I_{IN} = \frac{v_s}{R_1}\left(1 - \frac{R_2}{R_1 A + R_1 + R_2}\right)$$

$$= \frac{v_s}{R_1}\left(\frac{R_1 A + R_1}{R_1 A + R_1 + R_2}\right)$$

$\searrow R$

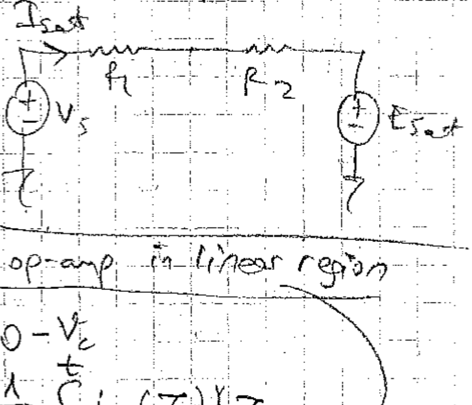
+SAT $$V_d = -\left(\frac{R_2 v_s + R_1 E_{sat}}{R_1+R_2}\right)$$

$$I_{IN} = \frac{v_s}{R_1} - \frac{R_2}{(R_1+R_2)}\frac{v_s}{R_1} - \frac{R_1}{(R_1+R_2)}\frac{E_{sat}}{R_1} \rightarrow ? ?$$

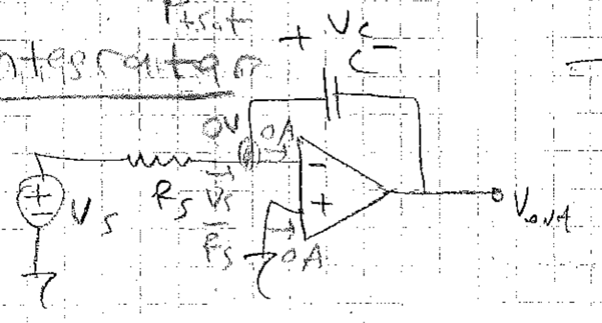
$$\boxed{I_{IN} = \frac{1}{R_1+R_2} v_s - \frac{1}{R_1+R_2} E_{sat}}$$

β_{+sat}

2^{nd} way:



Integrator



Ideal op-amp in linear region

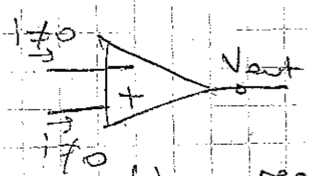
$$v_{out} = 0 - V_c$$

$$= -\frac{1}{C}\int_{-\infty}^{t} i_c(\tau)\, d\tau$$

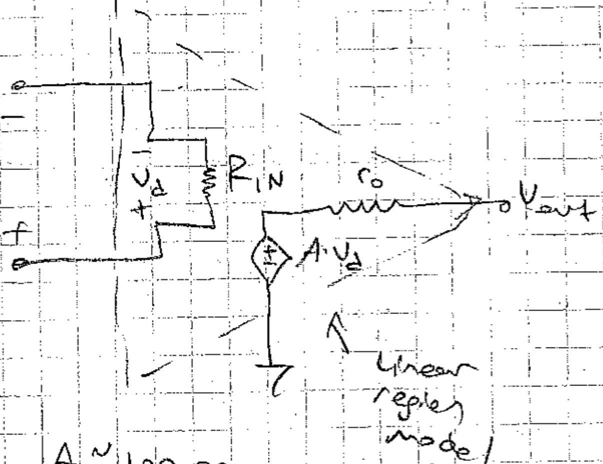
$$V_{out}(t) = -\frac{1}{R_S C}\int_{-\infty}^{t} v_s(\tau)\, d\tau$$

Op-amp's (cont'd)

Improved Model for op-amp : (A: finite, $r_o \neq 0$, $r_{in} \neq \infty$)

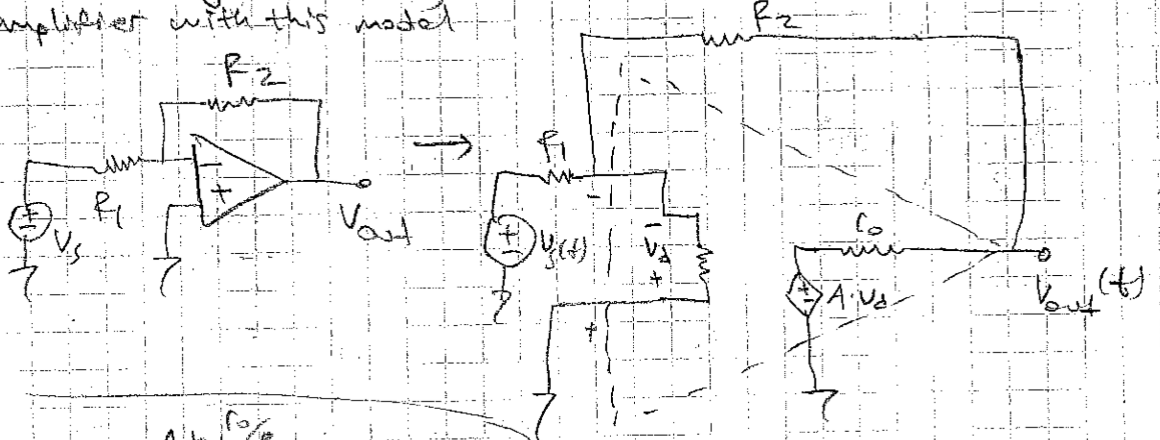


Linear region



$A \cong 100.000$
$R_{IN} \approx 1 M\Omega$
$r_o \approx 10 \Omega$

Let's analyze inverting amplifier with this model



$$V_{out} = \frac{-A + r_o/R_2}{\frac{R_1}{R_2}\left(1 + A + \frac{r_o}{R_{IN}}\right) + \left(\frac{R_1}{R_{IN}} + 1\right) + \frac{r_o}{R_2}} V_s$$

Linear region model
Finding V_{out} as a function of $V_s(t)$.

Let's check what happens as;

① $(A \to \infty) \to \left(V_{out} \to -\frac{R_2}{R_1} V_s\right)$

② A is finite ; and I would like to design an amplifier with gain -10, that is from ideal op-amp model $-\frac{R_2}{R_1} = -10$

$$\boxed{\frac{R_2}{R_1} = 10}$$

Design rule

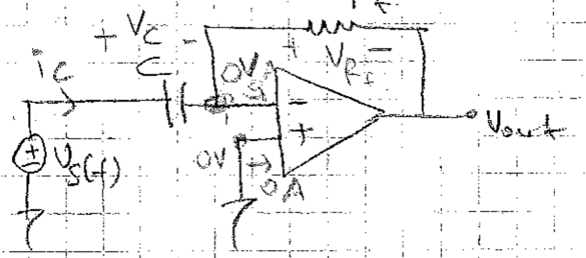
So, by selecting R_1 and R_2 as much larger than r_o and at the same time much smaller than R_{IN}, I can have a reliable design.

$k = \{1, 2\}$

$R_k >> r_o$
$R_k << R_{IN}$
Select $R_k \approx 10 k\Omega$

If $R_{IN} \approx 1 M\Omega$
$r_o \approx 10 \Omega$

Differentiator



(derivative)

$$C\cdot V_s(t)$$

$$i_c(t) = C\frac{d}{dt}V_c(t)$$

$$i_c(t) = C\frac{d}{dt}V_s(t) = C\cdot\dot{V}_s(t)$$

$$V_{R_f} = R_f\cdot i_c$$

$$V_{out} = 0 - V_{R_f} = -R_f\cdot C\frac{d}{dt}V_s(t)$$

Ideal op-amp, linear region

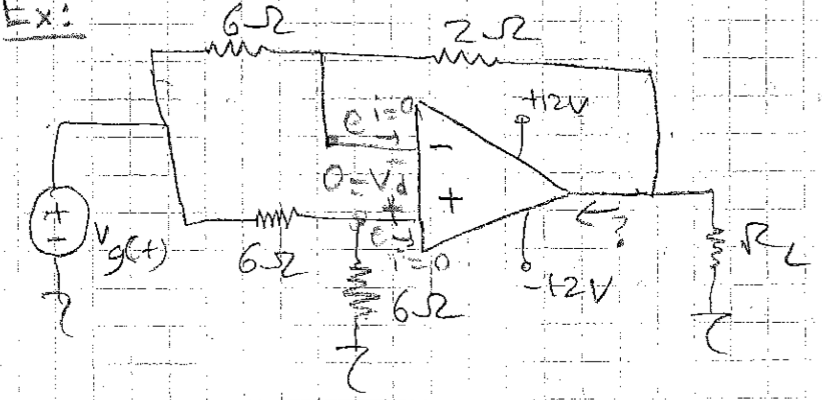
$$(D^2+D+1)\underbrace{V_{out}(t)}_{?} = \underset{\uparrow\ f(t)}{\boxed{V_s(t)}}$$

$$V_{out}(0)=0$$

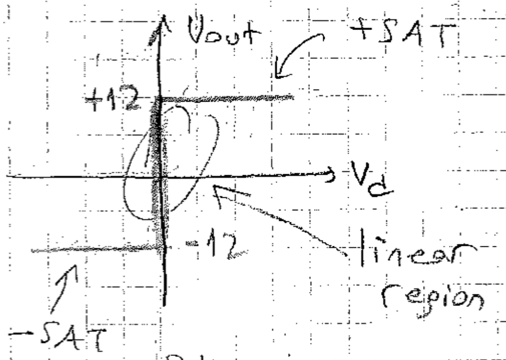
$$V_o(0)=0$$

Some examples for ideal op-amp analysis

Ex:



Find V_{out} (Assume ideal op-amp in linear region)



KCL at V_-: $\frac{e-V_g}{6}+\frac{e-V_{out}}{2}=0$

KCL at V_+: $\frac{e-V_g}{6}+\frac{e}{6}=0 \rightarrow \boxed{e=\frac{V_g}{2}}$

$$\xrightarrow{(*)} \overset{2V_g}{4e} - 3V_{out} - V_g = 0$$

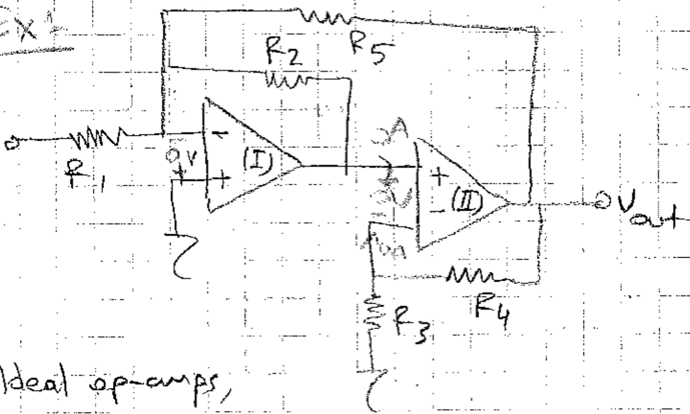
$$\rightarrow V_g = 3V_{out}$$

$$\boxed{V_{out}=\frac{V_g}{3}}$$

Caution: Do NOT write a KCL equation at Vout since with ideal op-amp model, you cannot calculate I_{out} since we do not know A or r_o values.

We almost always use negative feedback.

Ex:



Ideal op-amps, in linear region; Find V_{out}.

Don't write KCL at V_{out}.

KCL at V_- terminal of (II):

$$\frac{e}{R_3}+\frac{e-V_{out}}{R_4}=0 \rightarrow e=\frac{R_3}{R_3+R_4}V_{out}$$

KCL at V_- terminal of (I)

$$\frac{0-V_s}{R_1}+\frac{0-V_{out}}{R_5}+\frac{0-e}{R_2}=0$$

$$V_{out} = \frac{-R_5 R_2 (R_3+R_4)}{[R_2(R_3+R_4)+R_2R_5]R_1} V_s$$

This analysis is only valid when op-amp Ⓘ is in linear region!

$$|V_{out}^{\text{I}}| < E_{sat} \rightarrow |e| < E_{sat}$$

op-amp Ⓘⓘ is in linear region!

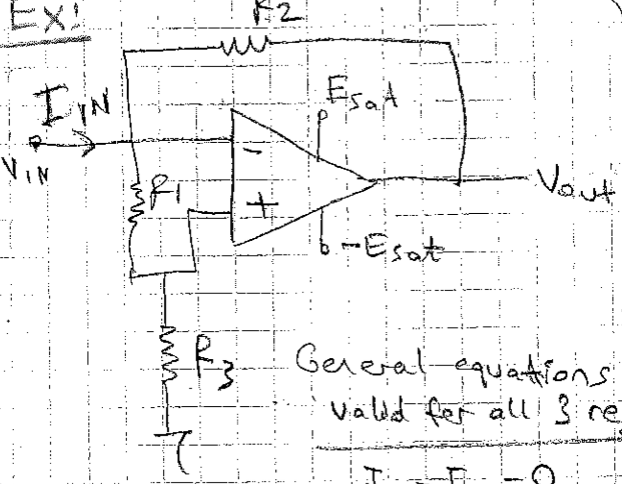
$$|V_{out}^{\text{II}}| < E_{sat} \rightarrow |V_{out}| \leq E_{sat}$$

$|V_s| < Y$

$|V_s| < X$

$|V_s| \leq \min(X,Y)$

Ex!



Assume ideal op-amp
Find V_{out} vs. V_{IN}, I_{IN} vs V_{IN} for all 3 regions of op-amps.

General equations valid for all 3 regions!

$I_+ = I_- = 0$

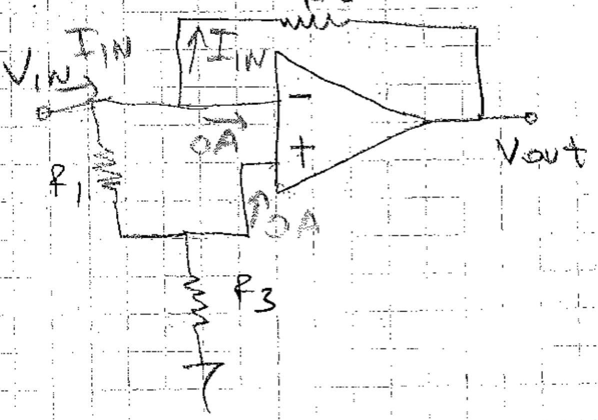
KCL at V_+!

$$\frac{V_+}{R_3} + \frac{V_+ - V_{IN}}{R_1} = 0 \rightarrow \boxed{V_+ = \frac{R_3}{R_3+R_1} V_{IN}} \qquad V_- = V_{IN}$$

① Linear region

$V_d = 0$
$|V_{out}| < E_{sat}$

$$\left.\right\} V_d = 0 \longrightarrow V_+ = V_- \longrightarrow \boxed{V_{IN} = 0} \; \boxed{A}$$



$$\boxed{V_{out} = -I_{IN} R_2}$$

Condition

$$|V_{out}| < E_{sat} \rightarrow \boxed{|I_{IN}| < \frac{E_{sat}}{R_2}} \; \boxed{C}$$

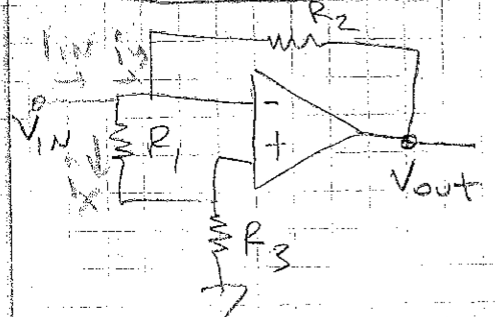
$\boxed{B}$

② +SAT region

$V_{out} = +E_{sat}$

$$V_d > 0 \longrightarrow V_+ > V_- \rightarrow \boxed{V_{IN} < 0} \; \boxed{D}$$

$$\boxed{V_{out} = +E_{sat}} \; \boxed{E}$$



$$I_{IN} = i_x + i_y$$
$$= \frac{V_{IN}}{R_1+R_3} + \frac{V_{IN} - E_{sat}}{R_2} \; \boxed{F}$$

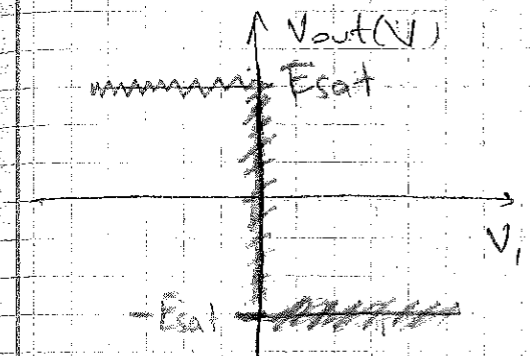
③ -SAT region

$$\left.\begin{array}{c} V_{out} = -E_{sat} \\ V_d < 0 \end{array}\right\}$$

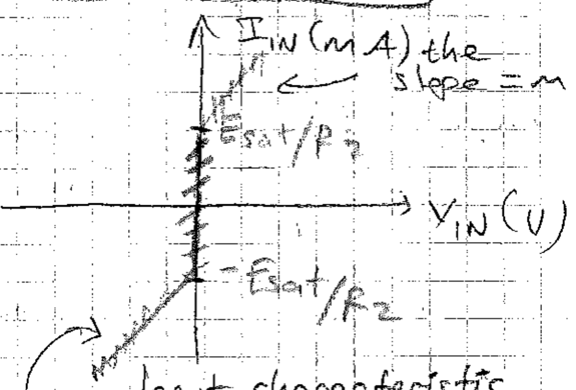
$\boxed{V_{out} = -E_{sat}}$ H

$\boxed{V_{IN} > 0}$ G

$$\boxed{I_{IN} = \frac{V_{IN}}{R_1+R_3} + \frac{V_{IN}+E_{sat}}{R_2}}$$ I



Voltage transfer characteristic



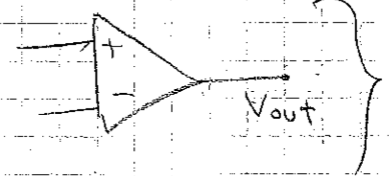
the slope $= m = \frac{1}{R_1+R_3} + \frac{1}{R_2}$

Input characteristic

$$m = \frac{1}{R_1+R_3} + \frac{1}{R_2} \sim \frac{1}{(R_1+R_3)//R_2}$$

Op-Amps (cont'd)

CMRR: (Common Mode Rejection Ratio)



Ideally, in linear region

$$\rightarrow V_{out} = A(\underbrace{V_+ - V_-}_{V_d})$$

A: open-loop gain/finite gain

open-loop means there is no feedback

But in practice

$$\rightarrow V_{out} = A_+V_+ - A_-V_-$$

A_-: finite gain of inverting terminal (V_-)

$$V_{com} = \frac{V_+ + V_-}{2} \qquad V_d = V_+ - V_- \rightarrow \begin{array}{l} V_+ = V_{com} + \frac{V_d}{2} \\ V_- = V_{com} - \frac{V_d}{2} \end{array}$$

$$\begin{aligned} V_{out} &= A_+V_+ - A_-V_- \\ &= A_+\left(V_{com} + \frac{V_d}{2}\right) - A_-\left(V_{com} - \frac{V_d}{2}\right) \\ &= V_{com}\underbrace{(A_+ - A_-)}_{A_{com}} + V_d \underbrace{\left(\frac{A_+ + A_-}{2}\right)}_{A_d} \end{aligned}$$

A_{com}: Common mode Volt gain

A_d: Differential mode gain

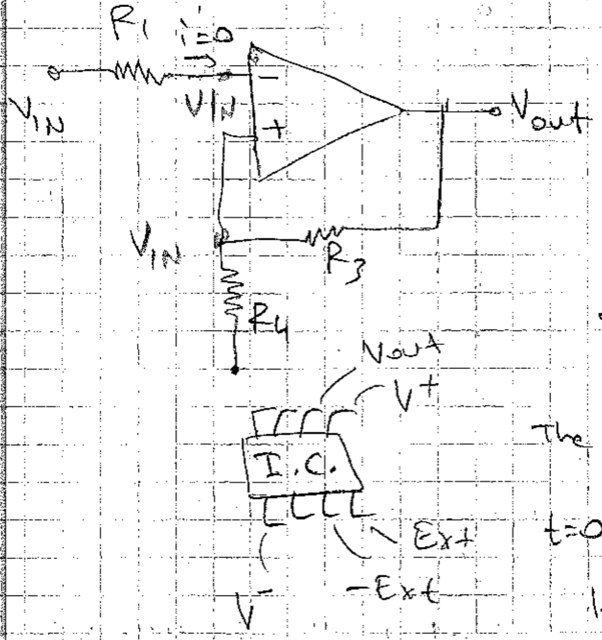
$$\underline{CMRR} \Rightarrow \left(\frac{A_d}{A_c}\right) \rightarrow \left(\frac{A_d}{A_c}\right)_{dB} = 20 \log_{10}\left(\frac{A_d}{A_c}\right)$$

$A_c = A_{com}$

Ideally, CMRR=∞, that is, $\underbrace{A_+ = A_-}_{=A} \Longrightarrow \left\{\begin{array}{l} A_{com} = 0 \\ A_d = A_+ = A_- = A \end{array}\right.$

dB = decibell Graham Bell

Positive and Negative Feedback



Assume linear region, ideal op-amp!

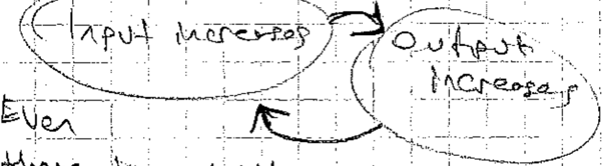
$$\boxed{V_{out} = \frac{R_3+R_4}{R_4} V_{IN}}$$

The difference with negative feedback is there isn't − (minus) sign.

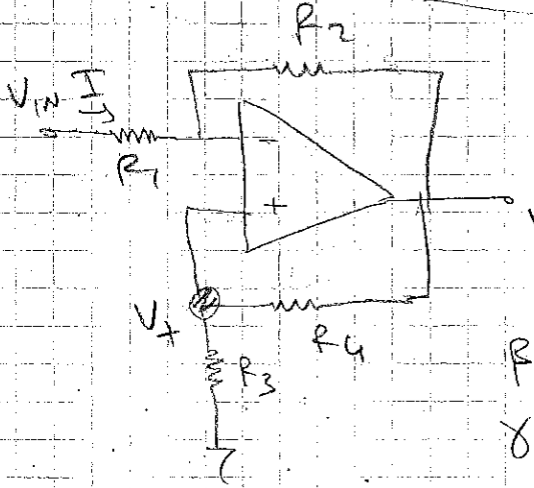
The propagation delay isn't zero!

$t=0$, $\underline{V_{IN}}$ 0.1 Volts → $\underline{V_{out}(0^+) = A.V_{IN}(0)}$

It amplifies until saturation



Even if there is a little noise.



$$V_+ = \underbrace{\frac{R_3}{R_3+R_4}}_{\beta} V_{out}$$

$$V_- = \frac{R_2}{R_1+R_2} V_{IN} + \underbrace{\frac{R_1}{R_1+R_2}}_{\gamma} V_{out}$$

β: Positive feedback coef.

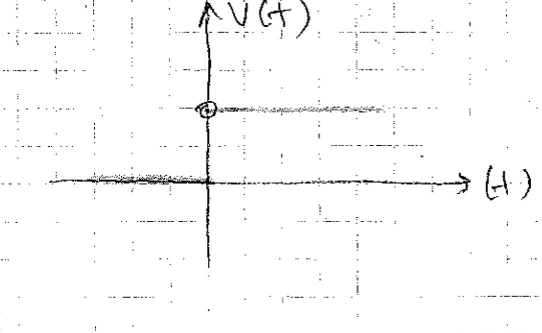
γ: Negative feedback coef.

Depending on the case $\beta > \gamma$, $\beta = \gamma$, $\beta < \gamma$

→ we have different V_{out} vs V_{IN} characteristic

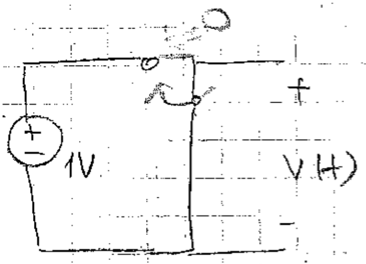
Waveforms:

① Unit Step Function



$$V(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$$

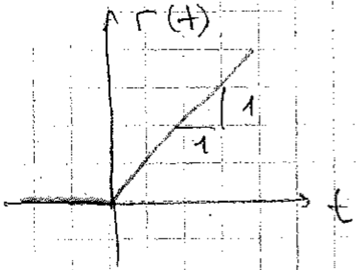
$V(0) \triangleq \frac{1}{2}$ (Not critical in EE201 but defined as $V(0)=\frac{1}{2}$ in EE301 for Fourier Series discussions)



$v(t) = u(t)$

Switch moves from A to B at t=0

(2) Ramp Function

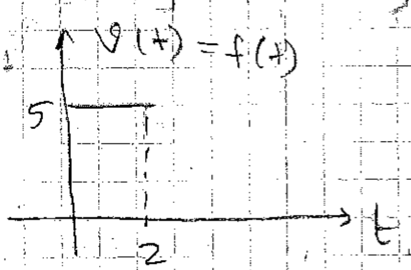


$$r(t) = \begin{cases} t & , t>0 \\ 0 & , t<0 \end{cases}$$

$$r(t) = t\,u(t)$$

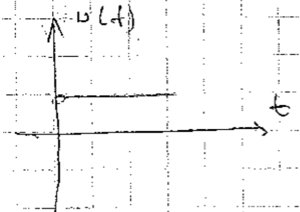
Combining elementary waveforms to synthesize other waveforms

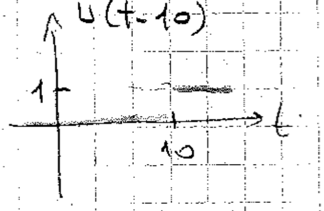
Ex 1

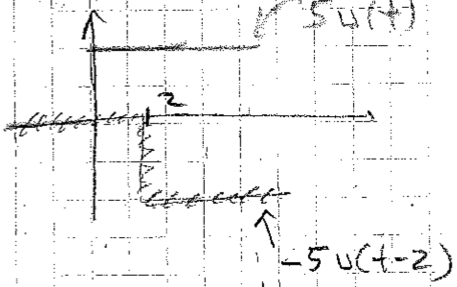


Express v(t) as a linear combination of u(t) and its shifted versions

$$f(t) \overset{?}{=} 10u(t) - 10u(t+1) - 20u(t-10)$$

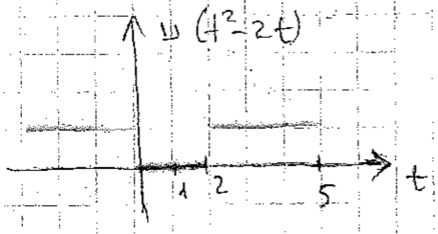






$$f(t) = 5u(t) - 5u(t-2) //$$

Ex 2: Sketch $u(t^2-2t)$ vs t.



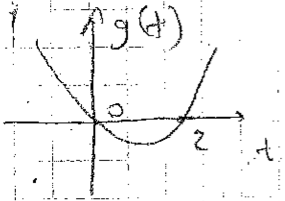
$$f\circ g(x) = f(g(x))$$

$$g(t) = t^2-2t = t(t-2)$$

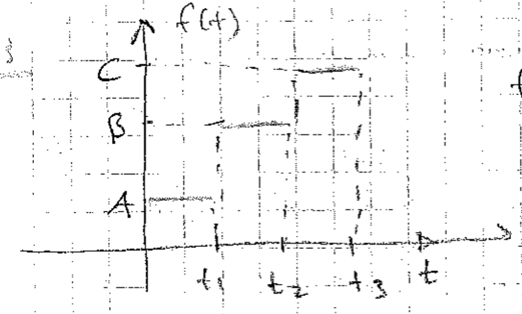
$$g(t) > 0 \text{ other ??}$$

$$g(t) = 0 \rightarrow t = \{0,2\}$$

$$g(t) < 0 \rightarrow t \in (0,2)$$



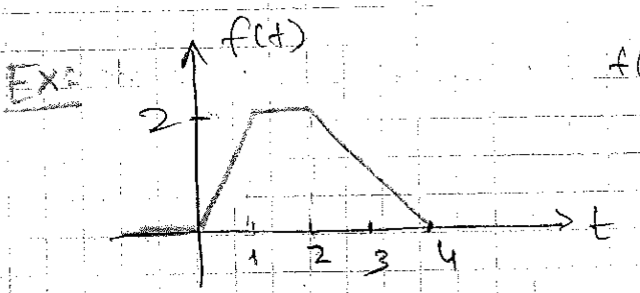
Ex 3

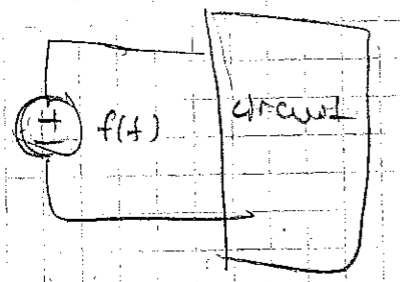


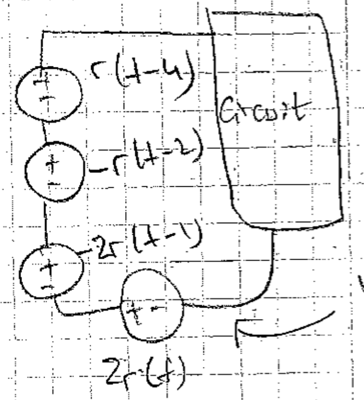
$$f(t) = A\,u(t) + (B-A)\,u(t-t_1) + (C-B)\,u(t-t_2) + (-C)\,u(t-t_3)$$

Ex:

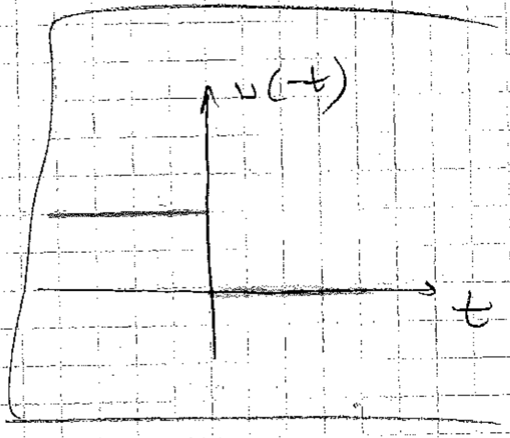
$f(t) = 2r(t) - 2r(t-1) - r(t-2) + r(t-4)$







we will use linearity later.

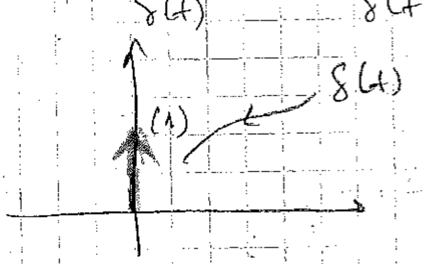


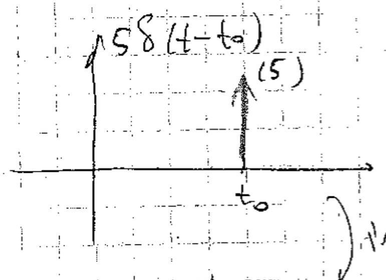
③ Impulse Function

(Generalized Function, not a regular calculus function)

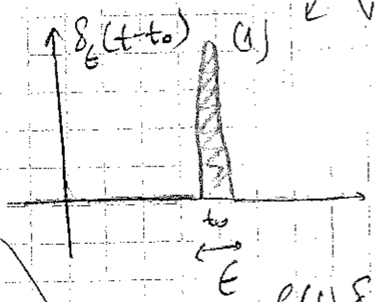
Definition: $$\int_{-\infty}^{\infty} f(t)\,\underbrace{\delta(t-t_0)}_{\text{impulse at } t=t_0}\,dt \triangleq f(t_0)$$ for all continuous f(t)

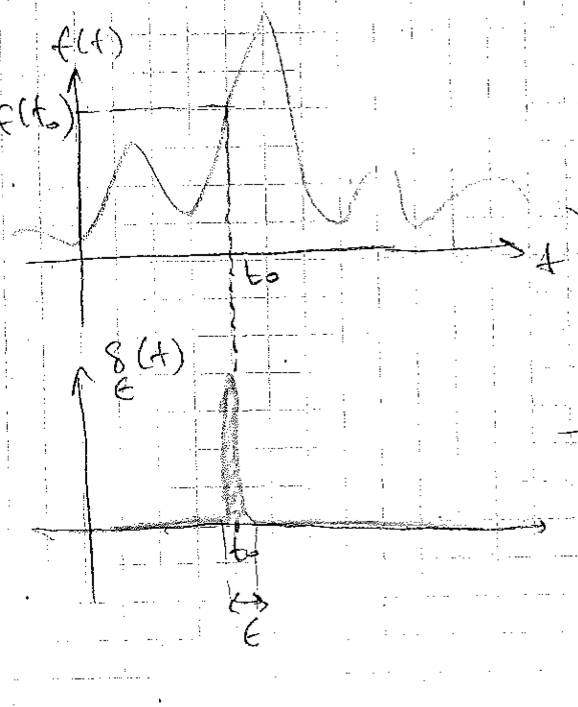
$\delta(t)$: Unit Weight Impulse Function

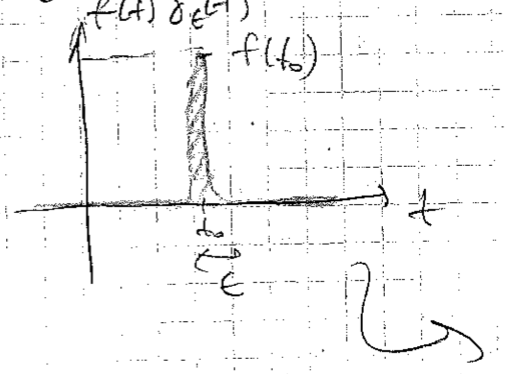




interpretation very narrow pulse at $t=t_0$







$$\rightarrow \int_{-\infty}^{\infty} f(t)\,\delta_\epsilon(t-t_0)\,dt \approx \int_{-\infty}^{\infty} f(t_0)\,\delta_\epsilon(t-t_0)\,dt = f(t_0)\cdot \underbrace{\int_{-\infty}^{\infty}\delta_\epsilon(t-t_0)\,dt}_{\text{area under } \delta_\epsilon(t)}$$

$$= f(t_0)\cdot 1 = f(t_0)$$

$$\int f(t)\,\delta_\epsilon(t-t_0)\,dt = f(t_0)$$

$$\hookrightarrow \lim_{\epsilon\to 0} \longrightarrow \lim_{\epsilon\to 0}\int f(t)\,\delta_\epsilon(t-t_0)\,dt = f(t_0)$$

$$\int f(t)\overset{\triangleq\,\delta(t)}{\boxed{\lim_{\epsilon\to 0}\delta_\epsilon(t-t_0)}}\,dt = f(t_0)$$

Note that impulse, $\delta(t)$ is not a regular calculus function.

$$\text{Since } \left.\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt\right|_{t_0=0} = \int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

Properties:

① $\int_{0^-}^{0^+}\delta(t)\,dt = 1$ ← Area under impulse (weight of impulse) is 1.

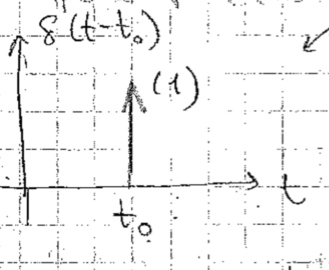
$0^- = 0 - \epsilon$ ← $\epsilon > 0$, but ϵ is arbitrary small $\epsilon \approx 10^{-12}$

$$\int_{-\infty}^{\infty} \underbrace{f(t)}_{1}\,\delta(t)\,dt = f(0)$$

$$\hookrightarrow \int_{-\infty}^{\infty}\delta(t)\,dt = 1$$

$$\hookrightarrow \int_{0^-}^{0^+}\delta(t)\,dt = 1$$

Impulse function (cont'd)



↙ It's weight is 1.

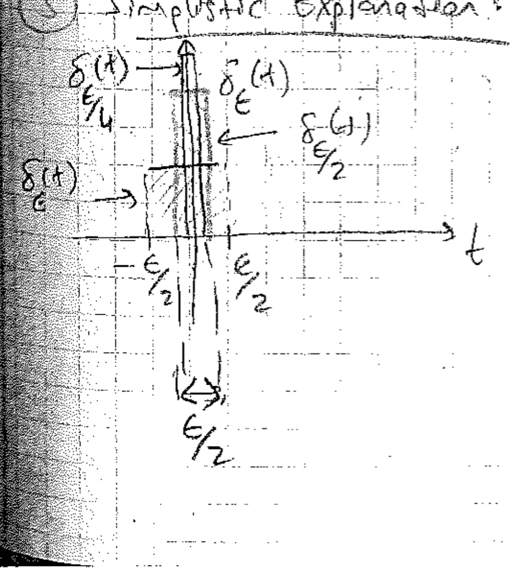
Properties

① $\delta(t) = 0 \Rightarrow t \neq 0$

$[\ \delta(t) = ?\quad t = t_0!\ ]$ ← more on this later...

② $\int_{-\infty}^{\infty}\delta(t)\,dt = \int_{0^-}^{0^+}\delta(t)\,dt = 1$

③ Simplistic Explanation:

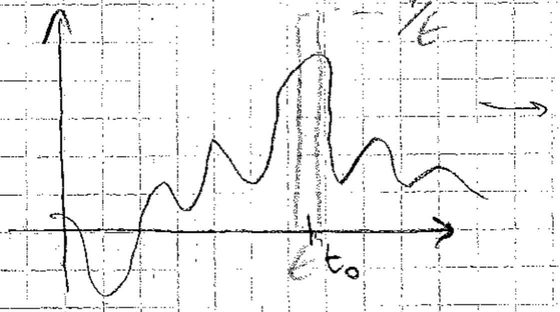


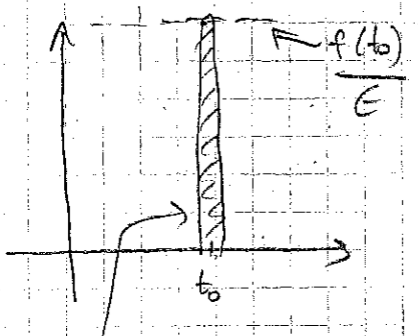
$$\delta_\epsilon(t) = \begin{cases} 1/\epsilon\ , & |t| < \epsilon/2 \\ 0\ , & \text{other} \end{cases}$$

$$\delta_\epsilon(t) \xrightarrow[\epsilon\to 0]{} \delta(t)$$

④ More Accurate Exploration:

$$\lim_{\epsilon \to 0}\left(\int_{-\infty}^{\infty} f(t)\,\delta_\epsilon(t-t_0)\,dt\right) = f(t_0)\ ;$$ f(t) is continuous at $t=t_0$





The area $\approx \frac{f(t_0)}{\epsilon}\cdot\epsilon$

Additional note:

$$\lim_{\epsilon \to 0} \underbrace{<\delta_\epsilon(t-t_0), f(t)>}_{\text{Inner product}} = f(t_0) \longrightarrow \delta_\epsilon(t) \to \delta(t) \longrightarrow ?, ?$$

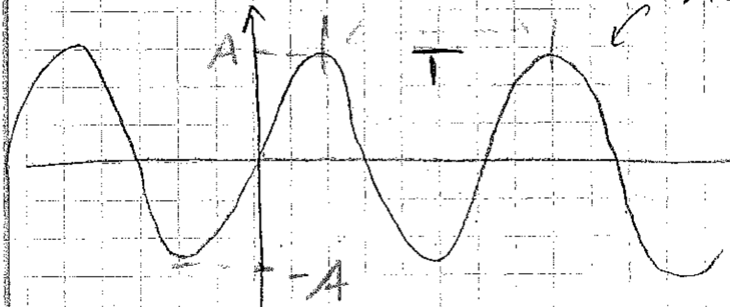
in the sense weak convergence.

Finally, $f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0)$ ← Don't forget this one

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} f(t_0)\,\delta(t-t_0)\,dt = f(t_0)$$

④ Sinosoid Function

$$A\cos(\overbrace{\omega t+\theta}^{\text{Argument}})$$



A: amplitude

ω: radial frequency (rad)

↑ omega (small case)

Capital Omega : Ω

Cosine function is a mapping from real numbers to real numbers. So argument of cosine is a real number.

$\cos(\pi) = -1$ $\quad \cos(180^\circ) = -1$ ← don't forget degree sign !!

$\pi \approx 3.14159$

$\omega = 2\pi f$

ω: rad/sec

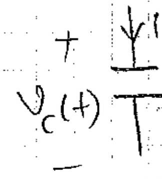
f: freq: Hz $\left(\frac{1}{sec}\right)$

$T \overset{(sec)}{=} \frac{1}{f}$

θ: phase of the sinosoid function

Capacitor

LTI capacitor $Q = C \cdot V$ Inverses of each other



$$i_c(t) = C \frac{d v_c(t)}{dt}$$

$$v_c(t) = v_c(0^-) + \frac{1}{C} \int_{0^-}^{t} i_c(\tau) d\tau, \quad t > 0^-$$

$$\frac{d v_c(t)}{dt} = \frac{1}{C} i_c(t)$$

$$\frac{d}{dt} \int_a^t f(\tau) d\tau = \frac{d}{dt} \left\{ F(t) - F(a) \right\}$$

$$\left(f(t) = t^2, F(t) = \frac{t^3}{3} \right)$$

Liebnitz's rule for differentiation under integral sign

General Capacitor Relation

$$Q_{cap}(t) = C(t) v(t)$$

total charge of capacitor

$$\frac{d}{dt}$$

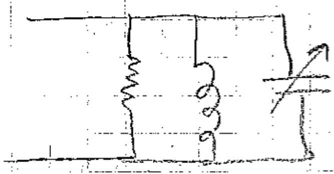
$$i_c(t) = \frac{d}{dt}\left(c(t) v(t) \right)$$

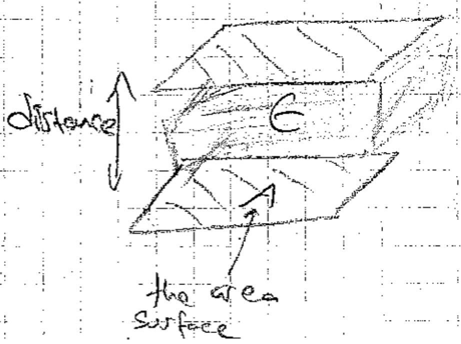
$$= \left(\frac{d}{dt} c(t) \right) v(t) + c(t) \frac{d v(t)}{dt}$$

$$Q_{cap}(t) = C(t) \cdot v_{cap}^3(t)$$

$$\frac{d}{dt} \rightarrow i_c(t) = \frac{d}{dt}\left\{ C(t) \cdot v_{cap}^3(t) \right\}$$

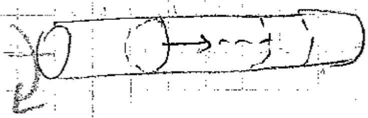
Analog radio:



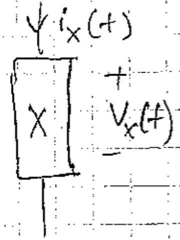


$$C = \epsilon \frac{A}{d}$$

Farads



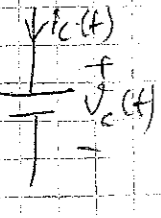
Power Calculation



$P_x(t) = V_x(t) \cdot i_x(t)$

absorbed power

Let's apply the power calculation on capaciter:



$P_{cap}(t) = i_c(t)\, v_c(t)$

instanteneous power

$= \boxed{\frac{C\, dv_c(t)}{dt} \cdot v_c(t)}$

$$\Delta E_{t_1 \to t_2} = \int_{t_1}^{t_2} P_x(t)\, dt$$

Energy change in between t_1 and t_2

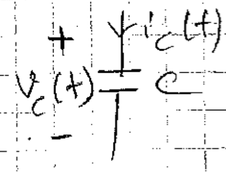
$$\Delta E_{t_1 \to t_2} = \int_{t_1}^{t_2} P_{cap}(t)\, dt$$

$v_c'(t) = \frac{d v_c(t)}{dt}$

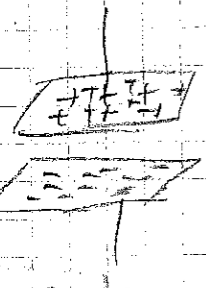
$$= \int_{t_1}^{t_2} C\, v_c(t)\, \underbrace{\dot{v}_c(t)\, dt}_{d v_c}$$

$$= C \int_{v_1 = v_c(t_1)}^{v_2 = v_c(t_2)} v\, dv = \frac{1}{2} C v^2 \Big|_{v_1 = v_c(t_1)}^{v_2 = v_c(t_2)} = \boxed{\underbrace{\frac{1}{2} C v_c^2(t_2)}_{\text{Stored energy of the cap at } t=t_2} - \underbrace{\frac{1}{2} C v_c^2(t_1)}_{\text{Stored energy at } t=t_1}}$$

Initial Condition Models for Capacitor



$v_c(0^-) = V_0$

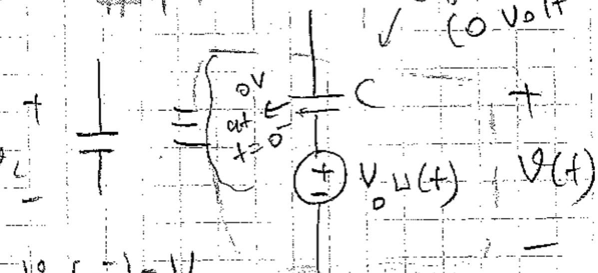


$$v_c(t) = v_c(0^-) + \frac{1}{C} \int_{0^-}^{t} i_c(\tau)\, d\tau$$

$$\boxed{v_c(t) = V_0 + \frac{1}{C} \int_{0^-}^{t} i_c(\tau)\, d\tau \;, t > 0} \quad (I)$$

Initial Condition Model #1:

empty capacitor (0 Volt at $t=0^-$) ✓



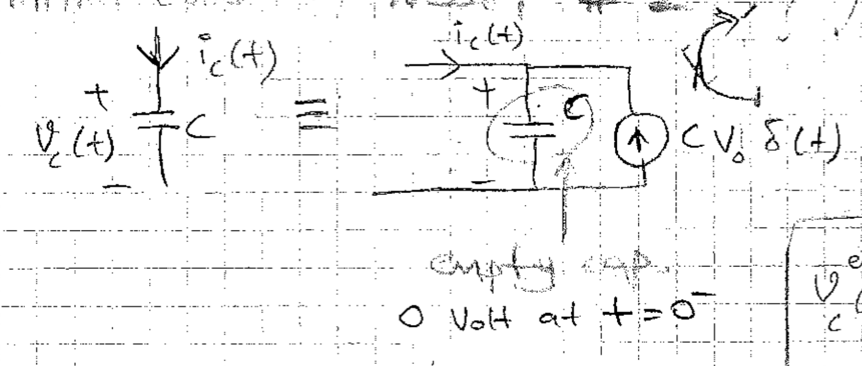
$v_c(0^-) = V_0$

$$v(t) = \underbrace{\left(0 + \frac{1}{C} \int_{0^-}^{t} i_c(\tau)\, d\tau\right)}_{V_{cap}^{empty}(t)} + V_0 u(t)$$

$$\boxed{v(t) = \frac{1}{C} \int_{0^-}^{t} i_c(\tau)\, d\tau + V_0 \,, t > 0} \quad (II)$$

actually the same

Initial Condition Model #2 ?!



$$i_c^{\text{empty}}(t) = i_c(t) + C V_0\,\delta(t)$$

$$V_c^{\text{empty}}(t) = 0 + \frac{1}{C}\int_{0^-}^{t} i_c^{\text{empty}}(\tau)\,d\tau\,,$$

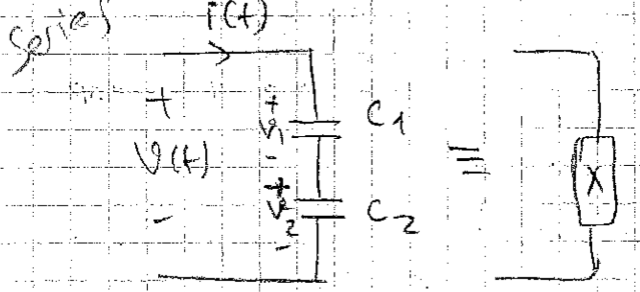
$t > 0$

$$\boxed{v_c^{\text{empty}}(t) = \frac{1}{\cancel{C}}\cdot\cancel{C}\cdot V_0 \underbrace{\int_{0^-}^{t}\delta(\tau)\,d\tau}_{1} + \frac{1}{C}\int_{0}^{t} i_c(\tau)\,d\tau}$$

(III)

Series and Parallel Combination of Capacitors

Series



$$v(t) = v_1 + v_2$$

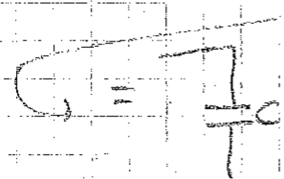
$$= \frac{1}{C_1}\int_{0^-}^{t} i(\tau)\,d\tau + \frac{1}{C_2}\int_{0^-}^{t} i(\tau)\,d\tau$$

$$\boxed{v(t) = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int_{0^-}^{t} i(\tau)\,d\tau}$$

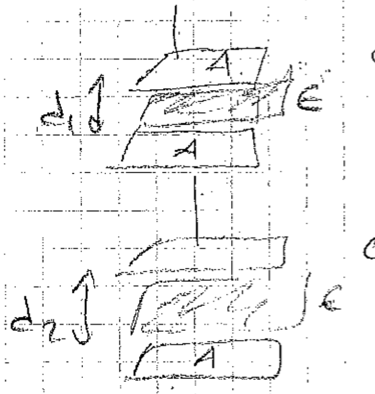
Assume C_1 and C_2 are uncharged (empty).

i-v relation of an empty capacitor with

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$



$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$$



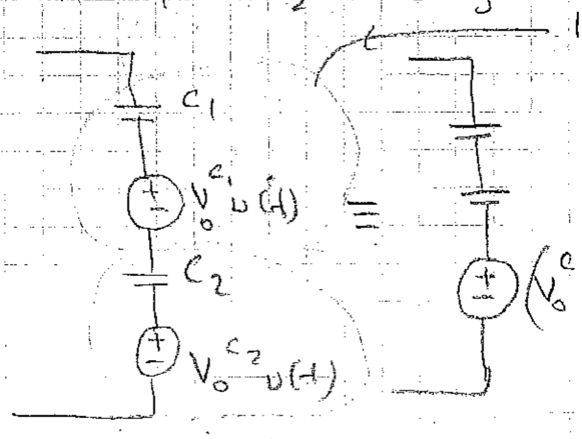
$$C_1 = \epsilon\frac{A}{d_1}$$

$$C_{eq} = \frac{\epsilon A}{d_1 + d_2} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$$

$$C_2 = \epsilon\frac{A}{d_2}$$

$$\frac{\epsilon A}{\epsilon A\left(\frac{1}{C_1} + \frac{1}{C_2}\right)} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$$

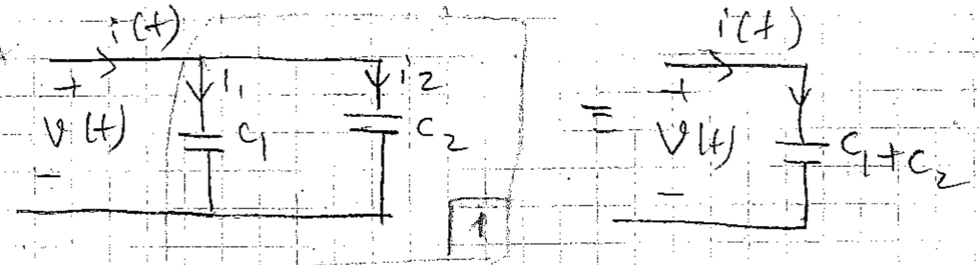
If C_1 and C_2 are charged:



So, if caps are charged, the equivalent comp. is a charged cap with initial voltage $= V_{C_1}(0^-) + V_{C_2}(0^-)$ and

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$$

Parallel



Assume C_1 and C_2 are empty

For $\boxed{1}$: $i(t) = i_1(t) + i_2(t)$

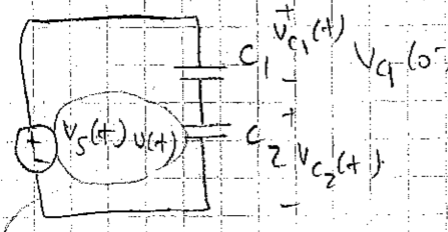
$$= C_1 \frac{d}{dt} v_{C_1}(t) + C_2 \frac{d}{dt} v_{C_2}(t)$$

$$i(t) = (C_1 + C_2) \frac{d v_C(t)}{dt}$$

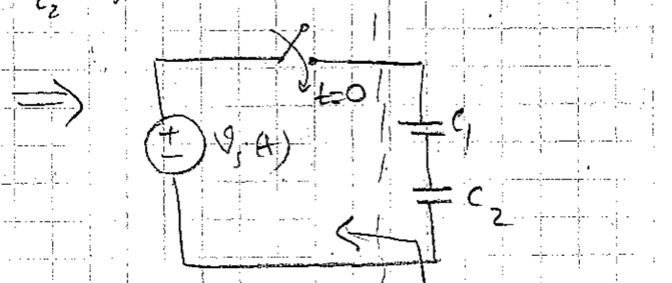
$$\boxed{C_{eq} = C_1 + C_2}$$

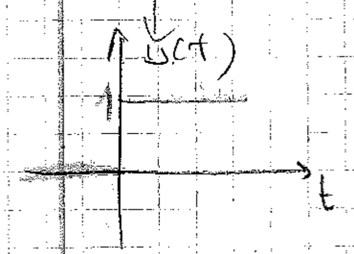
A terminal equation for $C_1 + C_2$ F cap.

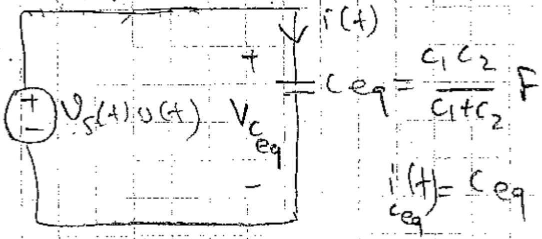
Voltage Division for Capacitors



$V_{c_1}(0^-) = V_{c_2}(0^-) = 0$ V







$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$ F

$V_{C_1}(0^-) = V_{C_2}(0^-) = 0$

$$i_{C_{eq}}(t) = C_{eq} \frac{d}{dt} v_{C_{eq}}(t)$$

$$= C_{eq}\left(\frac{d}{dt} v_s(t)u(t)\right)$$

From original circuit

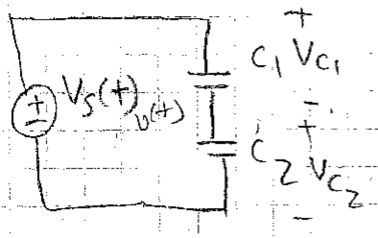
$$v_{c_1}(t) = \cancel{v_{c_1}(0^-)}^{0} + \frac{1}{C_2}\int_{0^-}^{t} i(\tau)\,d\tau$$

$$= \frac{1}{C_1} \cdot C_{eq}\, v_s(\tau)\,u(\tau) \Big|_{\tau=0^-}^{\tau=t}$$

$$= \frac{C_2}{C_1+C_2}\left[v_s(t)u(t) - v_s(0^-)\,\underbrace{\cancel{u(0^-)}}_{0}\right] = \frac{C_2}{C_1+C_2} v_s(t)u(t)$$

$$v_{c_2}(t) = v_{c_2}(0^-) + \frac{1}{C_2}\int_{0^-}^{t} i(\tau)\,d\tau = \frac{C_1}{C_1+C_2} v_s(t)\,u(t)$$

Note:

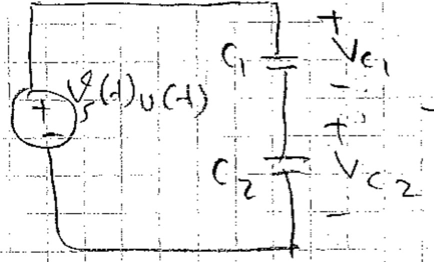


$$V_{C_1} = \frac{1/C_1}{1/C_1 + 1/C_2} V_S(t)$$

$$V_{C_2} = \frac{1/C_2}{1/C_1 + \frac{1}{C_2}} V_S(t)$$

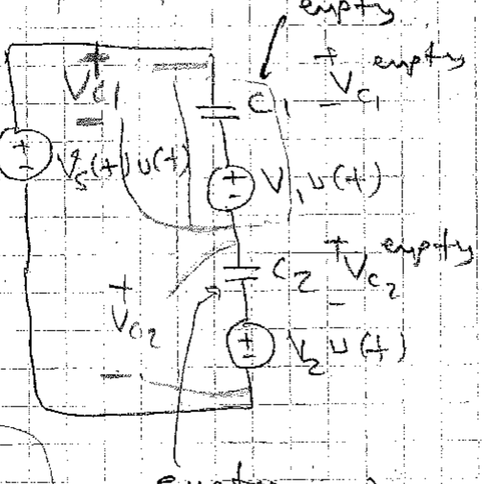
Vol. Div. for capacitors has proportionality constant of $\frac{1}{C_1}$ or $\frac{1}{C_2}$

Voltage Division for charged capacitors



$V_{C1}(0^-) = V_1$

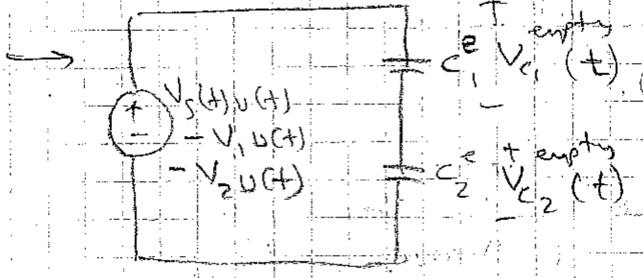
$V_{C_2}(0^-) = V_2$



$V_{C1}^{empty}(0^-) = 0$

$V_{C2}^{empty}(0^-) = 0$

empty (not the same with initial capacitor)

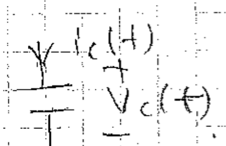


$$V_{C_1}^{empty} = \frac{C_2}{C_1+C_2}(V_S(t) - V_1 - V_2)u(t)$$

$$V_{C_1}(t) = \left[V_1 + \frac{C_2}{C_1+C_2}(V_S(t) - V_1 - V_2)\right]u(t)$$

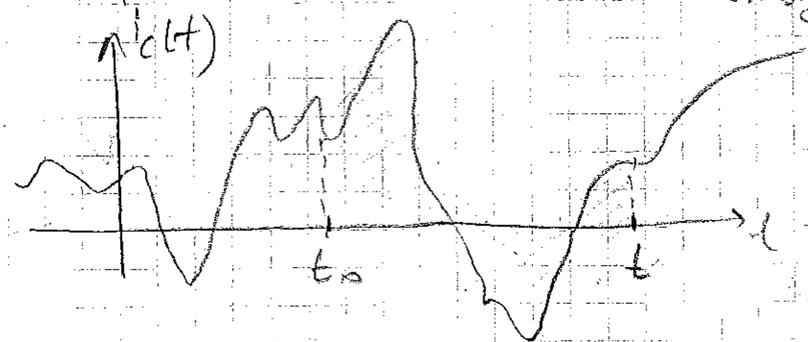
$$= \left[\frac{C_2}{C_1+C_2}(V_S - V_2) + \frac{C_1}{C_1+C_2}V_1\right]u(t)$$

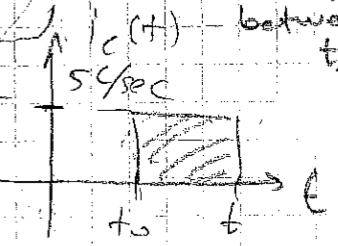
Application of external input on capacitors and calculation of $t=0^-$ and $t=0^+$ changes in the cap circuit variables



$$V_c(t) = V_c(t_0) + \underbrace{\frac{1}{C}\int_{t_0}^{t} i_c(\tau)\,d\tau}_{\text{Voltage change}}, \quad t > t_0$$

total # of charge transferred to the capacitor between t_0 and t





$$\frac{dq(t)}{dt} = i(t)$$

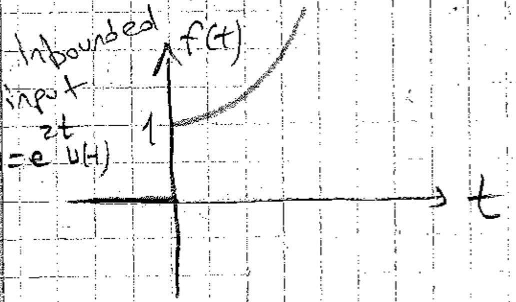
Question: Is $V_c(t)$ a continuous function of t?

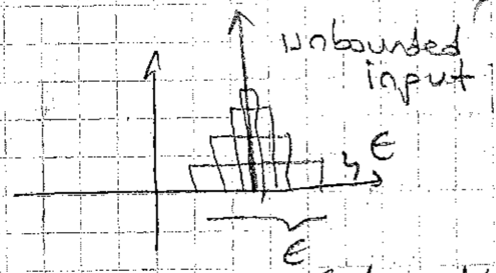
A: $V_c(t_0^+) = V_c(t_0^-) + \frac{1}{C}\int_{t_0^-}^{t_0^+} i_c(\tau)\,d\tau$

$(\exists M \in \mathbb{R})$

$V_c(t_0+) = V_c(t_0-)$ provided that there is no impulse.

So, for bounded inputs (all inputs such that $|r(t)| < M \quad \exists M$) that is for all practical inputs, we have the continuity of capacitor voltage.

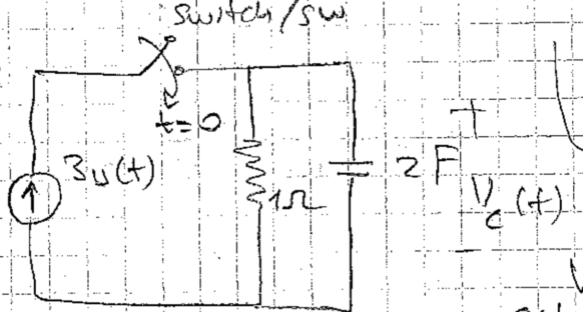




$$\delta_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & |t-t_0| < \frac{\epsilon}{2} \\ 0, & \text{otherwise} \end{cases}$$

and $\delta(t) \xrightarrow[\epsilon\to 0]{} \delta(t)$ ↑ unbounded input

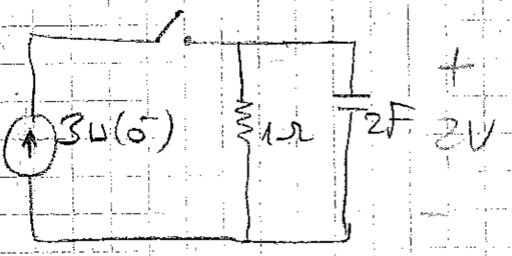
Ex:

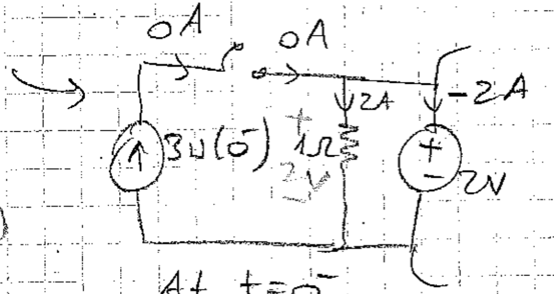


$V_c(0^-)=2V$ and sw is closed at $t=0$.
Find $V_c(0^+)$ and $\dot{V}_c(0^+)$.

↑ or $= V_c'(0^+) = \frac{dV_c(0^+)}{dt}$

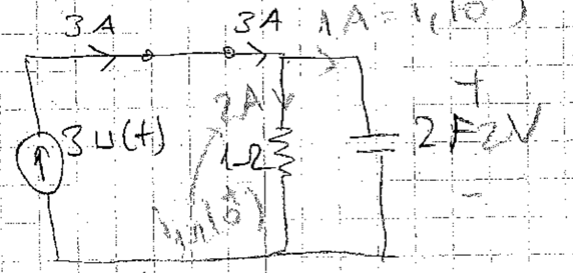
at $t=0^-$



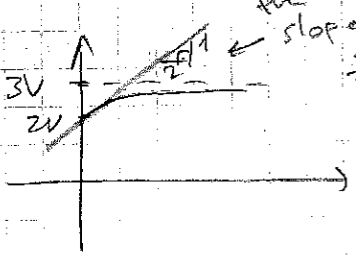


At $t=0^-$

At $t=0^+$:



Since input $3u(t)$ is bounded, $V_c(0^+) = V_c(0^-) = 2V$



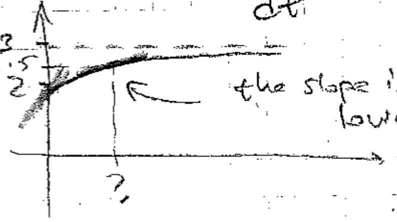
← the slope is $\frac{1}{2}$ at $t=0^+$

$i_c(0^+) = 1A \rightarrow C\cdot\dot{V}_c(0^+) = 1A$

$C\frac{d}{dt}V_c(0^+)$

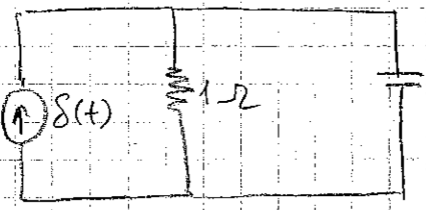
$\dot{V}_c(0^+) = \frac{1}{2}$ $\frac{\text{Volts}}{\text{sec}}$

2F

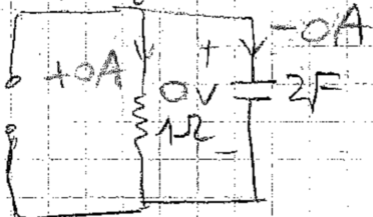


the slope is getting lower and lower

Ex:



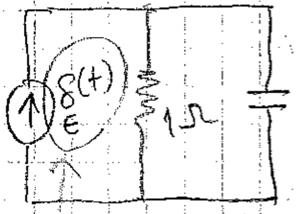
At $t=0^-$

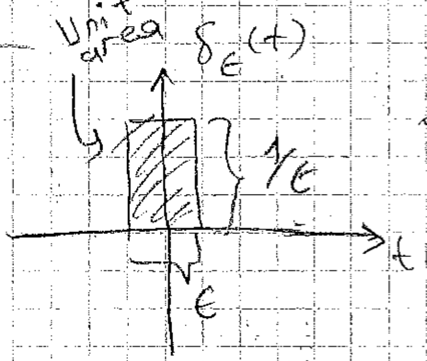


As in previous example

$V_c(0^-)=0\,V$

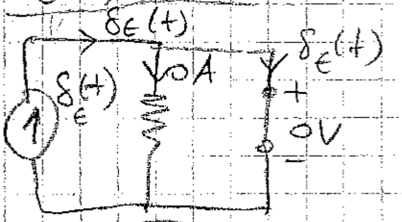
$0^- < t < 0^+$





Bounded function approximating impulse δ(t)

During the application of $\delta_\epsilon(t)$, $V_c(t)=V_c(0^-)=0V$



$i_c(t)=\delta_\epsilon(t)$, $0^-<t<0^+$

$$V_c(0^+)=\underbrace{V_c(0^-)}_{0V}+\frac{1}{C}\int_{0^-}^{0^+} i_c(\tau)\,d\tau$$

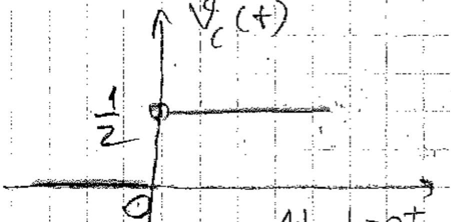
$$=\frac{1}{2}\underbrace{\int_{0^-}^{0^+}\delta_\epsilon(\tau)\,d\tau}_{1}=\frac{1}{2}\text{ Volts}$$

$V_c(0^+)$

independent of ε

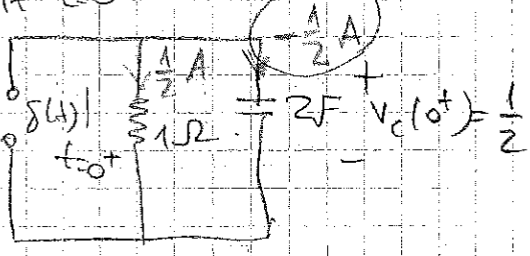
So, $v_c(0^+)=\frac{1}{2}$ Volts and

as $\delta_\epsilon(t)\xrightarrow[\epsilon\to 0]{}\delta(t)$, $\lim_{\epsilon\to 0} v_c(0^+)=\frac{1}{2}V$

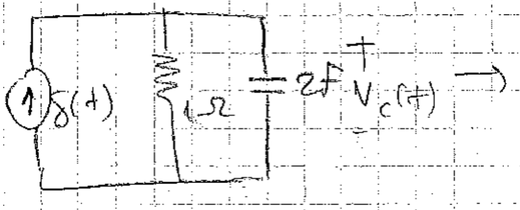


$C\dot{v}_c(0^+)=-\frac{1}{2}$

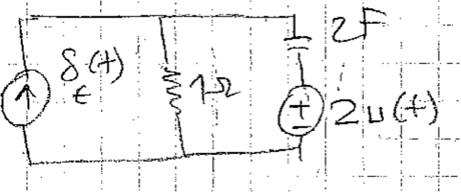
At $t=0^+$



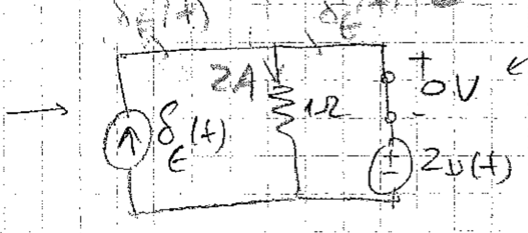
Ex:



$0^- < t < 0^+$



$V_c(0^-)=2V$



empty cap, remains at 0 Volts during $0^-<t<0^+$

??

$$\boxed{i_c(t)=\delta_\epsilon(t)-2,\ 0^-<t<0^+}$$

At $t=0^+$: $V_{C_{empty}}(0^+) = V_C(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_C(\tau)\,d\tau$

$$= 0 + \frac{1}{2}\Big[\underbrace{\int_{0^-}^{0^+}\delta_\epsilon(\tau)\,d\tau}_{1}\Big] + \frac{1}{2}\underbrace{\int_{0^-}^{0^+}(-2)\,d\tau}_{0}$$

$V_C^{empty}(0^+) = \frac{1}{2}$ Volts

$V_C(0^+) = V_C^{empty}(0^+) + 2u(t)$

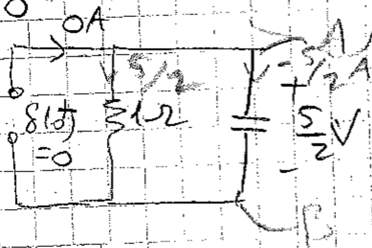
$= \frac{1}{2} + 2$

$= \frac{5}{2}$ Volts

$\int_{0^-}^{0^+} f(t)\,dt \leq 10 \cancel{\int_{0^-}^{0^+} 1\,dt} \rightarrow 0$

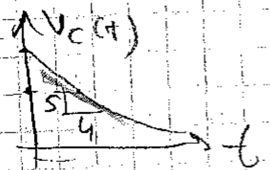
$|f(t)| < 10$

At $t=0^+$

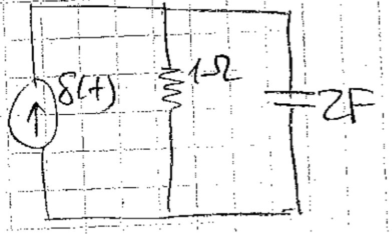


$\dot{v}_C(0^+) = \frac{-5}{4}$ $\frac{\text{Volt}}{\text{sec}}$?

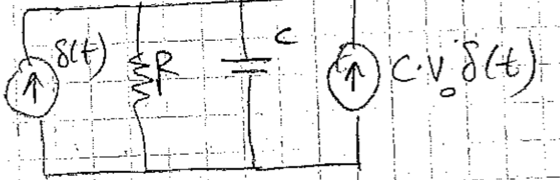
$q = vC$



Ex!

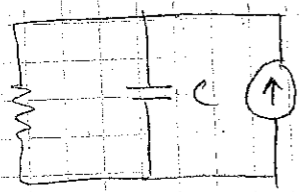


→ Let's use the following I.C. model for capa. :

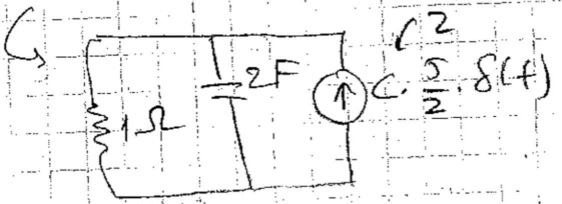


$V_C(0^-) = 2V$

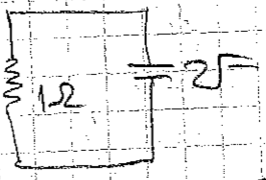
$0^- < t < 0^+$



$\overset{2}{C} \cdot \underset{2}{V_0}\,\delta(t) + \delta(t) = 5\,\delta(t)$



⟶

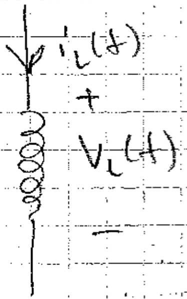


$V_C(0^+) = \frac{5}{2}$ Volts

Cuma 17:30 EE201

Ek Ders var Aynı yerde

Inductors:



Inductance

$$\phi(t) = L(t) \cdot i(t)$$

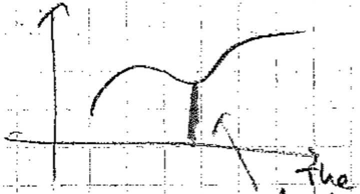
Flux Current (inductor current)

Faraday's Law

Review

$\to t = t_{ini} + \epsilon$

$$v_C(t) = v_C(t_{ini}) + \frac{1}{C}\int_{t_{ini}}^{t} i_C(\tau)d\tau$$



The Area of this line is zero (unless there is impulse)

For bounded currents, $V_C(t)$ is continuous.

(or $i_C(t)$)

$\frac{d}{dt}$

$$v_L(t) = \frac{d}{dt}\{L(t)I_L(t)\}$$

$$\boxed{v_L(t) = \dot{L}(t)I(t) + L(t)\dot{I}(t)}$$

$t > 0$ ← not for $t=0$

(derivative) $= \frac{dL(t)}{dt}$

LTI Inductor,

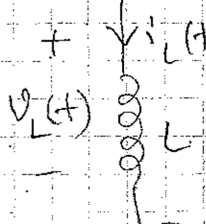
$L(t) = L$ ← constant

$\phi(t) = L\, i_L(t)$

$\frac{d}{dt} \to \boxed{v_L(t) = L\frac{di_L(t)}{dt}}$

$$\begin{tikzcd} \phi \arrow[d, leftrightarrow] & = L \arrow[d, leftrightarrow] & \cdot\, i \arrow[d, leftrightarrow] \\ Q & = C & \cdot\, V \end{tikzcd}$$

duals of each other.



$V_L(t) = L\frac{di_L(t)}{dt}$ ← differential form

$i_L(t) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} V_L(\tau)d\tau$ ← integral form

By duality, we can immediately say that $i_L(t)$ is a continuous function provided that $v_L(t)$ does not contain impulse function or $V_L(t)$ is bounded.

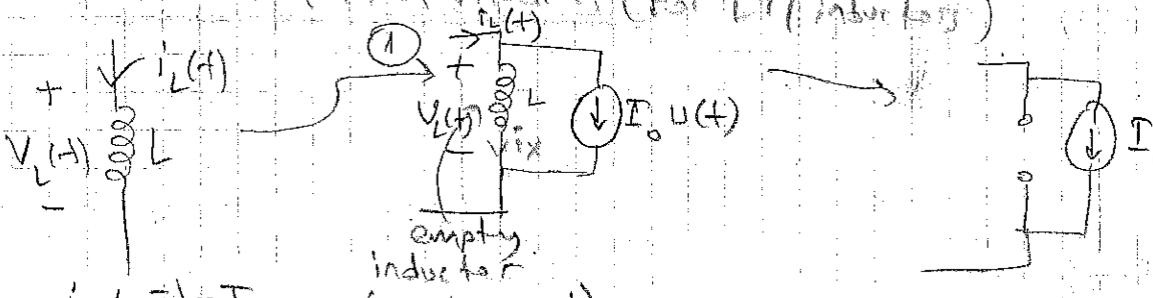
To remember the continuity of variables you may recall that

$E_{cap}(t) = \frac{1}{2}C\, v_C^2(t)$

$E_L(t) = \frac{1}{2}L\, I_L^2(t)$

← Energy function of the dynamic component is continuos if there is no impulse!

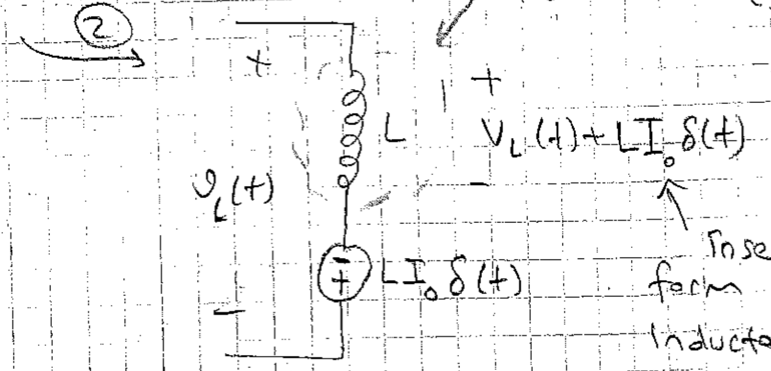
Initial Condition Models (For LTI inductors)



$i_L(0^-) = I_0$

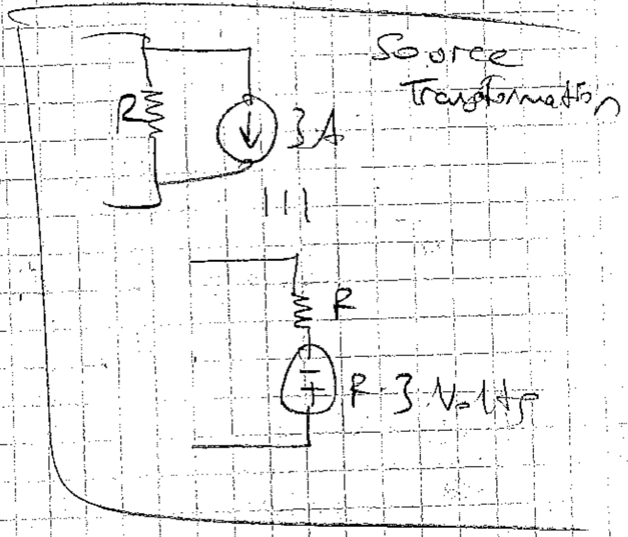
empty inductor (uncharged)

empty Inductor (uncharged)



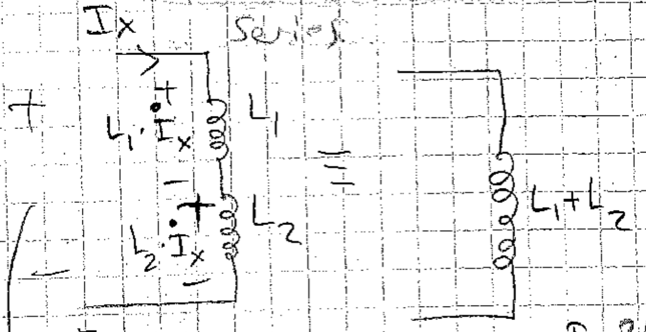
Insert this into integral form ($i_L(t)$ form) of Inductor relation

Source Transformation



Series and Parallel Combination of Inductors

Series

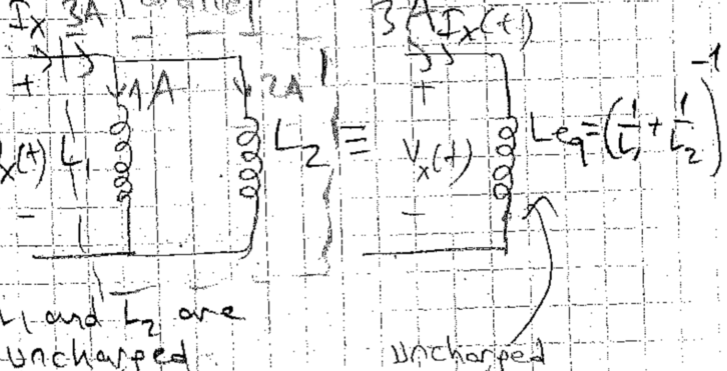


Inductors are uncharged

$(L_1+L_2)\,\dot{I}_x(t)$

?

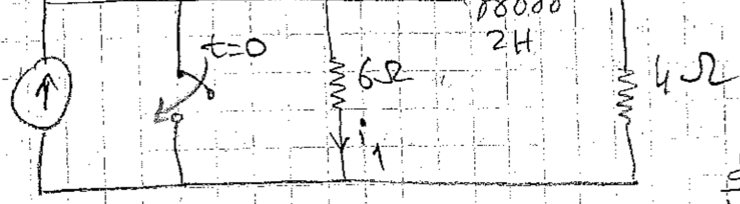
Parallel



L_1 and L_2 are uncharged

uncharged inductor

Ex1

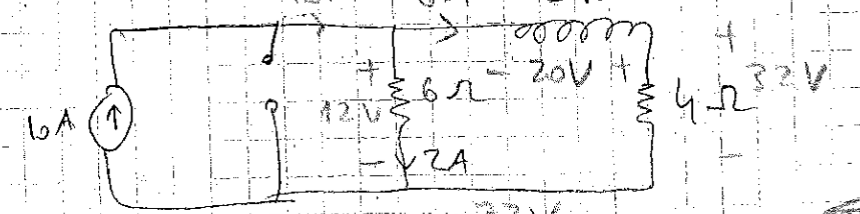


$i_1(0^-)=2A$

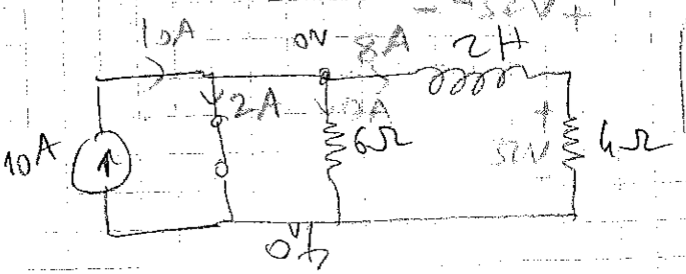
Find $i_L(0^+), i_L(0^+)$,

$\frac{d}{dt}i_L(t)\Big|_{t=0^+}$

At $t=0^-$



$i_L(0^-)=8A$



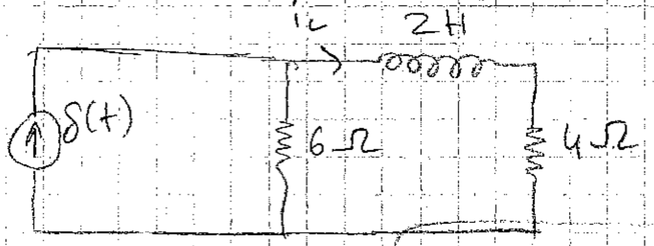
$I_L(0^+)=I_L(0^-)=8A$

(there are no impulses)

$I_1(0^+)=0A$

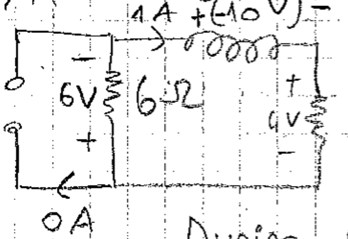
$\underset{-32}{v_L(0^+)} = \underset{2H}{L}\dot{I}_L(0^+) \longrightarrow \boxed{\frac{d}{dt}\, i_L(t)\Big|_{t=0^+} = -16\ A/sec}$

EX1

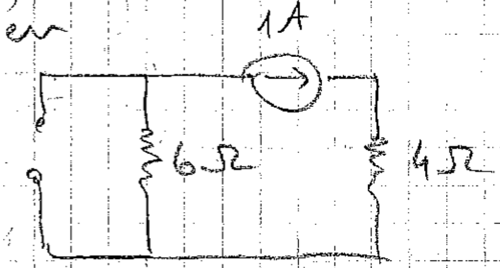


$i_L(0^-) = 1A$

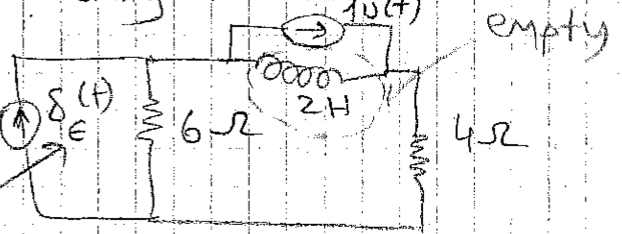
At $t = 0^-$



substitution theorem



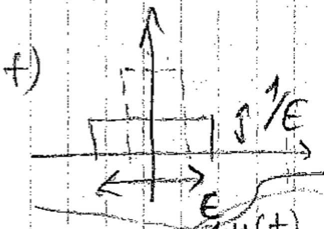
During $0^- < t < 0^+$

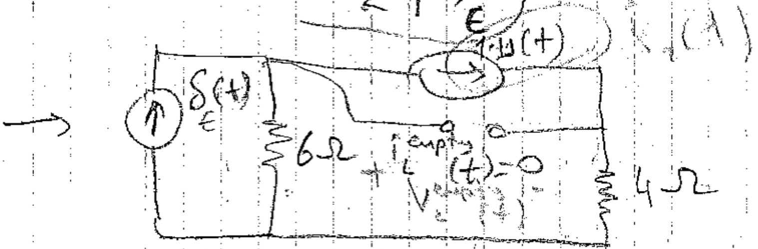


Approximation to impulse (not impulse itself)

Since input $\delta_\epsilon(t)$ is bounded (not an impulse)

$i_L^{empty}(t) = 0, \quad 0^- < t < 0^+$





$V_L^{empty}(t) = 6\delta_\epsilon(t) + \overset{?}{\ldots}\, 1u(t)$

??? $0^- < t < 0^+$

At $t = 0^+$

$$i_L^{empty}(0^+) = i_L^{empty}(0^-) + \frac{1}{L}\int_{0^-}^{0^+} V_L^{empty}(\tau)\,d\tau$$

$$= 0 + \frac{1}{2}\int_{0^-}^{0^+}\Big(6\,\delta_\epsilon(\tau) + \overset{?}{\ldots}\,\underbrace{1u(\tau)}_{i_{s2}(\tau)}\Big)d\tau$$

$$= 0 + \frac{1}{2}\,6\underbrace{\int_{0^-}^{0^+}\delta_\epsilon(\tau)\,d\tau}_{1} + \frac{1}{2}\int_{0^-}^{0^+}\overset{?}{\ldots}\,1(\tau)\,d\tau$$

$$\boxed{i_L^{empty}(0^+) = 3A}$$

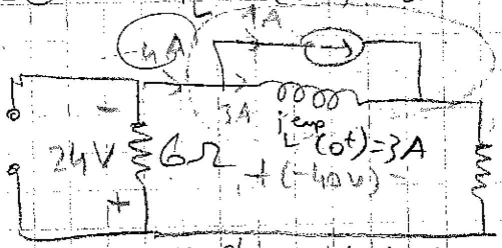
P. Dirac!

$\delta(t)$

↑

Dirac Delta function

At $t=0^+$ $\quad i_L^{emp}(0^+)=3A$



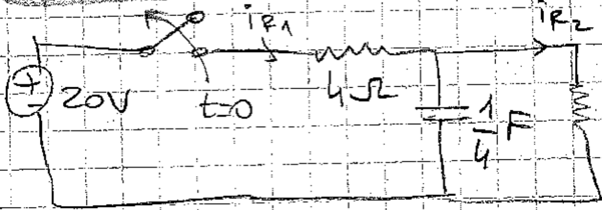
?? of original charged inductor

Charged inductor current:

$$i_L(0^+)=\underbrace{i_L(0^-)}_{1A}+3A=4A$$

Inductors and capacitors are all passive elements. They are energy storage elements.

Ex1



$i_{R_2}(0^-)=2A$

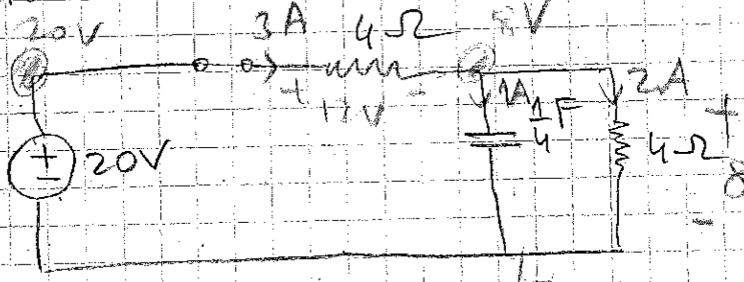
Find $Q(0^-)$, $Q(0^+)$

$i_R(0^-)$, $i_{R_1}(0^+)$

$i_c(0^-)$, $i_c(0^+)$

$V_c(0^-)$, $V_c(0^+)$

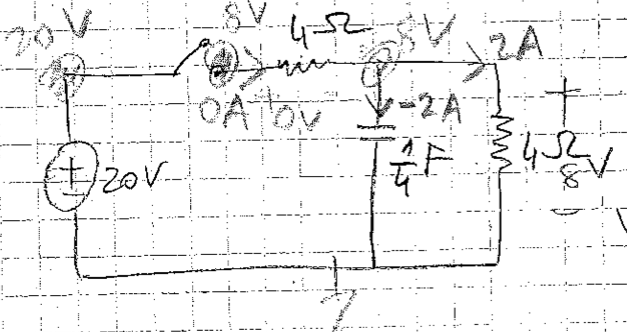
At $t=0^-$



$Q(0^-)=CV_c(0^-)=\frac{1}{4}8=2C$ $\quad i_{R_1}(0^-)=3A \quad i_{R_1}(0^+)=0A$

$Q(0^+)=2C$

$i_c(0^-)=1A \quad i_c(0^+)=-2A$

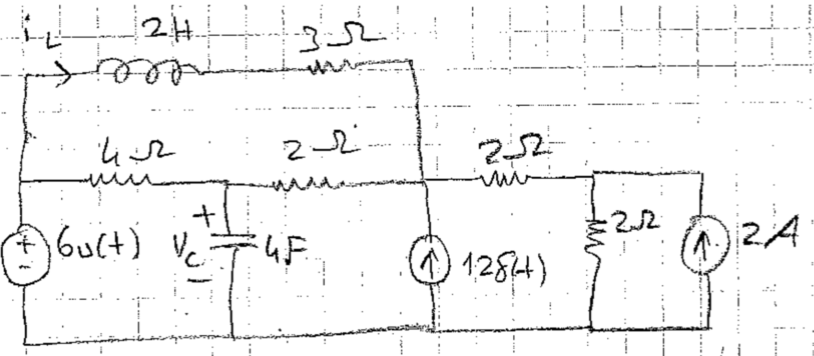


$V_c(0^-)=8V \quad V_c(0^+)=8V$

$V_c(0^+)=V_c(0^-)=8V$

$$C\dot{v}_c(0^+)=i_c(0^+)=-2A \longrightarrow \dot{v}_c(0^+)=\frac{-2}{\frac{1}{4}}=\frac{-8\text{ Volts}}{\text{sec}}$$

Ex.



$V_c(0^-) = 10V$

$I_L(0^-) = 5A$

Find $V_c(0^+), I_L(0^+)$

If we do not have any impulsive sources, then $V_c(t)$ and $I_L(t)$ should be continuous func. of time. (provided that there's no pathological switching).

$$V_c(0^+) = V_c(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_c(\tau)\,d\tau$$

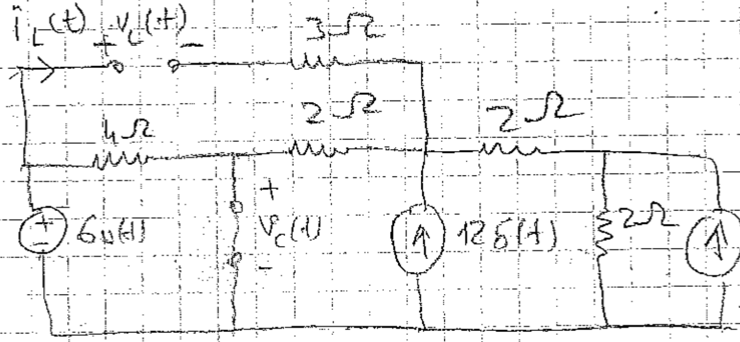
$$i_L(0^+) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{0^+} V_L(\tau)\,d\tau$$

Question is then finding the jump in $V_c(t)$ and $I_L(t)$ at $t=0$ due to the impulsive source.

During the application of impulse (Cap: short circuit
Ind: open circuit)

$0^- < t < 0^+$

Note! This circuit has no I.C. associated, the goal is to find the voltage/current jump.



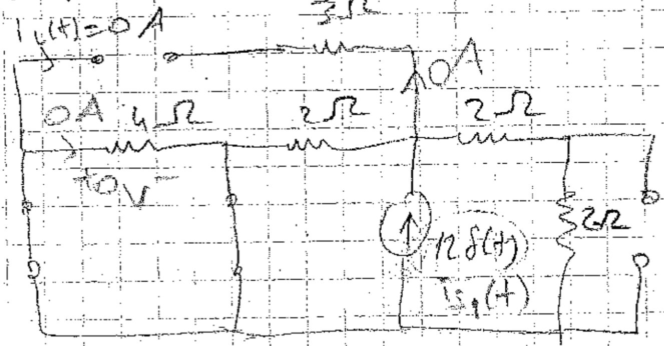
$i_c(t) = ?$

$V_L(t) = ?$

$0^- < t < 0^+$

$i_c(t) = \underline{4/6 ?}\ i_{s_1}(t) + \underline{?}\ i_{s_2}(t) + \underline{?}\ i_{s_3}(t)$

by superposition



$i_c(t) = 8\delta(t) + \underline{A}\ 2 + \underline{B}\ 6u(t)$

$$V_c(0^+) = V_c(0^-) + \frac{1}{C}\int_{0}^{0^+} i_c(\tau)\,d\tau$$

$$= 10 + \frac{1}{4}\int_{0^-}^{0^+}[8\delta(\tau) + \cancel{A\cdot 2 + B\cdot 6u(\tau)}]\,d\tau$$

$$= 10 + \frac{1}{4}\int_{0^-}^{0^+} \underbrace{8(\tau)d\tau}_{1}$$

$= 12$ Volts

$I_L(0^+) = ?$

$$I_L(0^+) = I_L(0^-) + \frac{1}{L}\int_{0^-}^{0^+} V_L(\tau)\,d\tau$$

$V_L(t) = \underline{\quad}\ i_{s_1}(t) + \underline{C}\ i_{s_2}(t) + \underline{D}\ i_{s_3}(t)$

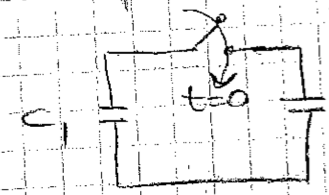
$$v_L(t) = -2i_C(t) = \underbrace{-16\delta(t)}_{\text{important component}} + \underbrace{\cdots i_{s_2}(t) + \cdots i_{s_3}(t)}_{\text{not important (they will be integrated from } 0^- \text{ to } 0^+)}$$

$$I_L(0^+) = 5 + \frac{1}{2}\int_{0^-}^{0^+} (-16\delta(\tau) + \cdots + \cdots)\,d\tau$$

$$= -3A$$

On Pathological Switching

Capacitor Case



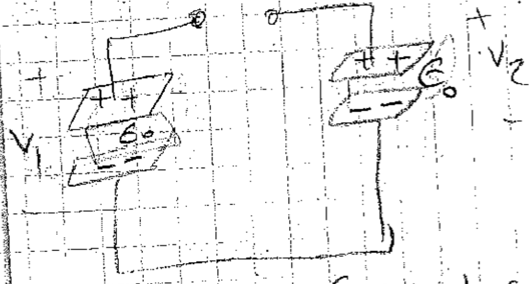
$v_{C_1}(0^-) = V_1$

$v_{C_2}(0^-) = V_2$

} Find $v_{C_1}(0^+)$ and $v_{C_2}(0^+)$

At $t=0^-$

$v_{C_1}(0^-) =$

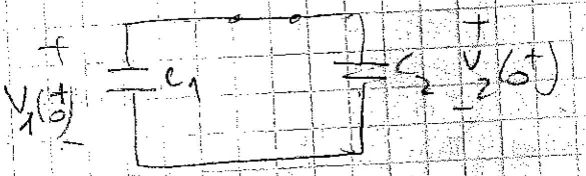


$q_{total}(0^-) = C_1V_1(0^-) + C_2V_2(0^-)$

$= C_1V_1 + C_2V_2$ (I)

total charge

At $t=0^+$



$V_1(0^+) = V_2(0^+) = V_{com}$

= common voltage

Claim: $Q^{total}_{before} = Q^{total}_{after}$

$Q_{total}(0^+) = C_1V_1(0^+) + C_2V_2(0^+)$

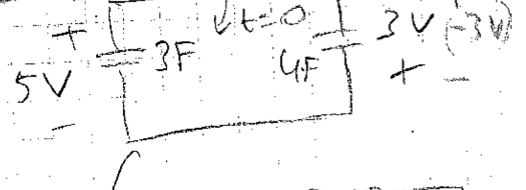
$= (C_1+C_2)V_{com}$ (II)

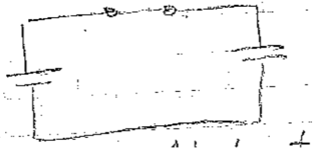
(I) = (II) by claim $C_1V_1 + C_2V_2 = (C_1+C_2)V_{com}$

$$V_{com} = \frac{C_1}{C_1+C_2}V_1 + \frac{C_2}{C_1+C_2}V_2$$

$V_{C_1}(0^+) = V_{C_2}(0^+)$

Ex:





$q_{before} = 5\cdot3 + 4(-3)$

$= 3\ C$

$q_{after} = 3C$

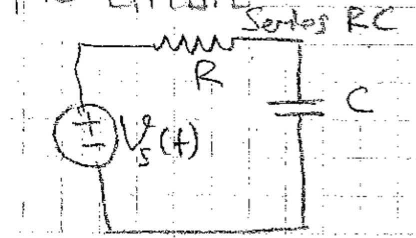
$V_{com} = \frac{q_{after}}{C_1+C_2}$

$= \boxed{\frac{3}{7}}$

2 souces of jumps in current / voltage → inpulsive source
→ pathological switch

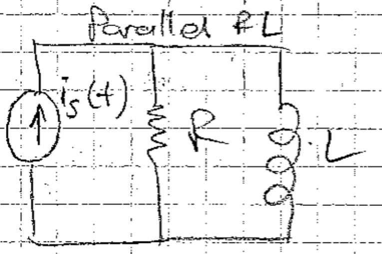
First order Circuits

RC circuit



$V_c(0^-) = V_0$ Find $V_c(t)$ $t \geq 0$

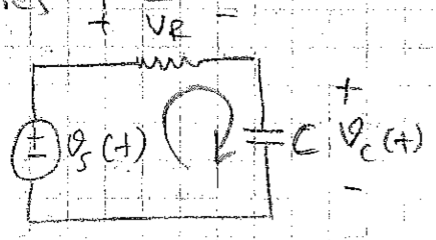
Dual



$I_L(0^-) = I_0$

Find $I_L(t)$, $t \geq 0$

Series RC



$v_c(0^-) = V_0$

~~KVL~~ $-v_s(t) + v_R(t) + v_c(t) = 0$, $t \geq 0$

$v_R = \boxed{i_R} R \to i_c(t)$

$i_c(t) = C\frac{dv_c(t)}{dt} = C\dot{v}_c(t) \to (= Cv_c'(t))$

$-v_s(t) + R \cdot \underbrace{i_c(t)}_{C\dot{v}_c(t)} + v_c(t) = 0$

$$\boxed{D \triangleq \frac{d}{dt}}$$

$RC\dot{v}_c(t) + v_c(t) = v_s(t)$

$\frac{1}{RC}$

$$\boxed{\begin{gathered}\left(D + \frac{1}{RC}\right)v_c(t) = \frac{1}{RC}v_s(t), \quad t \geq 0 \\ v_c(0^-) = V_0\end{gathered}}$$

→ 1^{st} Order Constant Coefficient Differential Equation (CCDE or ODE)

↑ Ordinary (Not partial)

Solutions:

Differential Equation

Homogeneous

$\left(D + \frac{1}{RC}\right)v_c^H(t) = 0$

Particular

$\left(D + \frac{1}{RC}\right)v_c^P(t) = \boxed{\frac{1}{RC}v_s(t)}$ → f(t): forcing term

$\boxed{RHS = f(t)}$ It is forcing the circuit to behave in a certain way.

Case 1: $v_s(t) = u(t)$

$\left(D + \frac{1}{RC}\right)v_c(t) = \frac{1}{RC}\boxed{v_s(t)} \leftarrow u(t)$

(I) $\left(D + \frac{1}{RC}\right)v_c^H(t) = 0$

$\left(D + \frac{1}{RC}\right)v_c^P(t) = \frac{1}{RC}u(t)$, $t \geq 0$ (II)

$= \frac{1}{RC}$

?

Guess: $V_c^H(t)=Ae^{\lambda t}$

?

Insert the "guess" into (I)

$$A(\lambda+\frac{1}{RC})e^{\lambda t}=0,\ t>0$$

$$\hookrightarrow A(\lambda+\frac{1}{RC})=0$$

$A=0 \to v_c^H(t)=Ae^{\lambda t}=0$ ↳ trivial solution

$\lambda=-\frac{1}{RC} \to$ non-trivial solution

$$\to \boxed{v_c^H(t)=Ae^{-\frac{1}{RC}t},\ t\geq 0}$$

(for homogeneous solution)

$$(D+\frac{1}{RC})v_c^P(t)=\frac{1}{RC},\ t\geq 0$$

Guess: $V_c^P(t)=K$ $\xrightarrow{\text{insert}}$ $\frac{K}{RC}=\frac{1}{RC},\ t\geq 0$

$$\downarrow$$

$$K=1$$

$$\downarrow$$

$$v_c^P(t)=1$$

$v_c^H+v_c^P$

$$(D+\frac{1}{RC})v_c(t)=\frac{1}{RC}v_s(t)$$

$$v^{complete}(t)=V_c^H(t)+v_c^P(t)$$
$$=(Ae^{-1/RC\,t}+1),\ t\geq 0$$

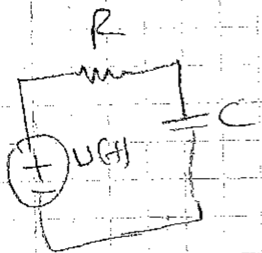
$V_c(0^-)=V_0 \longrightarrow V_c(0^+)=V_0$

Initial condition

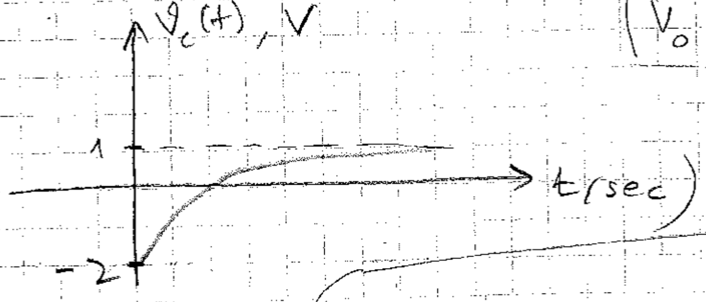
$$V_c^{comp}(t)\Big|_{t=0^+}=A+1=v_c(0^-)=v_c(0^+)=V_0 \longrightarrow A=V_0-1$$

$$\boxed{v_c^{comp}(t)=(V_0-1)e^{-t/RC}+1,\ t\geq 0}$$

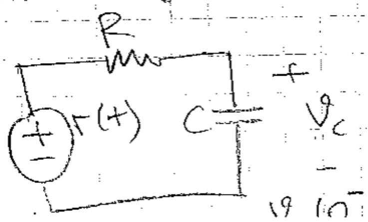
$\boxed{V_0=-2}$



$V_c(0^-)=V_0$



Case (2) $v_s(t)=r(t)$

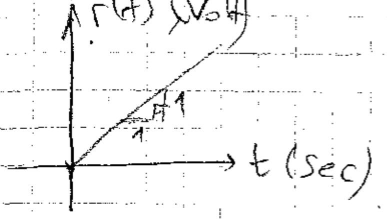


$v_c(0^-)=V_0$

t·u(t)

$$(D+\frac{1}{RC})v_c(t)=\frac{1}{RC}v_s(t)=\frac{1}{RC}\cdot t$$

$$v_c(0^-)=V_0$$



$$\left(D+\frac{1}{RC}\right)V_c(t)=\frac{1}{RC}\cdot t \quad ,\ t>0$$

$$V_c(0^-)=V_0$$

homogeneous

$$\boxed{V_c^H(t)=A\cdot e^{-1/RC\,t}}\ (I)$$

particular

$$\left(D+\frac{1}{RC}\right)V_c^P(t)=\frac{1}{RC}\cdot t\ (II)$$

Guess: $V_c^P(t)=\underset{?}{K}+\underset{?}{L}t$

insert into (II)

$$\frac{K}{RC}+\frac{Lt}{RC}+L=\frac{t}{RC}+0$$

$$L=1$$

$$\frac{K}{RC}+L=0$$

$$\frac{K}{RC}=-L=-1$$

$$K=-RC$$

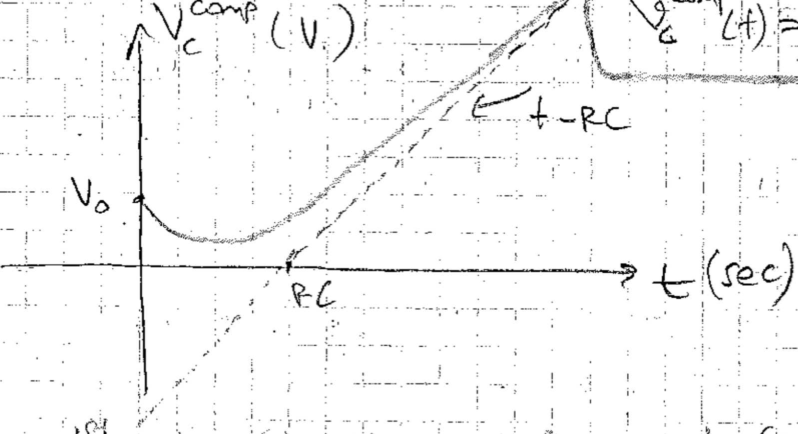
$$V_c^P(t)=-RC+t$$

$$\boxed{V_c^{comp}(t)=A\,e^{-1/RC\,t}-RC+t,\quad t>0}$$

Select A to meet I.C at $t=0^+$

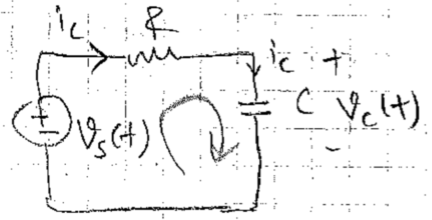
$$V_c(0^-)=V_c(0^+)=V_0\rightarrow A=RC+V_0$$

$$\boxed{V_c^{comp}(t)=(RC+V_0)\,e^{-\frac{1}{RC}t}-RC+t\ ,\ t>0}$$



Sol. of Dif. Eqn. for ramp input, meeting the I.C. at $t=0^+$

1st Order Circuits with DC input (recall)



$$i_c=C\dot{V}_c(t)\ \left(=C\frac{dV_c(t)}{dt}\right)$$

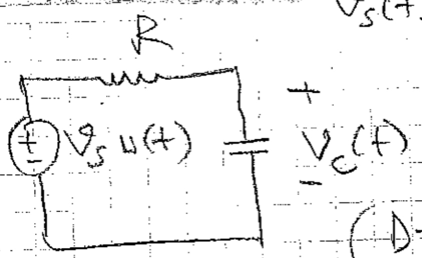
KVL: $-V_s(t)+R\,\underset{C\dot{V}_c}{i_c}+V_c=0$

$$\boxed{\left(D+\frac{1}{RC}\right)V_c(t)=\frac{V_s(t)}{RC}\qquad D=\frac{d}{dt}\qquad V_c(0^-)=V_0}$$

Soln: $V_c(t)=\underbrace{V_c^h(t)}_{\text{homogeneous}}+\underbrace{V_c^P(t)}_{\text{particular}}$

General form of the solution $\boxed{V_c(t)=A\,e^{\lambda t}+V_c^P(t)}$

Case of DC input $v_s(t) = V_s u(t)$



$V_s u(t)$

Homogeneous

$$\boxed{\left(D+\frac{1}{RC}\right)} v_c^h(t) = 0 \quad (1) \quad t>0$$

Particular

$$\left(D+\frac{1}{RC}\right) v_c^p(t) = \boxed{\frac{v_s(t)}{RC}, \; t>0} \quad (2)$$

$$= \frac{V_s}{RC}$$

Guess $v_c^h(t) = A\cdot e^{\lambda t}$ → insert into (1)

$$\rightarrow A\left(\lambda + \frac{1}{RC}\right) e^{\lambda t} = 0, \; t>0$$

$$A\left(\lambda + \frac{1}{RC}\right) = 0$$

$A = 0$ → $v_c^h(t) = 0$ trivial soln.

$A \in \mathbb{R}$, $\lambda = -\frac{1}{RC}$

$$\boxed{v_c^h(t) = A\cdot e^{-\frac{1}{RC}t}}$$

non-trivial soln.

Guess! $\boxed{v_c^p(t) = V_s}$ → satisfies (2)

$$\boxed{\begin{aligned} v_c^{\text{complete}}(t) &= v_c^h + v_c^p \\ &= A\cdot e^{-\frac{1}{RC}t} + V_s, \; t>0 \end{aligned}}$$

$$v_c^{\text{comp}}(0^+) = v_c^{\text{comp}}(0^-) = V_0 \rightarrow \left(\text{There is no impulsive current at } t=0\right)$$

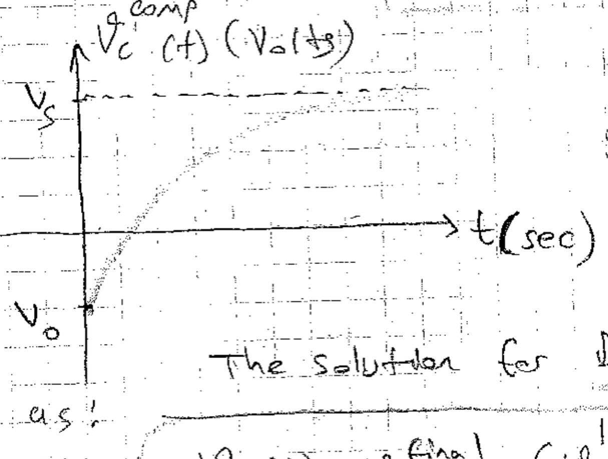
↑ Continuity of Capacitor Voltage

$$v_c^{\text{comp}}(0^+) = A + V_s \rightarrow A = V_0 - V_s$$

$$v_c(0^-) = V_0$$

$$\boxed{v_c^{\text{comp}}(t) = V_s + (V_0 - V_s)e^{-t/RC}, \; t \geq 0}$$

→ $t \geq 0$ is also true since the capacitor voltage is continuous.



The solution for DC input can also be expressed as:

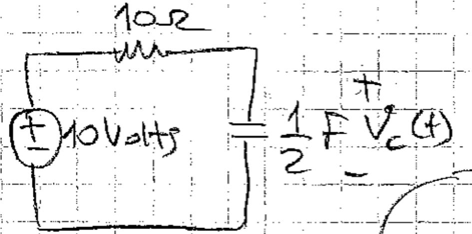
$$\boxed{v_c(t) = v_c^{\text{final}} + \left(v_c^{\text{initial}} - v_c^{\text{final}}\right) e^{-(t - t_{\text{initial}})/\tau}, \quad t > t_{\text{initial}}}$$

↑ General formula for first order circuits with DC-input

In our example,

$$\left.\begin{aligned} t_{initial} &= 0 \\ v_c^{initial} &= v_c(0^-) = V_0 \\ v_c^{final} &= v_c(\infty) = V_S \\ \tau &= RC \end{aligned}\right\}$$

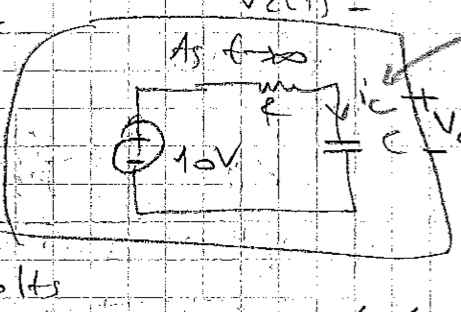
Ex:



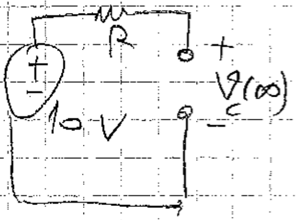
$v_c(\pi^-) = 3$ Volts. Find $v_c(t)$, $t \geq \pi$

$v_c(t) =$

$= C\dot{v}_c(t) = 0$ A



$V_c(t) = X \rightarrow$



$v_c(\pi^-) = 3$ Volts

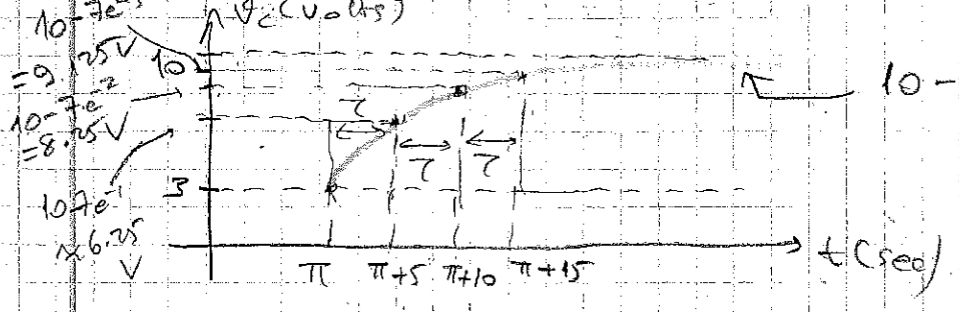
$v_c(\infty) = 10$ V

$$v_c(t) = 10 - 7e^{-(t-\pi)/\tau}, \quad t > \pi$$

Tau = Time constant

$\tau = R \cdot C = 5$ sec

Time constant of 1st order circuits describe how fast the circuit behaves or "reaches" the steady-state value.



$$10 - 7e^{-(t-\pi)/5}, \quad t \geq \pi$$

$\tau = 5$ sec

Bad approximation

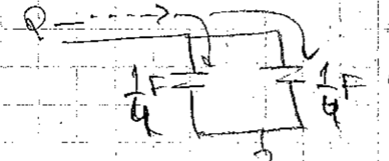
$e \approx 2$

In practice, we take 3-5 time constants (3τ-5τ) as the sufficient time to "reach" the final cap. state.

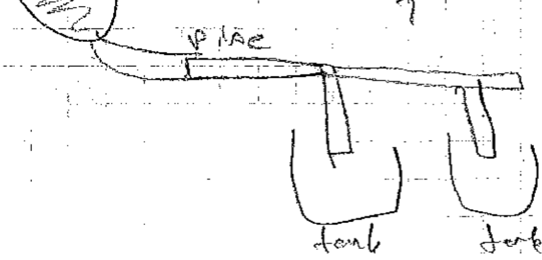
For this example, the time to reach steady state.

For $3\tau = 15$ sec

$5\tau = 25$ sec



Takes longer than only 1 $\frac{1}{4}$F capacitor



→ Takes longer than only 1 tank.

Google

>> 1kΩ × 10μF

10 msec

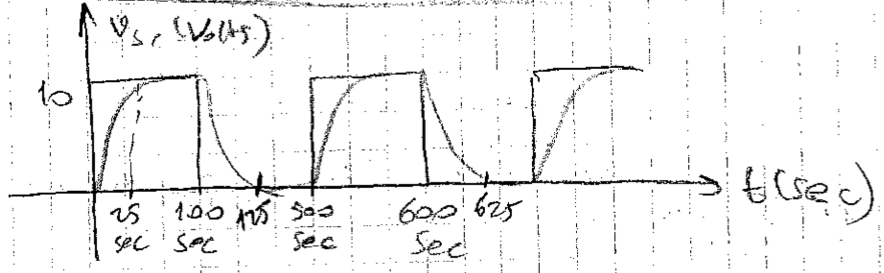
>> 10T in US dollar

31 dollars...

$Q = C \cdot V$

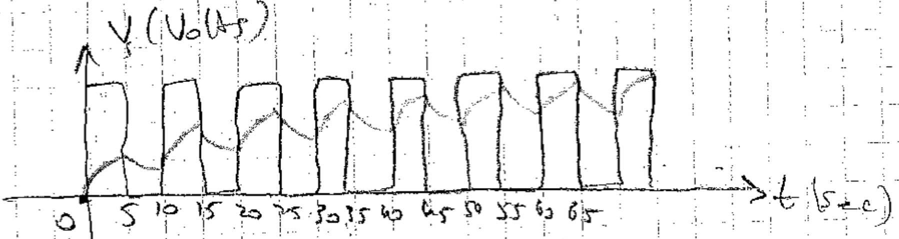
$\tau = R C \rightarrow \frac{\text{Charge}}{\text{Voltage}}$ $\quad \frac{\cancel{\text{Volt}}}{\frac{\cancel{Q}}{t}} = \frac{\cancel{Q}}{\cancel{\text{Volt}}} = t(\text{sec})$

$R \downarrow \frac{\text{Volt}}{\text{Current}}$



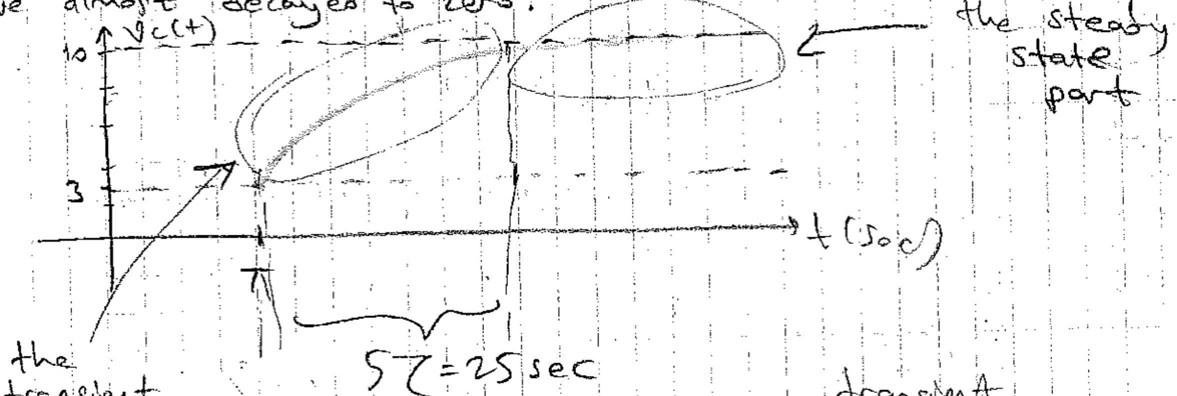
$5\tau = 25$ sec

$V_c(0^-) = 0V$



Transient and Steady-State Parts of the Solution

Roughly the steady-state solution is the solution for which the components in the solution due to the initial conditions have almost decayed to zero.

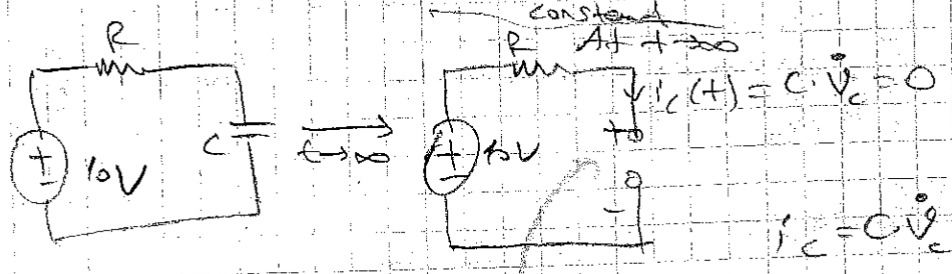


$(\tau) = 5$ sec

(the part which is the segment of the solution that is dominated by the initial conditions)

transient = in transit

Summary: Under DC input, the final value of the capacitor is found by setting $V_c(t) = K$, and this leads to:



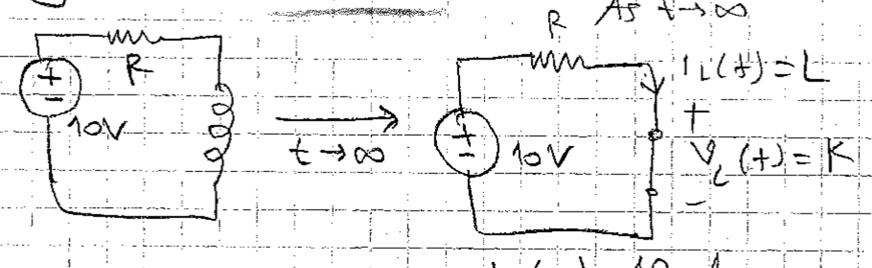
$i_c(t) = C \cdot \dot{V}_c = 0$

$i_c = C \dot{V}_c = C \frac{d}{dt}\{K\} = 0$

$V_c(\infty) = 10V$

i.e. cap. is open circuited for DC input as $t \to \infty$

Similarly for an inductor:



$v_L(t) = L \frac{di_L(t)}{dt}$

constant

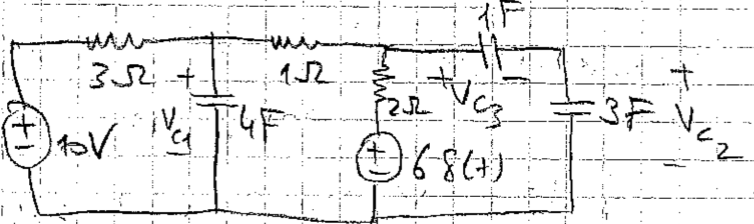
$= 0$

$i_L(\infty) = \frac{10}{R}$ A

, i.e. inductor is short circuited (since $v_L(t) = L \frac{d}{dt} i_L(t) = 0$ Volts)
for DC inputs as t → ∞

constant

EX:



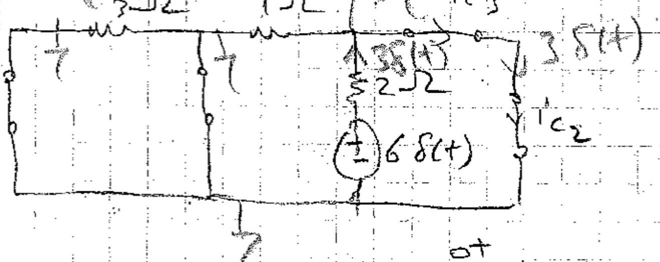
$v_{c_1}(0^-) = -2V$,
$v_{c_2}(0^-) = 4V$,
$v_{c_3}(0^-) = 3V$.

a) Find $v_{c_1}(0^+)$, $v_{c_2}(0^+)$, $v_{c_3}(0^+)$.

$0^- < t < 0^+$

$i_{c_1}(t) = 0\,\delta(t) + \ldots 10V$ $\quad 0^- < t < 0^+$

$i_{c_2}(t) = 3\,\delta(t) + \ldots 10V$ $\quad 0^- < t < 0^+$



$$v_{c_k}(0^+) = v_{c_k}(0^-) + \frac{1}{C_k} \int_{0^-}^{0^+} i_{c_k}(\tau)\, d\tau, \quad K = \{1, 2, 3\}$$

$v_{c_1}(0^+) = v_{c_1}(0^-) = -2V$

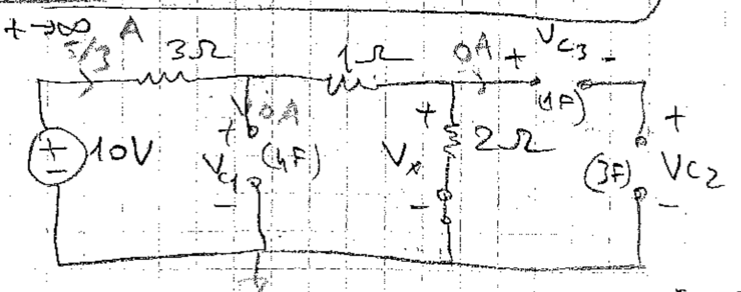
$v_{c_2}(0^+) = 4 + 3 \cdot \frac{1}{3} = 5V$

$v_{c_3}(0^+) = 3 + 3 \cdot 1 = 6V$

b) Find $v_{c_1}(\infty)$, $v_2(\infty)$, $v_{c_3}(\infty)$.

$q = C \cdot V$

$\frac{15}{6}$ $\frac{5}{6}$



$V_x = 2 \cdot 1 = \frac{10}{3}$ Volts

$v_{c_1}(\infty) = 3 \cdot 1 = 5$ Volts

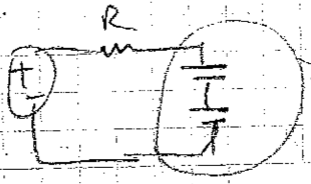
$v_{c_2}(\infty) + v_{c_3}(\infty) = V_x$

Voltage Division for Capacitors

$v_{c_2}(\infty) = \frac{1/C_3}{1/C_2 + \frac{1}{C_3}} V_x = \frac{5}{6}V$

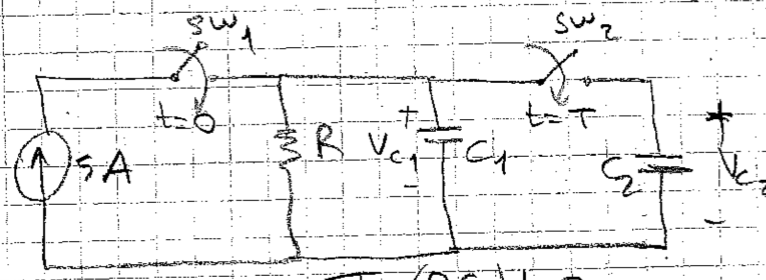
$v_{c_3}(\infty) = \frac{1/C_2}{[illegible]} V_x = \frac{15}{[illegible]} V$

★ Number of dynamic components = The order of circuit.



Only 1 dynamic component = 1^{st} order circuit

Ex.1



At t=0, sw_1 closes

At t=T, sw2 closes.

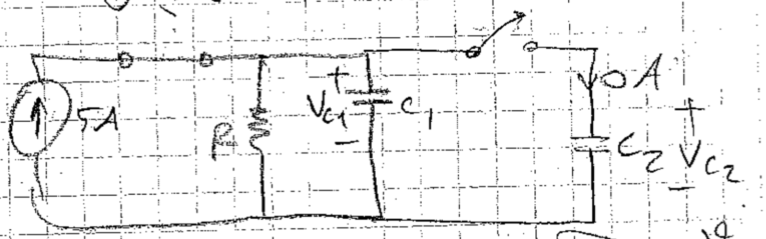
Find $V_{c_1}(t)$ and $V_{c_2}(t)$.

$T = (RC_1)\ln 3$

$V_{c_1}(0^-) = 2R$

$V_{c_2}(0^-) = 6R$ $V_{c_k}(0^+) = V_{c_k}(0^-)$ $k = \{1,2\}$, Since no impulsive current or pathological switching.

$0 \leq t < T$



$V_{c_1}(0^+) = 2R$

$V_{c_2}(0^+) = 6R$

$V_{c_2}(t) = V_{c_2}(0^+) + \frac{1}{C_2}\int_{0^+}^{t} \overset{0}{i_{c_2}(\tau)}\, d\tau$

$= 6R$ Volts

$V_{c_1}(t) = V_c^{final} + \left(V_c^{initial} - V_c^{final}\right)e^{-(t-t_{in})/\tau}$, $0 \leq t < T$

$= 5R + (2R - 5R)e^{-t/\tau}$,

$= 5R - 3Re^{-t/\tau}$, $(\tau = RC_1)$

$V_{c_1}(T^-) = 5R - 3Re^{-T/\tau} = 5R - 3Re^{-\ln 3} = 4R$.

$$\underbrace{\qquad\qquad}_{T^- < t < T^+}$$

When sw_2 closes, two capacitors are put in parallel and they have different voltages.

$V_{c_1}(T^-) = 4R$

$V_{c_2}(T^-) = 6R$

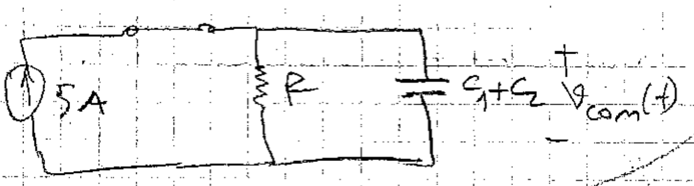
$\longrightarrow$ $q_{before} = q_{after}$

$C_1 V_{c_1}(T^-) + C_2 V_{c_2}(T^-) = (C_1 + C_2)V_{com}(T^+)$ common voltage of capacitor

$$\frac{C_1 4R + C_2 6R}{C_1 + C_2} = V_{com}(T^+)$$

$t > T$

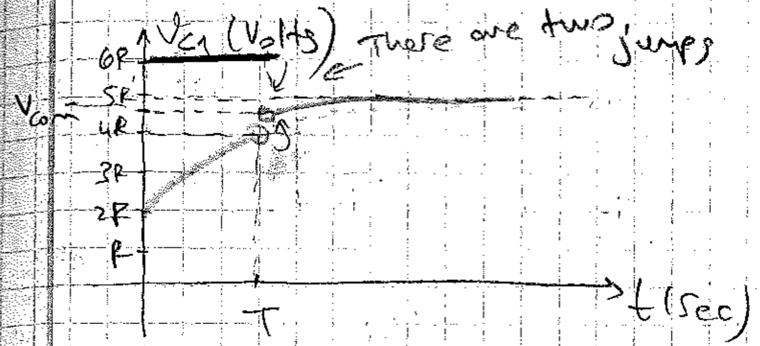
$V_{c_1}(T^+) = V_{c_2}(T^+) = \frac{4RC_1 + 6RC_2}{C_1 + C_2}$



$v_{com}(t) = v_c^{final} + (v_c^{ini} - v_c^{final}) e^{-(t-t_{ini})/\tau_2}$, $t > t_{ini}$

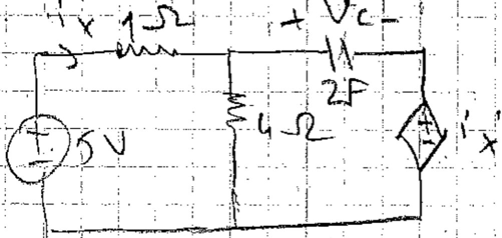
$= 5R + \frac{(4RC_1+6RC_2-5R)e^{-(t-T)/\tau_2}}{C_1+C_2}$, $t > t_{ini}'$

$\tau_2 = (C_1+C_2)R$



1st Order Circuits (cont'd)

Ex1:

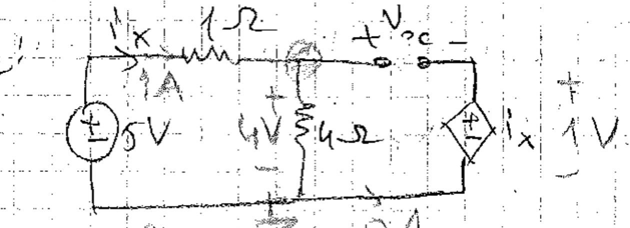


Find $v_c(t), i_x(t)$ for $t \geq 0$, ($v_c(0^-)=5V$)

Apply Thevenin Equivalent → $v_c(t)$

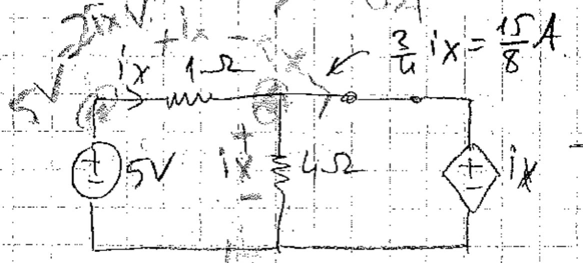
→ $i_x(t)$

V_{oc}:



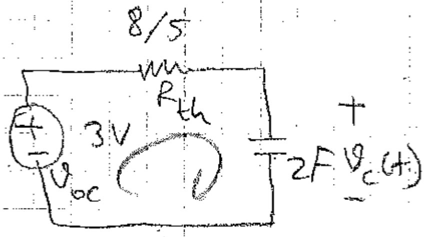
→ $v_{oc} = 4-1 = 3V$

i_{sc}:



→ $i_{sc} = \frac{15}{8} A$

$i_x = 2.5A$



$v_c(0^-) = 5V$

$-v_{oc} + R \cdot i_c(t) + v_c(t) = 0$

$C\dot{v}_c(t)$

$$\boxed{\dot{v}_c(t) + \frac{1}{RC} v_c(t) = \frac{v_{oc}}{RC}}$$

$$\boxed{v_c(0^-) = 5V}$$

$v_c(t) = v_c^{fin} + (v_c^{ini} - v_c^{final}) e^{-(t-t_{ini})/\tau}$, $t > t_{ini}$

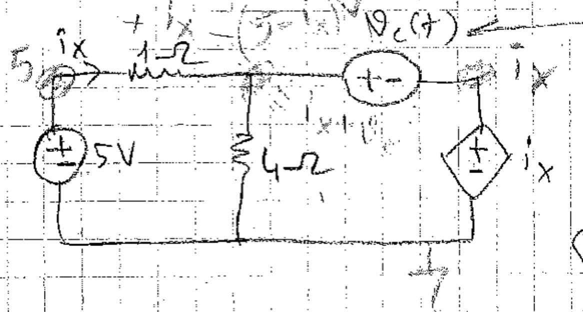
$= 3 + 2e^{-t/3.2}$, $t \geq 0$

→ $\tau = C R_{th} = \frac{16}{5}$ sec

(3.2 sec)

Q: Given your solution, can you immediately find $i_x(t)$?

A1:



We can consider the capacitor as a Voltage source.

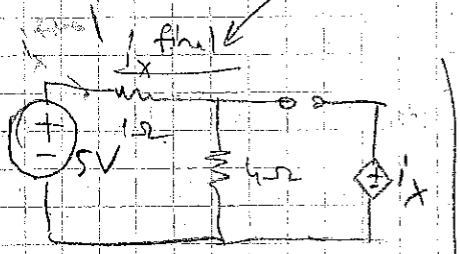
$$5 = 2i_x + v_c$$

$$\boxed{i_x = \frac{5-v_c}{2} = 1 - e^{-t/3.2},\ t > 0}$$

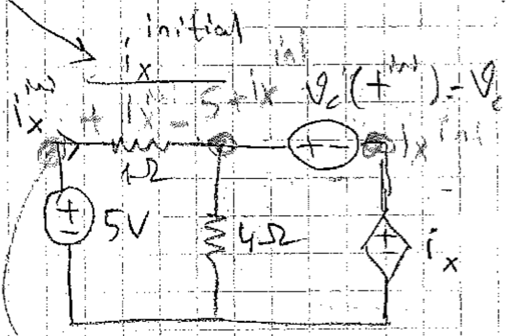
A2: I know that i_x also follows the function in the following form:

$$i_x(t) = i_x^{final} + (i_x^{ini} - i_x^{final})\, e^{-(t-t_{ini})/\tau},\ t \geq t_{ini}$$

??



$i_x^{final} = 1A$

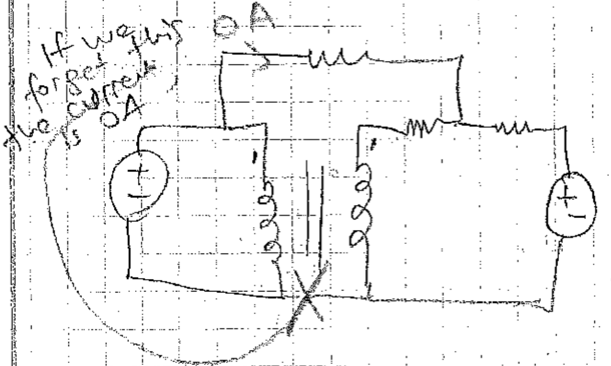


$v_c(t^{ini}) = v_c(0) = 5V$

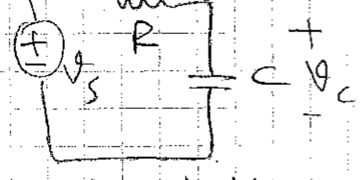
$$5 = (5 + i_x^{ini}) + i_x^{ini} \rightarrow i_x^{ini} = 0\ A$$

$$i_x(t) = 1 - e^{-t/3.2},\ t \geq 0$$

In MT1 exam:



Zero Input and Zero State Responses



$v_c(0^-) = V_0$

Zero input solution: This is the case for $v_s(t) = 0$ and $v_c(0^-) = V_0$

(So, input is zero; circuit response is due to initial conditions.)

Zero state solution: State is $v_c(t)$ [or $I_L(t)$] function and if this function is given to us (as in the previous example), all circuit branch voltages / currents can be found immediately.

Zero-state solution is the case for the initial conditions to be all zero; (initial conditions are in general state variables)

$\rightarrow V_c(0^-)=0$

$\dot{v}_c(0^-)=0$

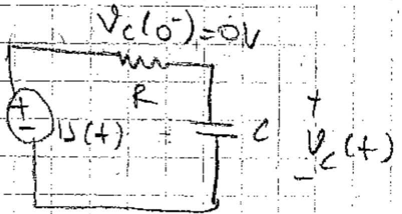
$I_L(0^-)=0$ and input $v_s(t)\neq 0$.

Complete Solution $\boxed{v_c^{comp}(t) = \underset{\text{zero-input}}{v_c^{zi}(t)} + \underset{\text{zero-state solution}}{v_c^{zs}(t)}}$

Zero-State Response

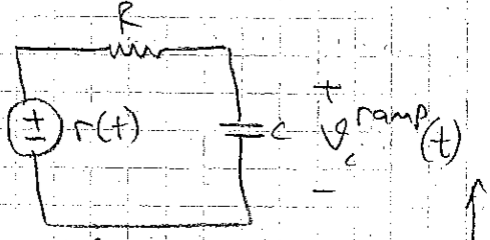
① Unit Step Response

Caution: All zero state responses are defined by definition at zero-state, i.e. zero initial conditions. A+

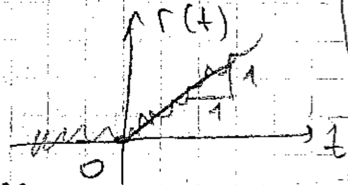


$v_c^{step}(t)=? \rightarrow \boxed{v_c^{step}(t)=1-1e^{-t/\tau},\ t\geq 0}$

② Ramp Response



$V_c^{ramp}(0^-)=0V$



$$\boxed{\begin{gathered}\left(D+\frac{1}{RC}\right)v_c^{ramp}(t)=\frac{r(t)}{RC},\ t>0\\ v_c^{ramp}(0^+)=v_c^{ramp}(0^-)=0V\end{gathered}}$$

$\rightarrow v_c^{ramp}(t)=\underbrace{\textcircled{A}e^{-t/\tau}}_{\text{homogeneous}}+\underbrace{Kt+L}_{\text{particular}}$

? ? ?

Insert particular solution into differential equation!

$$K+\frac{1}{RC}(Kt+L)==\frac{t}{RC},\ t>0$$

$$K+\frac{Kt}{RC}+\frac{L}{RC}=\frac{t}{RC}\qquad K=1$$

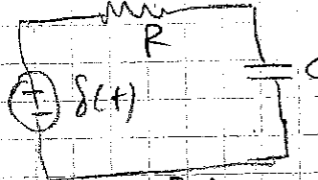
$$\boxed{\left(K+\frac{1}{RC}L\right)\underset{f_1(t)}{\textcircled{1}}+\left(\frac{K}{RC}\underset{f_2(t)}{\textcircled{t}}\right)=\left(\frac{1}{RC}\cdot t\right),\ t>0}\qquad -K=\frac{L}{RC}$$

$$L=-RC$$

$\downarrow t=0\ \downarrow t=3\ \downarrow t,$

$$\boxed{v_c(t)=\underbrace{RC}_{A=RC\text{ so that }v_c(0^+)=0}e^{-t/\tau}+t-RC,\ t\geq 0}$$

③ Impulse Response



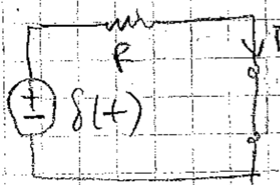
$v_c^{Imp}(0^-)=0$

$$\boxed{\left(D+\frac{1}{RC}\right)v_c^{imp}(t)=\frac{\delta(t)}{RC},\ t>0 \qquad v_c^{imp}(0^-)=0}$$

$$\boxed{\left(D+\frac{1}{RC}\right)v_c^{imp}(t)=0,\ t>0 \qquad v_c^{imp}(0^+)=?}$$

Q: what is $v_c(0^+)=?$

$0^-<t<0^+$



$i_c(t)=\frac{\delta(t)}{R},\ 0^-<t<0^+$

$$v_c(0^+)=\overset{0}{\cancel{v_c(0^-)}}+\frac{1}{C}\int_{0^-}^{0^+}\underbrace{i_c(\tau)}_{\frac{\delta(t)}{R}}d\tau = \frac{1}{RC}$$

$$\boxed{\left(D+\frac{1}{RC}\right)v_c^{imp}(t)=0,\ t>0 \qquad v_c^{imp}(0^+)=\frac{1}{RC}}$$

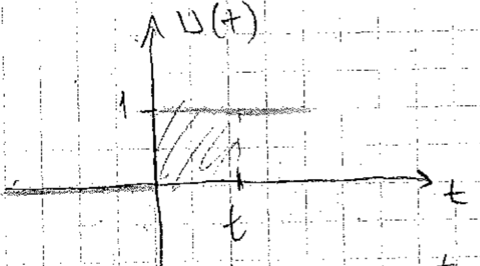
$$\underset{h(t)}{v_c^{imp}(t)}=\underbrace{\frac{1}{RC}e^{-t/\tau}}_{\text{homogeneous}}+\underbrace{0}_{\text{particular}},\ t>0$$

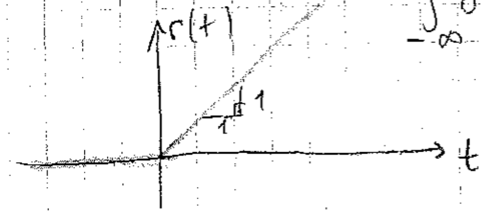
Note: We denote impulse response by $h(t)$ in general.

Some general Comments

$$\begin{tikzcd}
\delta(t) \arrow[r, shift right, "D^{-1}"'] & u(t) \arrow[l, shift right, "D"'] \arrow[r, shift right, "D^{-1}"'] & r(t) \arrow[l, shift right, "D"']
\end{tikzcd}$$

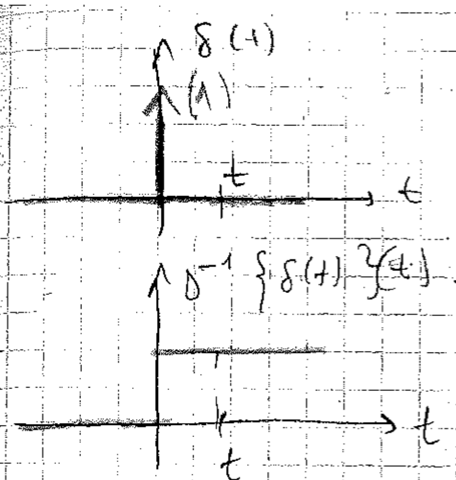
$$\boxed{D=\frac{d}{dt} \qquad D^{-1}=\int_{-\infty}^{t}(\ )\,d\tau}$$

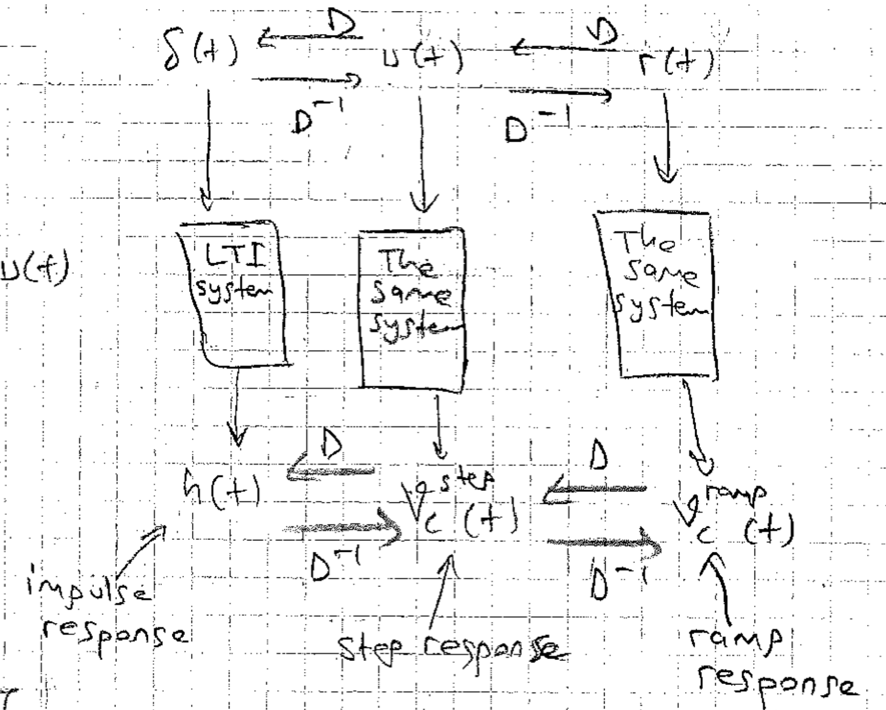




$$\int_{-\infty}^{t}u(\tau)d\tau = D^{-1}\{u(t)\}(t) \overset{\vee}{=} r(t)$$

"of t" not multiplication





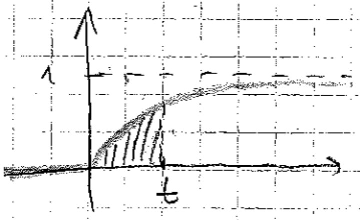
$$v_c^{ramp}(t) = D^{-1}\{v_c^{step}(t)\}$$

$$= \int_{-\infty}^{t} v_c^{step}(\tau)\,d\tau$$

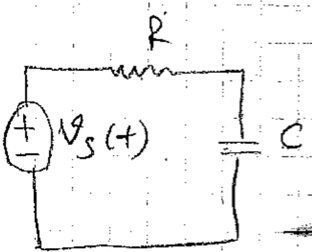
$$= \int_{-\infty}^{t} (1-e^{-\tau/RC})\,u(\tau)\,d\tau$$

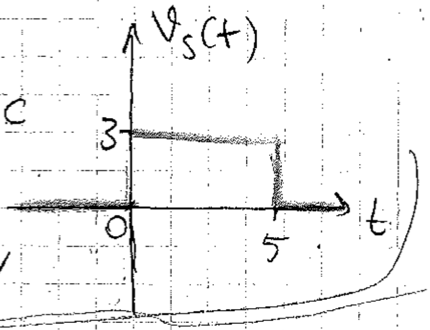
$$= \int_{0}^{t} (1-e^{-\tau/RC})\,d\tau$$

$$= \tau - \frac{e^{-\tau/RC}}{-1/RC}\Bigg|_{\tau=0}^{\tau=t} = \underbrace{(t+RCe^{-t/RC}-RC)}_{v_c^{ramp}(t)\ \checkmark},\ t \geq 0$$



Ex:





$v_c(0^-) = 10V$

Method 1:

$0 < t < 5^-$

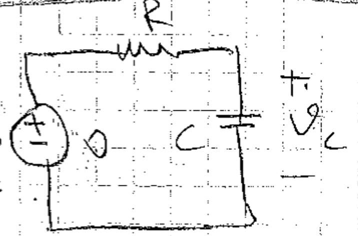
$v_c(0^+) = 10V$

$v_c(t) = V_c^{fin} + (V_c^{ini} - V_c^{fin})e^{-t/\tau}$

$= 3 + 7e^{-t/\tau}\ V$

$v_c(5^-) = 3 + 7e^{-5/\tau},\ \tau = RC$

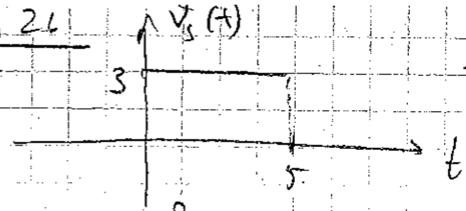
$t > 5$



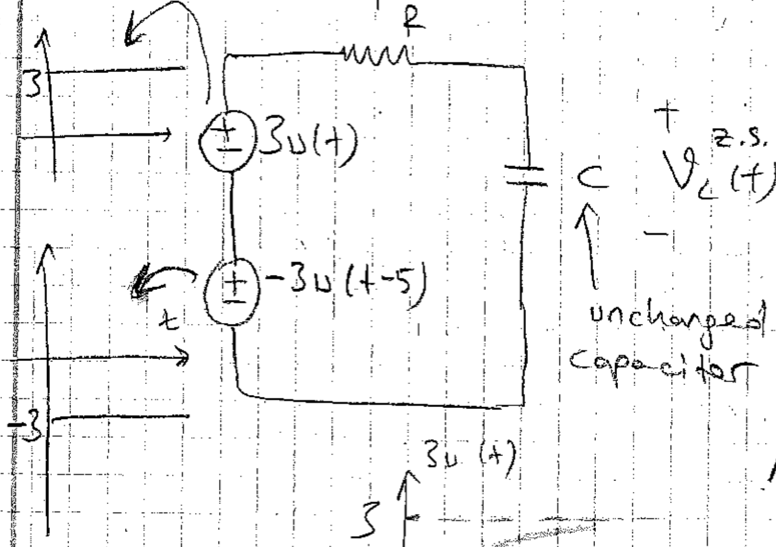
$v_c(5^+) = v_c(5^-) = V_5$

$v_c(t) = 0 + V_5 e^{-(t-5)/\tau},\ t > 5$

Method 2(

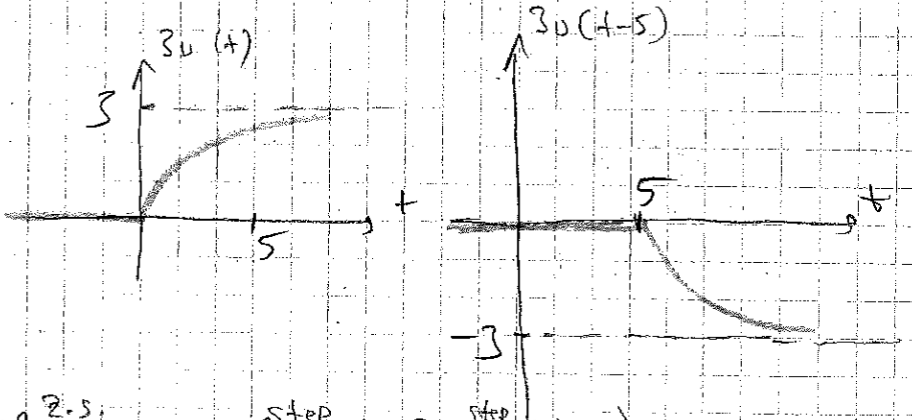


$\rightarrow v_s(t) = 3u(t) - 3u(t-5)$



Assume cap. is uncharged that

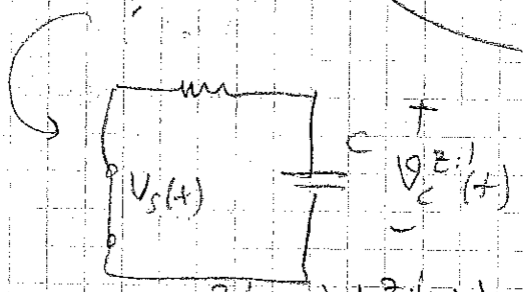
$$\boxed{v_c^{z.s}(t) = 3v^{step}(t) - 3v^{step}(t-5)}$$



Then $v_c^{z.s.}(t) = 3v^{step}(t) - 3v^{step}(t-5)$

$= 3(1-1e^{-t/\tau})u(t) - 3(1-1e^{-(t-5)/\tau})u(t-5)$

$$v_c^{comp}(t) = \underbrace{v_c^{z.i}(t)}_{?} + \underbrace{v_c^{z.s.}(t)}_{\checkmark}$$



$$\longrightarrow \boxed{v_c^{z.i}(t) = 10e^{-t/\tau},\ t \geq 0}$$

$v_c^{z.i}(0^-) = v^{z.i}(0^+)$
$= 10V$

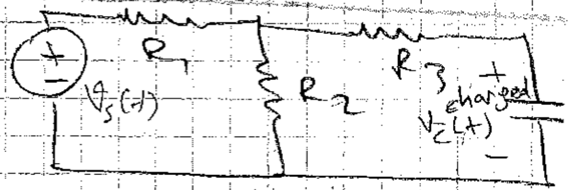
$v_{comp}(t) = 10e^{-t/RC}u(t) +$
$3(1-e^{-t/\tau})u +$
$-3(1-e^{-(t-5)/\tau})u(t-5)$

Zero-state / Zero-input Decomposition and its application in LTI circuit analysis

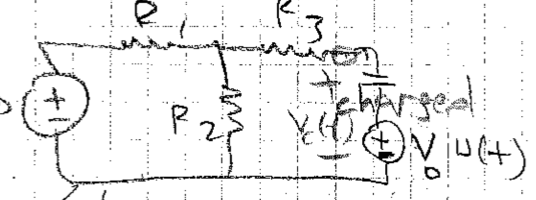
Claim: A dynamic circuit has a solution which can be decomposed as

$$\boxed{V_c^{Comp}(t) = \underbrace{V_c^{z.s}(t)}_{\text{zero state response}} + \underbrace{V_c^{z.i}(t)}_{\text{zero input response}}}$$

proof:

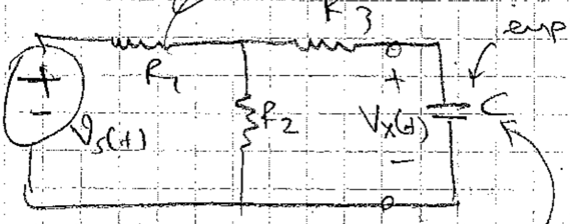


$V_c(0^-) = V_0$

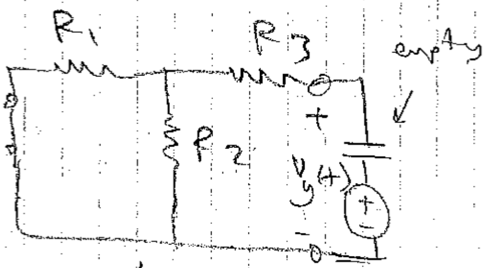


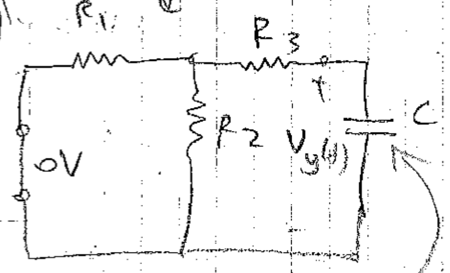
$V_c^{charged}(t) = V_x(t) + V_y(t)$

Super position



$V_x(t)$: zero-state response since this is the response to input when I.C. = 0

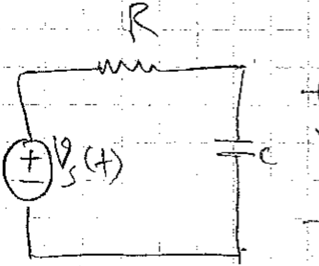




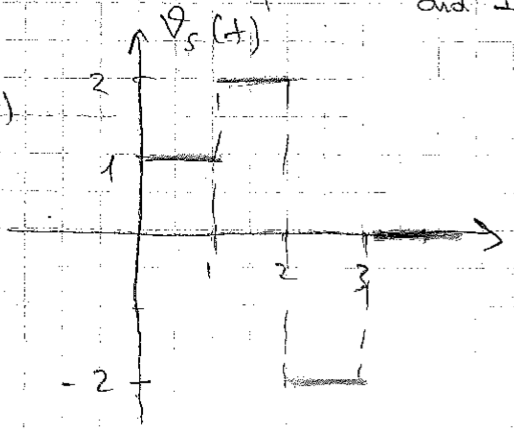
charged cap. with I.C. V_0

$V_y(t)$: zero-input is the response when $V_s(t) = 0$ (input = 0) and I.C. ≠ 0

EX:



$V_c(0^-) = V_0$



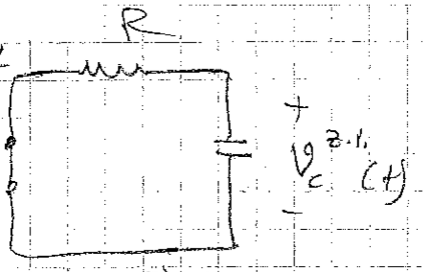
Find $V_c(t)$.

$v_c(t) = v_c^{z.i.}(t) + v_c^{z.s.}(t)$

$$\boxed{\begin{aligned} v_c^{z.i}(t) &= V_0 e^{-t/\tau}, \quad t \geq 0 \\ &= V_0 e^{-t/\tau} u(t) \end{aligned}}$$

$(\tau = RC)$

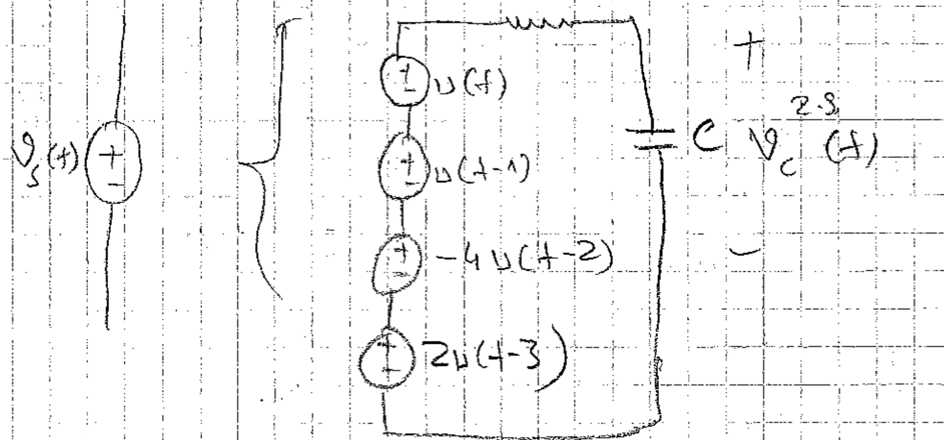
z.i.



$v_c(0^-) = V_0$

z.s.

$v_s(t) = u(t) + u(t-1) - 4u(t-2) + 2u(t-3)$

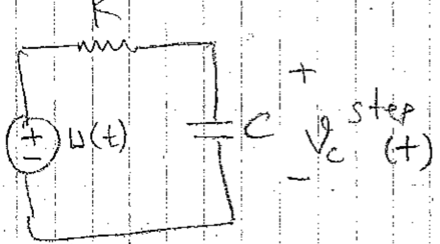


$V_c(0^-) = 0$

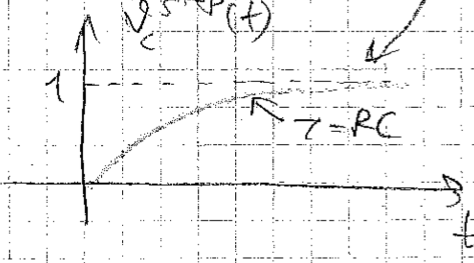
$v_c^{z.s.}(t) = \underbrace{v_c^{step}}_{\text{step response of the circuit}} + v_c^{step}(t-1) - 4v_c^{step}(t-2) + 2v_c^{step}(t-3)$

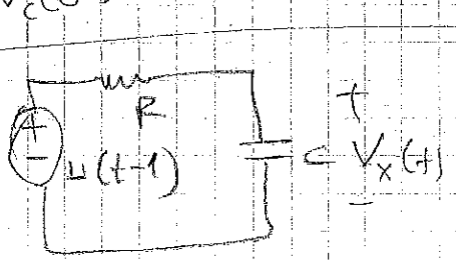
$\tau = RC$

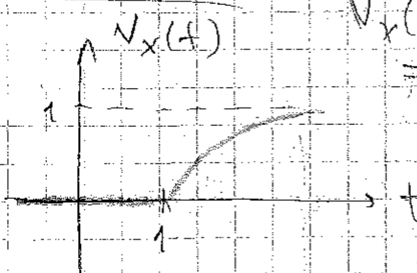
$v_c^{step}(t) = (\underset{v_c^{step}(\infty)}{1} - \underset{v_c^{step}(0^+) - v_c^{step}(\infty)}{1} e^{-t/\tau}) \quad t \geq 0 \rightarrow \boxed{v_c^{step}(t) = (1 - e^{-t/\tau}) u(t)}$



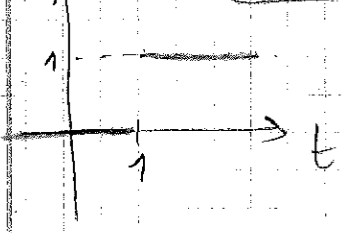
$v_c(0^-) = 0$







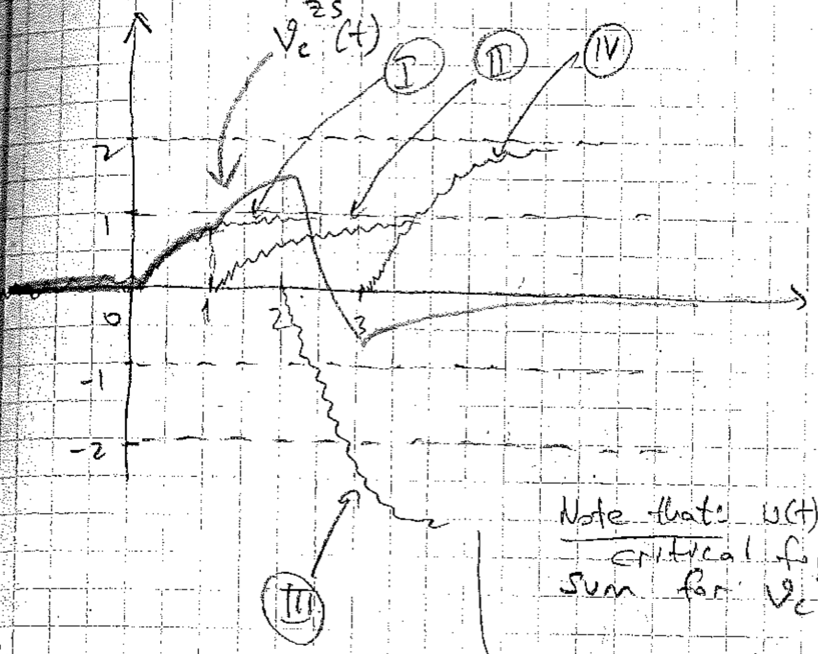
$V_x(t) = v_c^{step}(t-1) = (1 - e^{-(t-1)/\tau}) u(t-1)$

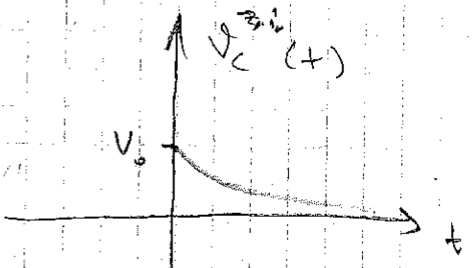


$V_c(0^-) = 0$

$$v_c^{z.s.}(t) = \underbrace{(1-e^{-t/\tau})u(t)}_{I} + \underbrace{(1-e^{-(t-1)/\tau})u(t-1)}_{II} \underbrace{-4(1-e^{(t-2)/\tau})u(t-2)}_{III}$$

$$+ \underbrace{2(1-e^{-(t-3)/\tau})u(t-3)}_{IV}$$

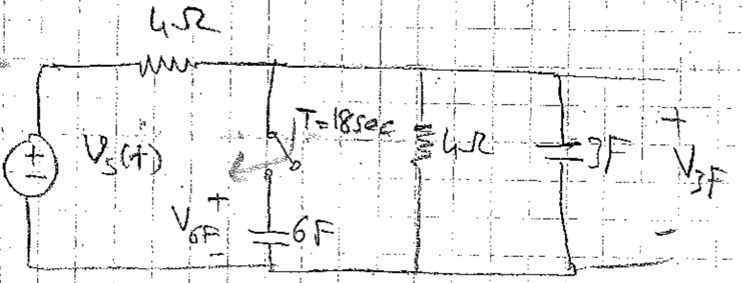


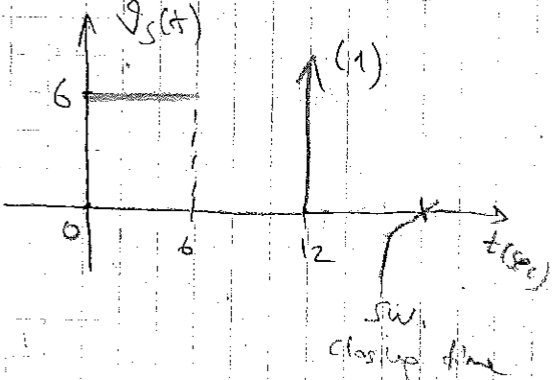


<u>Note that:</u> u(t) functions given in (I)-(IV) is critical for the correctness of superposition sum for $v_c^{z.s.}(t)$.

$$\to v_c^{comp}(t) = v_c^{z.i.}(t) + v_c^{z.s.}(t)$$

Ex:

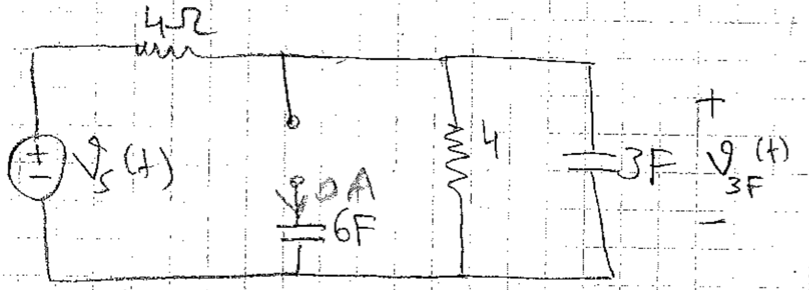




$v_{6F}(0^-) = -2V$

$v_{3F}(0^-) = 7V$

<u>0 < t < 18:</u>



$$v_{3F}(t) = v_{3F}^{z.i.}(t) + v_{3F}^{z.s.}(t)$$

$$\boxed{\begin{aligned} v_{3F}^{z.i.}(t) &= 7e^{-t/\tau}, u(t), \\ \tau &= 3\times(4 \parallel 4) \\ &= 6\,sec \end{aligned}}$$

$$v_{6F}(t) = v_{6F}(0^+) = v_{6F}(0^-) = -2, \quad 0 < t < 18$$

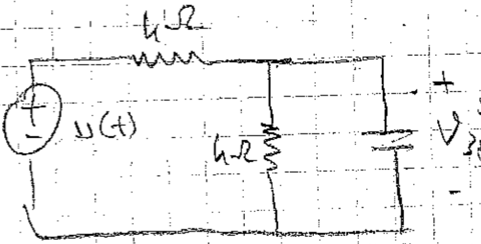
$$v_c(0^+) + \frac{1}{c}\int_{0^+}^{t} i_c(\tau)\,d\tau$$

0A

$$v_s(t) = [6u(t) - 6u(t-6) + 1\delta(t-12)$$

$$\hookrightarrow v_{3F}^{z.s.}(t) = 6v_{3F}^{step}(t) - 6v_{3F}^{step}(t-6) + 1\cdot\underbrace{v_c^{impulse}(t-12)}_{h(t-12)}$$

$v_C^{step}(t) = ?$



$$\boxed{v_{3F}^{step} = \left(\frac{1}{2} - \frac{1}{2} e^{-t/\tau}\right) u(t)}$$

$v_{3F}(0^-) = 0V$

$h(t) \longrightarrow h(t) = \frac{d}{dt} v_{3F}^{step}(t) = \frac{1}{2} \frac{d}{dt} \left\{ (1 - e^{-t/\tau}) u(t) \right\}$

$$= \frac{1}{2}\left(\frac{1}{2} e^{-t/\tau} u(t) + (1 - e^{-t/\tau}) \delta(t)\right)$$

$$h(t) = \frac{1}{2}\left[\frac{1}{6} e^{-t/\tau} u(t) + \underbrace{(1 - e^{-t/6})}\delta(t)\right.$$

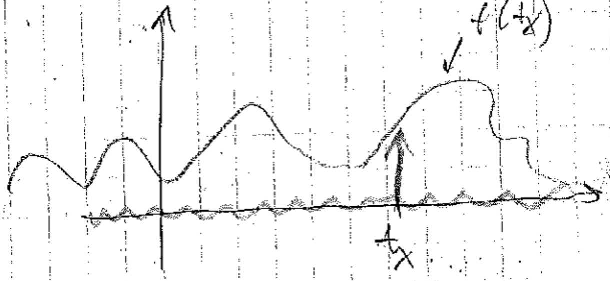
$$\boxed{f(t)\,\delta(t) = f(0)\,\delta(t)}$$

or

$$f(t)\,\delta(t - t_x) = f(t_x)\,\delta(t - t_x)$$

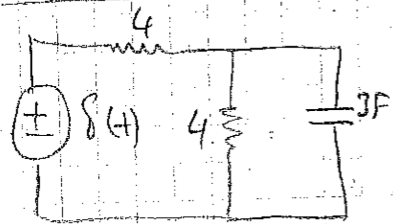
$$\boxed{h(t) = \frac{1}{12} e^{-t/\tau} u(t)} + (1 - e^{0/6})\Big|_{t=0} \delta(t)$$

$$= 0 \cdot \delta(t)$$
$$= 0$$



$v_C(0^-) = 0$

What is h(t)?



$$\boxed{h(t) = \frac{d}{dt} v_C^{step}(t)}$$

$$\boxed{0 < t < 18^-}$$

$$v_C^{comp}(t) = 7e^{-t/6} u(t) + 3(1 - e^{-t/\tau}) u(t) - 3(1 - e^{-(t-6)/\tau}) u(t-6)$$
$$+ \frac{1}{12} e^{-(t-12)} u(t-12)$$

$v_{3F}(18^-) = 2.14$ Volts

$\longrightarrow 18^- < t < 18^+$ At $t = 18$ sec.

Two cap. are forced by the switch operation to a common voltage level.

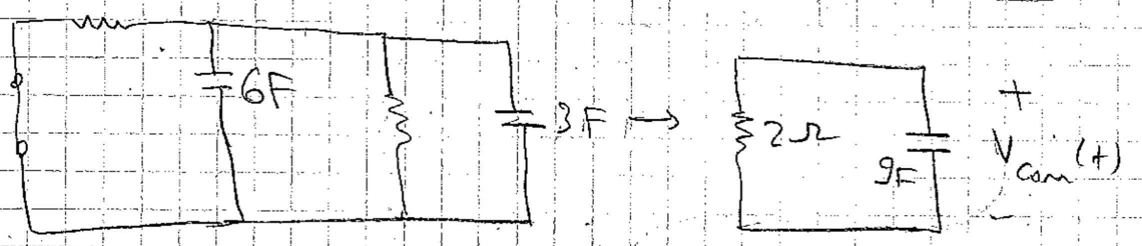
$q_{before} = q_{after}$

$$3\underbrace{V_{3F}(18^-)}_{2.14} + 6\underbrace{V_{6F}(18^-)}_{-2} = 9V_{com} \rightarrow V_{com} = -0.62 \text{ Volts}$$

$$\uparrow$$

$$V_{3F}(18^+) = V_{6F}(18^+)$$

$t > 18$



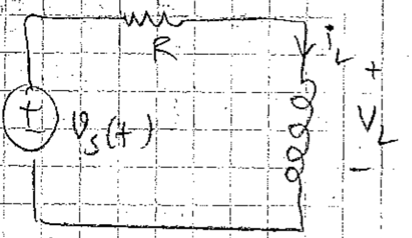
$V_{com}(18^+) = -0.62V$

$$\boxed{v_{com}(t) = -0.62 \times e^{-(t-18)/\tau_2}}$$

$t > 18$

$\tau_2 = 9F \times 2\Omega = 18sec$

On RL Circuits



$$-v_s(t) + Ri_L(t) + \underbrace{v_L(t)}_{L\frac{d}{dt}i_L(t)} = 0$$

$$\boxed{\begin{aligned} &\left(D + \frac{R}{L}\right) i_L(t) = \frac{v_s(t)}{L}, \quad t > 0 \\ &i_L(0^-) = I_0 \end{aligned}}$$

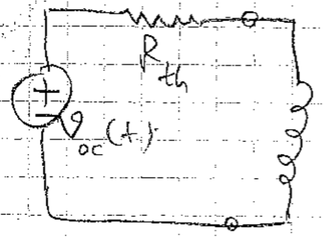
$$\boxed{i_L(t) = A \cdot e^{-\frac{Rt}{L}}} + \left(\begin{matrix}\text{Particular solution} \\ \text{to} \\ v_s(t)/L\end{matrix}\right) \quad t > 0 \quad \boxed{\tau = L/R}$$

Let's remember $v_c(t) = A\,e^{-\frac{1}{RC}t} + (\text{Particular solution}), \; t > 0$ $\boxed{\tau = RC}$

Duality
$R \leftrightarrow G$
$C \leftrightarrow L$

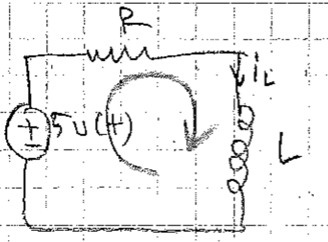
Comments

① If we have a complicated circuit with a single inductor, then apply Thevenin eq. to the part of the circuit seen by the inductor given us!



The solution of this circuit is mathematically the same solution of RC circuits and only difference is the change in time constant.

Ex1

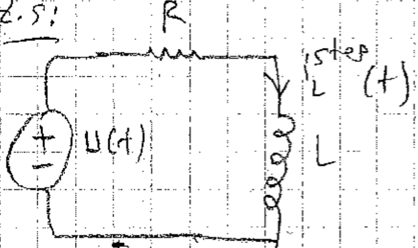


$$i_L(t) = i_L^{zi}(t) + i_L^{zs}(t)$$

z.i: $$\boxed{i_L^{z.i}(t) = 10e^{-t/\tau}u(t), \tau = \frac{L}{R}}$$

$i_L(0^-) = 10 A$

z.s:

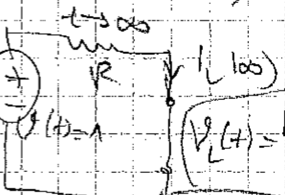


$i_L(0^-) = 0A$

$$i_L^{step}(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-t/\tau}, t \geq 0$$

$$\boxed{i_L^{z.s.}(t) = 5 i_L^{step}(t) = \frac{5}{R}(1-e^{-t/\tau})u(t)}$$

$i_L(\infty) = \frac{1}{R}$ since



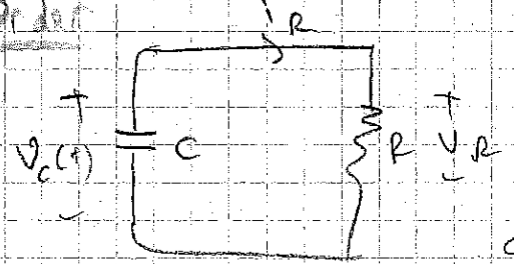
$$\boxed{v_L(t) = L\frac{di_L(t)}{dt}}$$

$$\boxed{i_L^{comp}(t) = i_L^{zi}(t) + i_L^{z.s.}(t)}$$

Inductor acts as a short circuit as $t \to \infty$.

Time Invariant - Time Varying - Non Linear Circuits

1st Order



a) R ⇒ 1Ω → Time Interval

b) R ⇒ $R = \frac{1}{1+0.5\cos t}$ Ω → Time Varying

c) R ⇒ $i_R = V_R^2$ → Non-Linear

$V_c(0^-) = 1V$

a) $V_c(t) = \underbrace{V_c(0^+)}_{1V} e^{-t/\tau}, t \geq 0$ $(\tau = RC = (1\Omega) \times (1F) = 1sec$

b) $$\underset{\underset{1F}{\uparrow}}{C}\dot{v}_c(t) + \frac{V_c}{R(t)} = 0 \longrightarrow \underset{\underset{\frac{d}{dt}v_c(t)}{\nearrow}}{\dot{v}_c}(t) = (1+0.5\cos t)v_c(t)$$

$$\frac{dv_c(t)}{v_c(t)} = (-1+0.5\cos t)dt$$

$$\ln(v_c(t) = -(t+0.5\sin t) + k$$

$$v_c(t) = e^{-(t+0.5\sin t)}\underbrace{e^{k}}_{k'}$$

$$v_c(t) = k'e^{-(t+0.5\sin t)} V$$

↑ Time-varying circuit case

DC input $\longrightarrow v_C(t) = v_C(t_{fin}) + \left(v(t_{in}) - v_{fin}\right) e^{-t/\tau}$

c) $\underset{1F}{C}\,\dot{v}_C(t) + \underbrace{i_R}_{= v_R^2 = v_C^2} = 0$

$$\frac{d\,v_C(t)}{dt} = -v_C^2(t)$$

$$\downarrow$$

$$\frac{d\,v_C(t)}{v_C^2(t)} = -dt$$

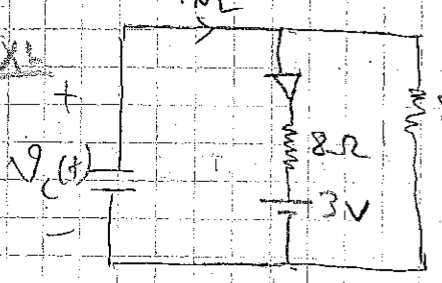
$\int \leftarrow$ indefinite integral

$$-\frac{1}{v_C(t)} = -t + K \longrightarrow v_C(t) = \frac{1}{t-K} \longrightarrow \boxed{v_C(t) = \frac{1}{t+1}}$$

non-linear resistor caracteristic ?

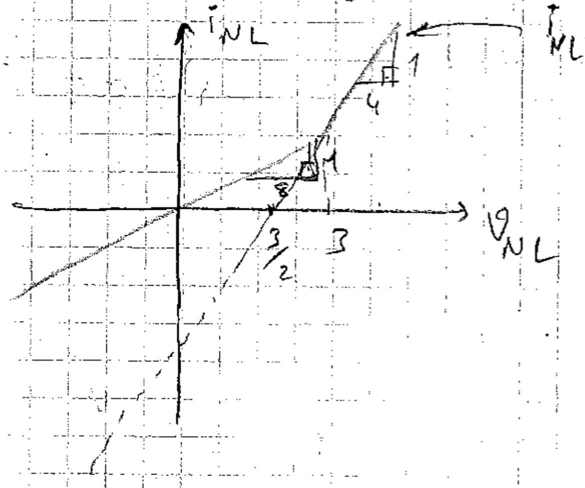
$v_C(0^+) = v_C(0^-) = 1 \to K = -1$

EX1

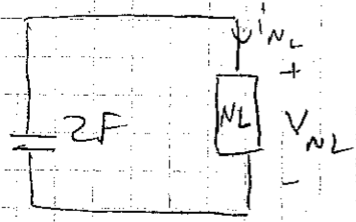


$v_C(0^-) = 10V$

Find $v_C(t)$



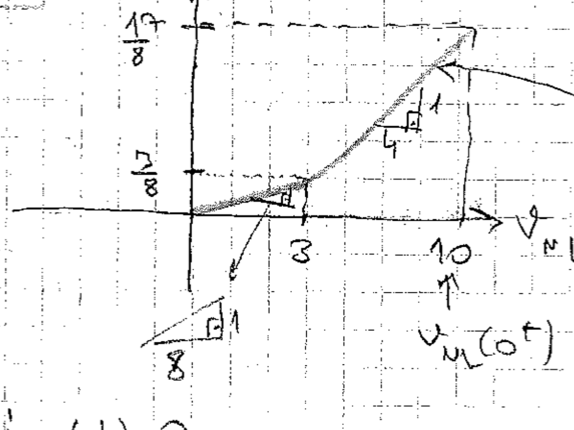
$i_{NL} = \frac{1}{4}v_{NL} - \frac{3}{8}$



$v_C(0^-) = 10V$

$v_C(0^+) = 10V \to$ at $t = 0^+$ D:ON

$V_{NL}(0^+)$



$i_{NL} = \frac{1}{4} v_{NL} - \frac{3}{8}$

KCL!

$\underset{2F}{C}\,\dot{v}_C(t) + i_{NL}(t) = 0$

$$\boxed{\dot{v}_C(t) = \frac{-i_{NL}(t)}{2},\ t \geq 0}$$

$$\dot{v}_C(0^+) = \frac{-i_{NL}(0^+)}{2} = \frac{-17/8}{2} = \frac{-17}{16}$$

$\to v_C(t)$ is decreasing at time $t=0$

The discharge of the capacitor continue until the diode turns off which is the case of $\vartheta_c(t)$ reaches 3 Volts or since D : ON at $t=0^+$:

$$\dot{\vartheta}_c(t) = \frac{-i_m(t)}{2} \leftarrow \frac{1}{4}\vartheta_c - \frac{3}{8}$$

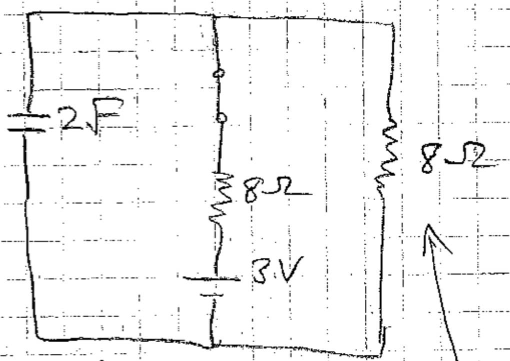
$$\dot{\vartheta}_c(t) = -\frac{1}{8}\vartheta_c + \frac{3}{16} \qquad \text{Valid for}$$

$$\vartheta_c(0^+) = 10 \qquad \vartheta_c > 3$$

$$\vartheta_c(t) = A \cdot e^{-\frac{1}{8}t} + \frac{3}{2} = \frac{17}{2} e^{-\frac{1}{8}t} + \frac{3}{2}$$

$$\frac{17}{2}$$

Another method: D: ON:



$\vartheta_c(0^+) = 10$

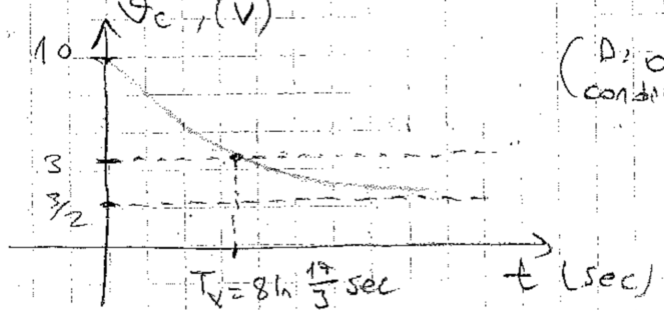
$$\rightarrow \vartheta_c(t) = \frac{3}{2} + \frac{17}{2} e^{-t/\tau}$$

$$\tau = C \cdot R_{th} = (2F) \times (8 // 8)$$

$$\tau = 8 \text{ sec}$$

Valid for $\vartheta_c(t) > 3$

(D: ON condition)

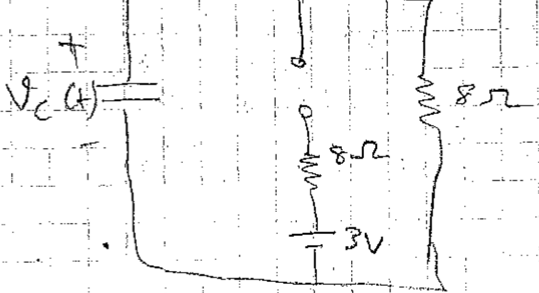


$$\vartheta_c(T_x) = 3 \rightarrow T_x = ?$$

$$T_x = 8 \ln\frac{17}{3}.$$

$t > T_x$: D : OFF and the cap. discharges

(or T = time)



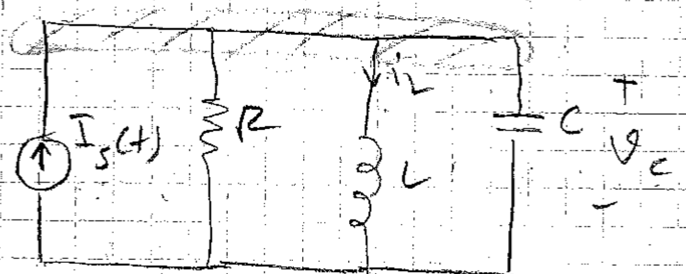
$\vartheta_c(T_x^+) = 3V$

$$\vartheta_c(t) = 3 \cdot e^{-(t-T_x)/\tau_2}, \; t > T_x$$

$$\tau_2 = 2 \times 8 = 16 \text{ sec}$$

2^{nd} Order Circuits

Parallel RLC:



$I_L(0^-) = I_0$

$v_c(0^-) = V_0$

KCL: $-I_s(t) + i_R + i_L + i_C = 0 \quad , \; t \geq 0$

$i_R \to \dfrac{v_c}{R}$, $\quad i_C \to C\dfrac{dv_c}{dt}$

$i_L \to i_L(t) = i_L(0^-) + \dfrac{1}{L}\int_{0^-}^{t} v_c(\tau)\,d\tau$

$$-I_s(t) + \frac{1}{R}v_c + \left(I_0 + \frac{1}{L}\int_{0^-}^{t} v_c(\tau)\,d\tau\right) + C\frac{d}{dt}v_c(t) = 0$$

$$-I_s(t) + I_0 + \left(\frac{1}{R} + \frac{1}{L}D^{-1} + CD\right)v_c(t) = 0 \quad (*)$$

integral-differential equations

$D = \dfrac{d}{dt}$

$D^{-1} = \int_{0^-}^{t} (\ldots)\,d\tau$

$D\{t^2\} = 2t$

$D^{-1}\{1\} = \int_{0^-}^{t} (1)\,d\tau = t$

Apply $\frac{1}{C}D = \frac{1}{C}\frac{d}{dt}\{\;\}$ to both sides of $(*)$

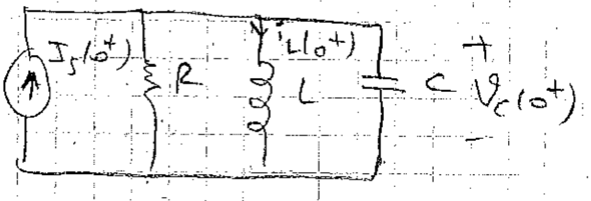
$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)v_c(t) = \frac{d}{dt}I_s(t)\cdot\frac{1}{C} = \frac{\dot{I}_s(t)}{C} \quad , \; t > 0$$

$v_c(0^+) = \;?$

$\dot{v}_c(0^+) = \;?$

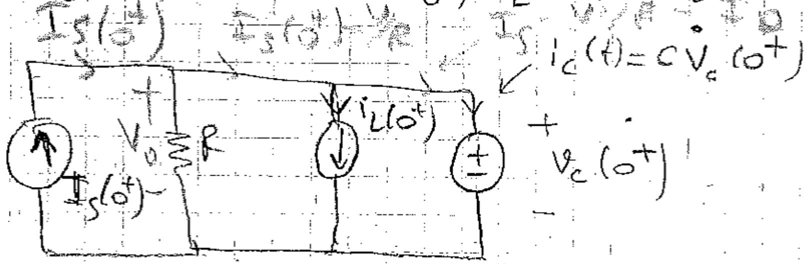
We need I.C. for $v_c(t)$ and its derivative at $t=0$ for the solution.

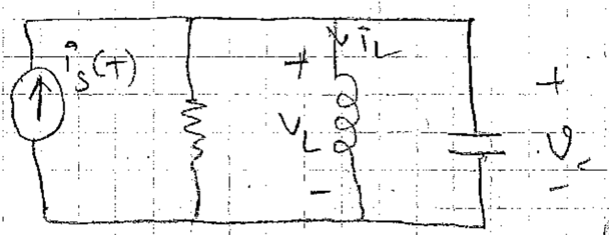
At $t=0^+$



Assume $I_s(t)$ isn't $\delta(t)$

$v_c(0^+) = v_c(0^-) = V_0 \;/\; I_L(0^+) = I_L(0^-) = I_0$





v_c $v_L = L\frac{d}{dt} i_L(t)$

$$\left(D^2+\frac{1}{RC}D+\frac{1}{LC}\right) \underset{\uparrow \; LD\{I_L(t)\}}{v_c(t)} = \frac{D\{I_s(t)\}}{C}$$

$$\left(D^2+\frac{1}{RC}+\frac{1}{LC}\right) D\, i_L(t) = \frac{D\{I_s(t)\}}{C}$$

Apply D^{-1} to both sides

$$\left(D^2+\frac{1}{RC}D+\frac{1}{LC}\right) i_L(t) = \frac{I_s(t)}{LC}$$

↑ Dif. eqn. for the variable $i_L(t)$.

I.C.'s

$$i_L(0^+) \underset{\substack{\uparrow \\ \text{(no impulsive input)}}}{=} i_L(0^-) = I_0$$

$$\underset{\substack{\uparrow \\ L\dot{I}_L(0^+)}}{v_L(0^+)} = v_c(0^+) = v_c(0^-) = V_0$$

$$\boxed{\dot{I}_L(0^+) = \frac{V_0}{L}}$$

cons. coef. dif eq.

Solution of 2nd Order CCDE

① Zero-input Solution

Parallel RLC $\left(D^2+\frac{1}{RC}D+\frac{1}{LC}\right) v_c(t) = \frac{1}{C} D\{I_s(t)\} = 0$

$v_c(0^-)$

$\dot{v}_c(0^-) = \dot{V}_0$

General form for 2^{nd} order CCDE: $\boxed{\left(D^2+\underbrace{2\alpha}D+w_0^2\right)v_c(t)=0}$ (*)

Parallel RLC $\Rightarrow 2\alpha = \frac{1}{RC}$ α: Damping coefficient

$w_0^2 = \frac{1}{LC}$ w_0: resonance frequency

$v_c(t) = A\cdot e^{\lambda t}$ ↙ Guess for zero-input solution

↳ insert into (*): $A(\lambda^2+2\alpha\lambda+w_0^2)e^{\lambda t}=0$

$A = 0$

$\downarrow$

$v_c(t) = A e^{\lambda t}$

$= 0!$

trivial solution

$\forall A, \ \lambda^2 + 2\alpha\lambda + w_0^2 = 0$

$$\lambda_{1,2} = \frac{-2\alpha \mp \sqrt{4\alpha^2 - 4w_0^2}}{2}$$

$$\boxed{\lambda_{1,2} = -\alpha \mp \sqrt{\alpha^2 - w_0^2}}$$

$$V_c^{z.i.}(t) = \underset{?}{A_1} e^{\lambda_1 t} + \underset{?}{A_2} e^{\lambda_2 t}$$

α: damping coefficient

A_1, A_2: are the unknowns to be set to satisfy the I.C.

λ: Natural frequencies of the circuit ← Very important

Please pay attention to λ is the property of the circuit and isn't related with the input. (Actually, we've derived λ_1, λ_2 from zero-input response!)

$$\underbrace{\lambda^2 + 2\alpha\lambda + w_0^2}_{\text{characteristic polynomial}} = 0 \leftarrow \text{characteristic equation}$$

The roots of characteristic polynomial are called natural frequency.

There are 3 possible classes of solution for the char. equation,

① λ_1, λ_2 are real and distinct $(\Delta > 0)$ $\alpha > w_0$ overdamped

② $\lambda_1 = \lambda_2$ roots are real valued but identical $(\Delta = 0)$ $\alpha = w_0$ Critically damped

③ λ_1, λ_2 are complex valued but distinct, $(\Delta < 0)$ $\alpha < w_0$ Underdamped

$$\boxed{\begin{array}{l}\text{Parallel RLC} \\ \alpha = \dfrac{1}{2 \cdot RC} \\ w_0 = \sqrt{\dfrac{1}{LC}}\end{array}}$$

$\rightarrow \alpha > w_0$

$$\frac{1}{2Rc} > \frac{1}{LC} \rightarrow R < \frac{1}{2}\sqrt{\frac{L}{C}}$$

↑ condition for overdamped case (Parallel RLC)

$$\longrightarrow V_c^{z.i.}(t) = A_1 e^{-2t} + A_2 e^{-3t}$$

Let $\lambda_1 = -2$

$\lambda_2 = -3$

$\rightarrow \alpha = w_0$

$$V_c^{z.i.}(t) = A_1 e^{-\lambda t} + A_2 t e^{-\lambda t}$$

$$\lambda = \lambda_1 = \lambda_2 = -\alpha$$

$\hookrightarrow \alpha < \omega_0 \longrightarrow v_c^{z.i.}(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$

$\lambda_2 = \lambda_1^{*}$ ⟵ complex conjugate

$(\lambda^2 + \underset{\alpha=3}{6}\lambda + \underset{\omega_0=4}{16} = 0)$

$\boxed{j = i = \sqrt{-1}}$

$\lambda_{1,2} = -3 \mp \sqrt{9-16} = -3 \mp j\sqrt{7}$

$v_c^{z.i.}(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ $\quad \lambda_1 = -3 + j\sqrt{7}$

$= A_1 e^{(-3+j\sqrt{7})t} + A_2 e^{(-3-j\sqrt{7})t}$ $\quad \lambda_2 = -3 - j\sqrt{7}$

$= A_1 e^{-3t} \cdot e^{j\sqrt{7}t} + A_2 e^{-3t} \cdot e^{-j\sqrt{7}t})$

$= e^{-3t}(A_1 e^{j\sqrt{7}t} + A_2 e^{-j\sqrt{7}t}$

$v_c^{z.i.}(0^+) = V_0$

$\dot{v}_c^{z.i.}(0^+) = \dot{V}_0$

I.C. are utilized to find A_1, A_2

$\rightarrow A_1^{*} + A_2^{*} = V_0^{*}$

$(-3 \mp j\sqrt{7})A_1^{*} + (-3 \pm j\sqrt{7})A_2^{*} = \dot{V}_0$

$A_1 = A_2^{*}$

$v_c^{z.i.}(t) = e^{-3t}(A_1 e^{j\sqrt{7}t} + A_1^{*} e^{-j\sqrt{7}t})$

$= e^{-3t}(\|A_1\| e^{j\measuredangle A_1} e^{j\sqrt{7}t} + \|A_1\| e^{-j\measuredangle A_1} e^{-j\sqrt{7}t})$

$= e^{-3t}\|A_1\|(e^{j(\sqrt{7}t + \measuredangle A_1)} + e^{-j(\sqrt{7}t + \measuredangle A_1)})$

$= e^{-3t}\|A_1\| \cdot 2\cos(\sqrt{7} + \measuredangle A_1)$ = [illegible]

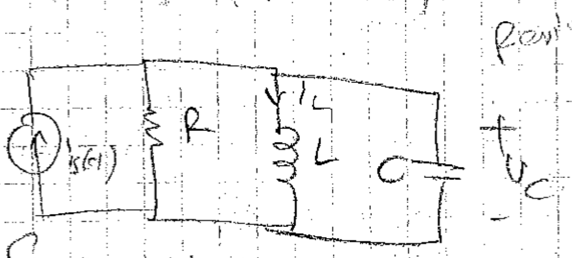
$$\boxed{v_c^{z.i.}(t) = K e^{-3t}\cos(\sqrt{7}t + L)}$$

$K = 2\|A_1\|$ $\qquad L = \measuredangle A_1$

2nd order Circuits (Ch 4 3)

Reminder of last lecture

Parallel RLC:



I.C. $\begin{cases} v_c(0^-) = V_0 \\ I_L(0^-) = I_0 \end{cases}$

Dif Eqn: $(D^2 + \frac{1}{RC}D + \frac{1}{LC})\, v_c(t) = \frac{1}{C} D\{i_s(t)\},\ t > 0$

$v_c(0^+) = \ldots ?$

$\dot{v}_c(0^+) = \ldots ?$ ⟶

→ Zero-input response ($i_s(t)=0$)

Guess: $v_c^{z.i.}(t) = Ae^{\lambda t}$

For the guess to be correct

$$A\left(\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC}\right) = 0 \rightarrow A = 0$$

↳ $A \neq 0$, $\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC} = 0$

Char Poly: $\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC}$

Char Eqn: $\lambda^2 + \underbrace{\frac{1}{RC}}_{2\alpha}\lambda + \overbrace{\frac{1}{LC}}^{\omega_0^2} = 0$

ω_0: resonant frequency

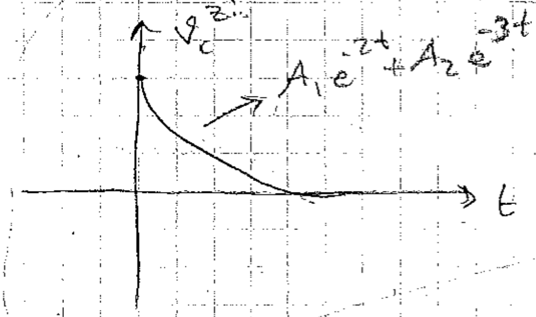
α: damping constant

↳ Roots of char. poly. → $\lambda_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$, $\lambda_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$ } $\lambda_{1,2}$: natural freq.

Cases: ① $\alpha > \omega_0$ (overdamped) → $v_c^{z.i.}(t) = A_1e^{-2t} + A_2e^{-3t}$

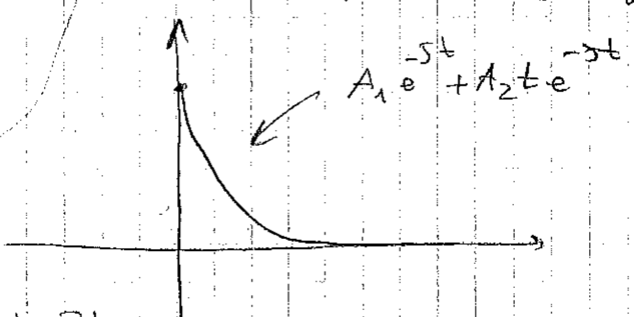
② $\alpha = \omega_0$ (critically damped) → $v_c^{z.i.}(t) = A_1e^{-5t} + A_2te^{-5t}$

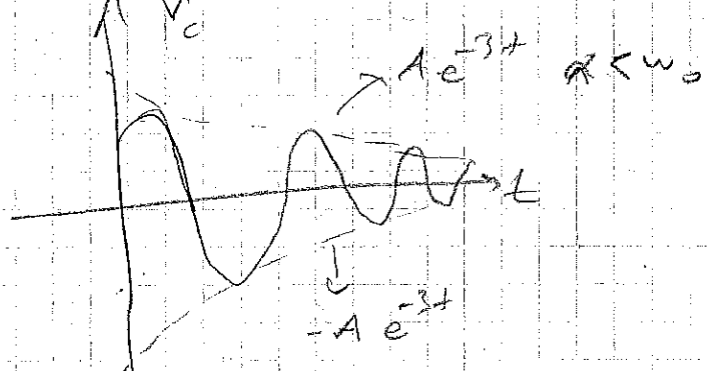
③ $\alpha < \omega_0$ (Underdamped) → $v_c^{z.i.}(t) = Ae^{-3t}\cos(8t + \phi)$



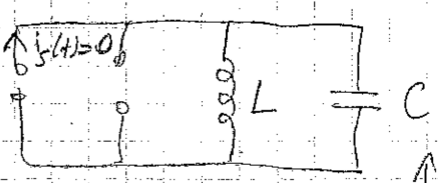
Overdamped
↓
(Too much friction)
mechanical analogy

Example for cases



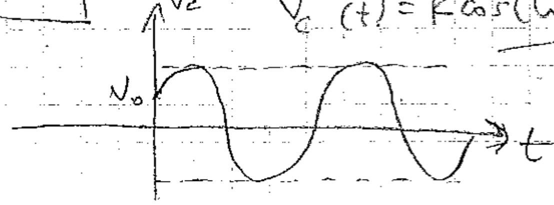


Case ④ ($\alpha = 0$) i.e. for parallel RLC $R = \infty$



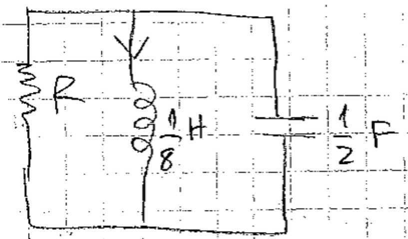
$$\left(D^2 + \frac{1}{LC}\right) v_c^{z.i.}(t) = 0$$

$v_c^{z.i.}(t) = K\cos(\omega_0 t + \theta)$



case of lossless circuit
(i.e. no friction → no energy loss)

Ex:



$i_L(0^-) = -4A$

$V_c(0^-) = 5V$

a) $R = \frac{1}{5}$, $V_c(t) = ?$ $t > 0$

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) V_c^{z.i.}(t) = 0$$

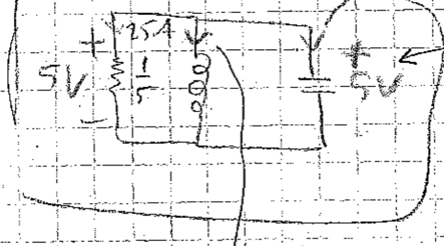
$$\left.\begin{aligned} w_0 &= \sqrt{\frac{1}{LC}} = 4 \text{ rad/sec} \\ \alpha &= \frac{1}{2RC} = 5 \left(\frac{1}{sec}\right) \end{aligned}\right\} \alpha > w_0 \to \text{overdamped}$$

$$\lambda_{1,2} = -\alpha \mp \sqrt{\alpha^2 - w_0^2} = \{-8, -2\}$$

$$V_c^{z.i.}(t) = \underset{?}{A_1} e^{-8t} + \underset{?}{A_2} e^{-2t} \quad , t > 0$$

Find A_1 and A_2 s.t. I.C. at $t = 0^+$ is satisfied.

At $t = 0^+$



$-21A = C\dot{V}_c(0^+)$

$V_c(0^+) = V_c(0^-)$

$V_c(0^+) = 5V$

$\dot{V}_c(0^+) = -42$ Volts/sec

$-4A = I_L(0^+) = I_L(0^-)$

$$V_c(0^+) = A_1 e^{-8t} + A_2 e^{-2t} \Big|_{t=0^+} = 5$$

$$\dot{V}_c(0^+) = -8A_1 e^{-8t} - 2A_2 e^{-2t} \Big|_{t=0^+} = -42$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 \\ -8 & -2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -42 \end{bmatrix}$$

$$\downarrow$$

$A_2 = -1/3$

$A_1 = 16/3$

$$\longrightarrow \boxed{V_c^{z.i.}(t) = -\frac{1}{3}e^{-2t} + \frac{16}{3}e^{-8t} , t \geq 0}$$

b) $R = \frac{1}{4}$ $v_c(t) = ?$ $t > 0$

$$\left.\begin{array}{l} \alpha = \frac{1}{2RC} = 4 \\ \omega_0 = 4 \end{array}\right\} \longrightarrow \alpha = \omega_0 \text{ critically damped}$$

$$\lambda_{1,2} = 4$$

$$v_c^{z.i.}(t) = A_1 e^{-4t} + A_2 t e^{-4t}, \quad t > 0$$

$$v_c^{z.i.}(0^+) = 5V$$

$$\dot{v}_c^{z.i.}(0^+) = -32 \text{ Volts/sec}$$

$$v_c^{z.i.}(t) = A_1 e^{-4t} + A_2 t e^{-4t} \to v_c^{z.i.}(0^+) = 5 = A_1$$

$$\dot{v}_c^{z.i.}(0^+) = -32 = -4A_1 + A_2$$

$$\boxed{v_c^{z.i.}(t) = 5e^{-4t} - 12te^{-4t} \text{ Volts}}$$

c) $R = \frac{1}{3}$, Find $v_c(t)$, $t > 0$

$$\left.\begin{array}{l} \alpha = 3 \\ \omega_0 = 4 \end{array}\right\} \alpha < \omega_0 \to \text{underdamped}$$

$$\lambda_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$

$$= -3 \mp j\sqrt{7}$$

$\lambda_{1,2}$: natural frequencies

From $t = 0^+$ Analysis

$$v_c^{z.i.}(0^+) = 5 \text{ Volts}$$

$$\dot{v}_c^{z.i.}(0^+) = -22 \text{ Volts/sec}$$

$$v_c^{z.i.}(t) = \underset{?}{A_1} e^{\lambda_1 t} + \underset{?}{A_2} e^{\lambda_2 t}, \quad t > 0 \quad \text{I.C. at } t = 0^+ \text{ given}$$

$$\left.\begin{array}{l} v_c^{z.i.}(0^+) = A_1 + A_2 = 5 \\ \dot{v}_c(0^+) = \lambda_1 A_1 + \lambda_2 A_2 = -22 \end{array}\right\} \begin{bmatrix} 1 & 1 \\ 3+j\sqrt{7} & 3-j\sqrt{7} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -22 \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} 3-j\sqrt{7} & -1 \\ -3-j\sqrt{7} & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -22 \end{bmatrix}$$

$$\Delta = -j2\sqrt{7}$$

$$A_1 = \frac{5}{2} + j\frac{\sqrt{7}}{2}$$

$$A_2 = \frac{5}{2} - j\frac{\sqrt{7}}{2}$$

$$v_C(t) = \underbrace{\left(\frac{5}{2} + j\frac{\sqrt{7}}{2}\right)}_{A_1} e^{(\underbrace{-3+j\sqrt{7}}_{\lambda_1})t} + \underbrace{\left(\frac{5}{2} - j\frac{\sqrt{7}}{2}\right)}_{A_2} e^{(\underbrace{-3-j\sqrt{7}}_{\lambda_2})t}$$

Euler's formula: $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$ $\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

Taylor series ⟶ $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

$$v_c(t) = z + z^* \;;\; z = \left(\frac{5}{2} + j\frac{\sqrt{7}}{2}\right)e^{(-3+j\sqrt{7})t}$$

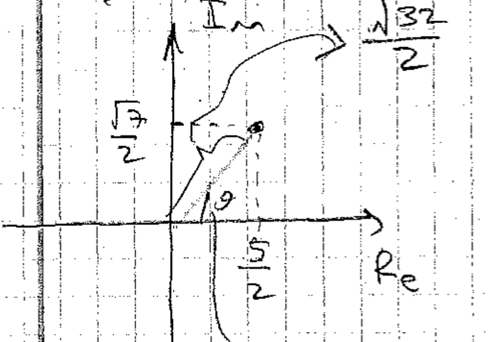
$$= 2R_e\{z\}$$

$$= 2\,\mathrm{Re}\left\{\left(\frac{5}{2} + j\frac{\sqrt{7}}{2}\right)e^{(-3+j\sqrt{7})t}\right\}$$

$$= \cancel{2}\,\mathrm{Re}\left\{\frac{\sqrt{32}}{\cancel{2}}\,e^{-j\measuredangle \tan^{-1}\sqrt{7}/5}\,e^{-3t}\,e^{j\sqrt{7}t}\right.$$

$$= \sqrt{32}\,e^{-3t}\,\mathrm{Re}\left\{e^{j(\sqrt{7}t + \tan^{-1}\frac{\sqrt{7}}{5})}\right\}$$

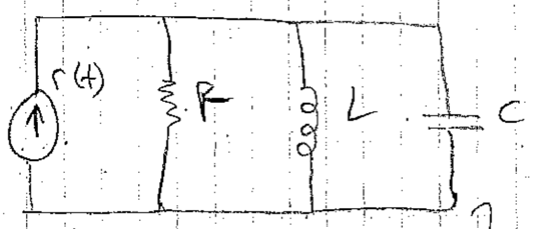
$$\boxed{v_c(t) = \sqrt{32}\,e^{-3t}\cos\left(\sqrt{7}t + \tan^{-1}\left(\frac{\sqrt{7}}{5}\right)\right)\text{Volts}}$$



$$\theta = \tan^{-1}\left(\frac{\sqrt{7}}{5}\right)$$

zero-state response

Ramp Response



$v_c(0^-) = V_o = 0$
$I_L(0^-) = I_o = 0$

} Zero-state response requires zero I.C. (zero initial energy)

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)v_c(t) = \frac{1}{C}\frac{d}{dt}\{\underbrace{i_s(t)}\},\quad t>0$$

$r(t) = t\,u(t) = t \;, t>0$

$$= \frac{1}{C}\cdot 1$$

$$V_c^{ramp}(t) = \underbrace{A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}}_{\text{Homogeneous solution}} + \underbrace{(\text{Particular Solution})}_{P}$$

Generic Solution for ramp input. Any A_1 and A_2 of this soln.

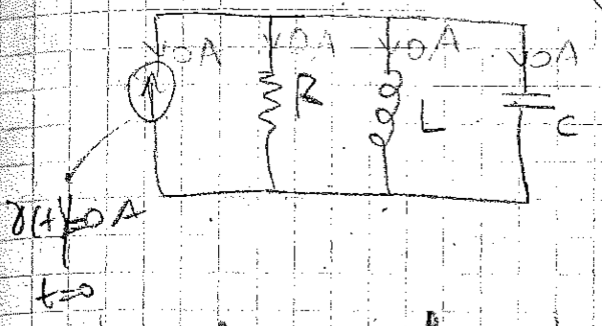
$$= \underbrace{A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}} + P \qquad t>0$$

? ? ⟵ Insert into dif. equation to find

A_1 and A_2 are to be found from $t=0^+$ I.C.'s

$$P\cdot\frac{1}{LC} = \frac{1}{C}\cdot 1 \rightarrow P = L$$

(At $t=0^+$)

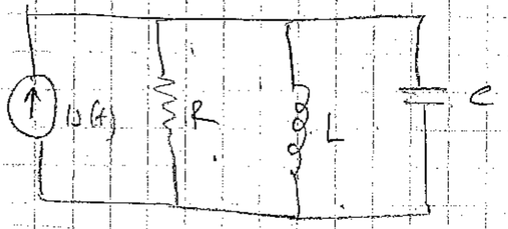


$v_c^{Ramp}(0^+)=0$

$\dot{v}_c^{Ramp}(0^+)=0$

$\rightarrow$ Select A_1 and A_2 in $A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t}+P$ s.t. $t=0^+$ cond. are satisfied

Zero-State Response for step input (Step response) :



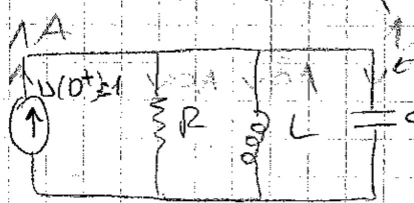
$v_c(0^-)=0$

$I_L(0^-)=0$

At $t=0^+$, $v_c(0^+)=0$

$I_L(0^+)=0$ $\longrightarrow$

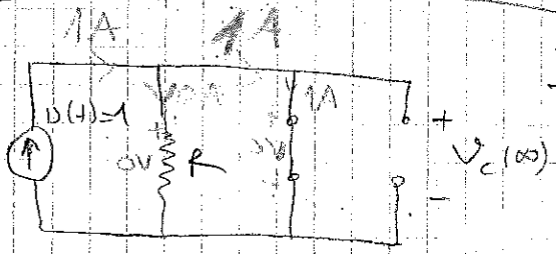
At $t=0^+$



$C\dot{v}_c(0^+)=1$

$\rightarrow \dot{v}_c(0^+)=\frac{1}{C}$

$\rightarrow v_c(0^+)=0$



$t\to\infty$

$v_c(\infty)=0$

$v_c^{step}(t)=A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t}+v_c(\infty)\rightarrow 0$

$v_c^{step}(t)=A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t},\ t>0$

A_1 and A_2 are selected to meet I.C. at $t=0^+$

For the same circuit

$I_L(t)=D_1e^{\lambda_1 t}+D_2e^{\lambda_2 t}+\underbrace{I_L(\infty)}$

$=D_1e^{\lambda_1 t}+D_2e^{\lambda_2 t}+\overset{1}{1}$

D's are found from $I_L(0^+)$, $\dot{I}_L(0^+)$

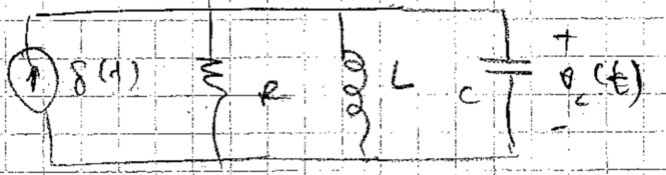
$I_L(0^+) \Downarrow =I_L(0^-)=0$

$L\dot{I}_L(0^+)=v_L(0^+)=0$

$\dot{I}_L(0^+)=0$

Zero-State Solutions (cont'd)

Impulse Response



Let's find the response to $\delta(t)$ for the output $v_c(t)$, i.e. $v_c^{\text{impulse}}(t)$.

$$\left(D^2+\frac{1}{RC}D+\frac{1}{LC}\right)v_c(t)=\frac{1}{C}\frac{d}{dt}\left\{i_s(t)\right\}=\frac{1}{C}\underset{\nwarrow \text{doublet} ☆}{\dot{\delta}(t)}$$

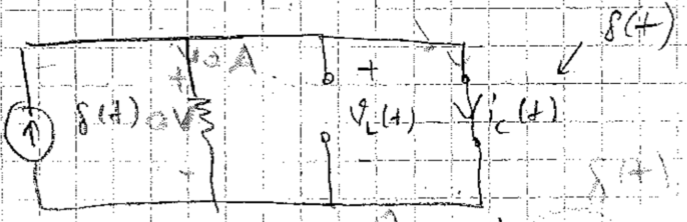
$$v_c(0^-)=0 \qquad I_L(0^-)=0$$

$$\left(D^2+\frac{1}{RC}D+\frac{1}{LC}\right)v_c(t)=0, \quad t>0$$

$$v_c(0^+)=\frac{1/C}{}$$

$$\dot{v}_c(0^+)=\underline{\;?\;}$$

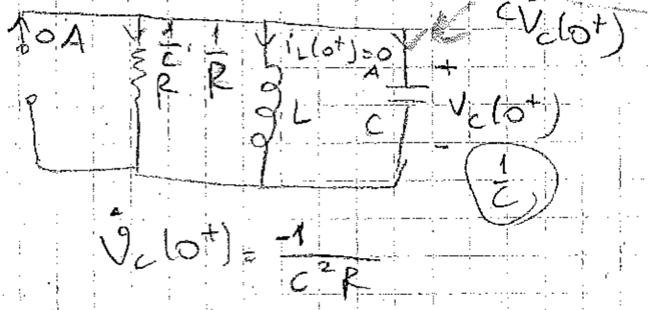
$$0^- < t < 0^+$$



$$v_c(0^+)=\overset{0}{v_c(0^-)}+\frac{1}{C}\int_{0^-}^{0^+}\overset{\delta(t)}{i_c(\tau)}\,d\tau=\frac{1}{C}$$

$$i_L(0^+)=\overset{0}{i_L(0^-)}+\frac{1}{L}\int_{0^-}^{0^+}\underset{\uparrow\,0}{v_L(\tau)}\,d\tau=0$$

At $t=0^+$



$$\dot{v}_c(0^+)=\frac{-1}{C^2R}$$

then for $t>0$

$$\left(D^2+\frac{1}{RC}D+\frac{1}{LC}\right)v_c^{\text{impulse}}(t)=0, \quad t>0$$

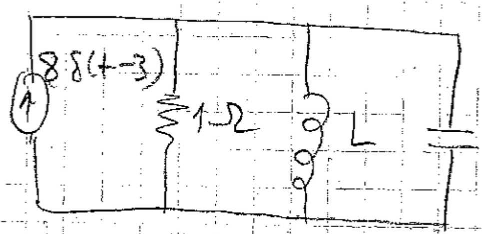
$$v_c(0^+)=\frac{1}{C}$$

$$\dot{v}_c(0^+)=-\frac{1}{C^2R}$$

$$\boxed{v_c^{\text{impulse}}(t)=A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t}, \quad t>0}$$

☆ Set A_1 and A_2 to match initial conditions

EX: ZPS – VII
Problem ⑫



$\vartheta_c(0)=0$
$I_L(0)=0$

Given $\vartheta_c(3^+)=1V$ and the circuit its critically damped:
a) Find L, C
b) Find $\vartheta_c(t)$, $t>0$.

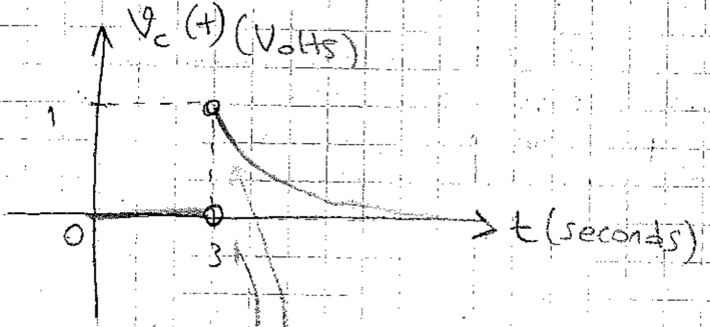
Critically damped

$$\left(\begin{matrix}\alpha=\omega_0\\ \vartheta_c(t)=A_1e^{\lambda t}+A_2te^{\lambda t}\end{matrix}\right)\longrightarrow\left(D^2+\underset{\overset{\uparrow}{2\alpha}}{\frac{1}{RC}}D+\underset{\overset{\uparrow}{\omega_0^2}}{\frac{1}{LC}}\right)\vartheta_c(t)=\frac{1}{C}\frac{d}{dt}\left[\int i_s(t)\right]$$

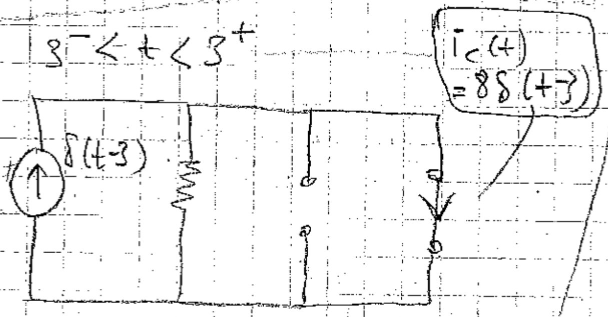
For $\alpha=\omega_0\Rightarrow\frac{1}{2RC}=\sqrt{\frac{1}{LC}}$

$1\Omega\longrightarrow R$

$\longrightarrow L=4C$



$3^-<t<3^+$



Don't forget the circles.

$$V_c(3^+)=\vartheta_c(3^-)+\frac{1}{C}\int_{3^-}^{3^+}i_c(\tau)d\tau=\frac{8}{C}\nwarrow\vartheta_c(3^+)=1\longrightarrow C=8F$$

$\downarrow$

$L=32H\longrightarrow$

$$\longrightarrow 2\alpha=\frac{1}{RC}=\frac{1}{8}$$

$$\omega_0^2=\frac{1}{256}\longrightarrow\omega_0=\frac{1}{16}\longrightarrow\left(D^2+\frac{1}{8}D+\frac{1}{256}\right)\vartheta_c(t)=0,\ t>3$$

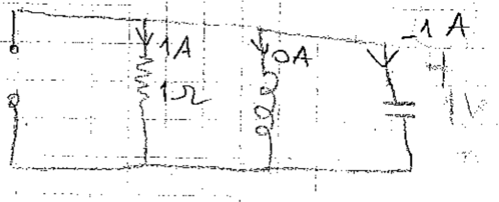
$$\boxed{\begin{matrix}\vartheta_c(t)=A_1e^{-\frac{t}{16}}+A_2te^{-\frac{t}{16}},\ t>3\\ \vartheta_c(3^+)=1\\ \dot{\vartheta}_c(3^+)=-\frac{1}{8}\text{ Volts/sec}\end{matrix}}$$

$$\lambda^2+\frac{1}{8}\lambda+\frac{1}{256}=0$$

$$\lambda=\left\{-\frac{1}{16}\right\}$$

$c\cdot\dot{\vartheta}_c(3^+)$

At $t=3^+$



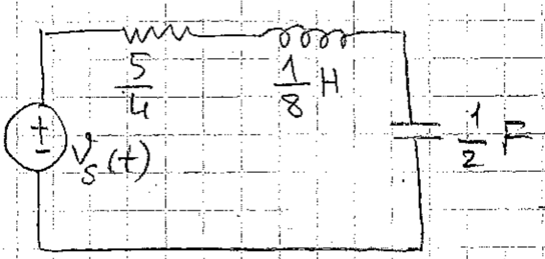
$$v_c(t) = B_1 e^{-\frac{(t-3)}{16}} + B_2 (t-3) e^{-\frac{(t-3)}{16}}, \quad t>3$$

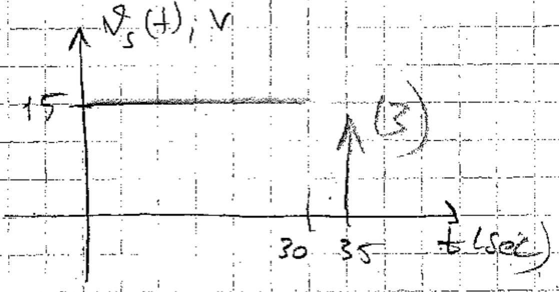
$$v_c(3^+) = 1 \longrightarrow B_1 = 1$$

$$\dot{v}_c(3^+) = \boxed{-\frac{1}{8}} \longrightarrow -\frac{1}{16} B_1 + B_2 = \boxed{-\frac{1}{8}} \longrightarrow B_2 = -\frac{1}{16}$$

$$\boxed{v_c(t) = 1 e^{-(t-3)/16} - \frac{1}{16}(t-3) e^{-(t-3)/16}, \quad t>3}$$

Ex:





$v_c(0^-) = -3V$

$i_L(0^-) = 0 A$

Find $v_c(t)$

Solution method: zero-state

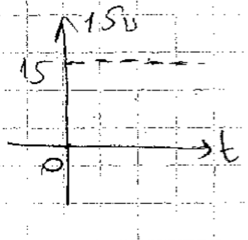
① Decompose $v_s(t)$ into elementary functions and find zero-state response to each elementary function and find the sum of the zero-state response.

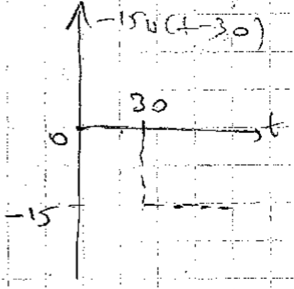
zero-input ② Find zero-input solution

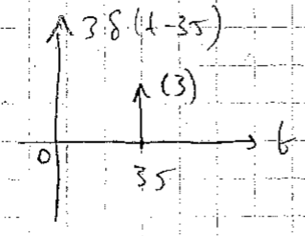
③ $v_c^{complete}(t) = v_c^{z.i.}(t) + v_c^{z.s.}(t)$

$$v_s(t) = 15u(t) - 15u(t-30) + 3\delta(t-35)$$

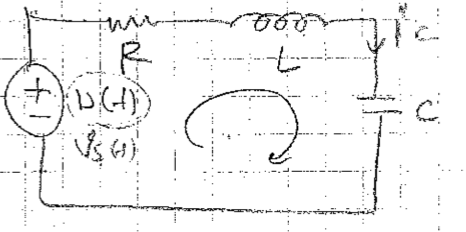
①







Step Response:



$$-v_s(t) + R\cdot i_c + v_L(t) + v_c(t) = 0$$

$$i_c = C\dot{v}_c$$

$$v_L(t) = L\frac{di_c(t)}{dt} = LC\ddot{v}_c(t)$$

$$[(LC)D^2 + (RC)D + 1]\, v_c(t) = v_s(t)$$

2α $\qquad$ ω_0^2

$$\boxed{\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) v_c(t) = \frac{v_s(t)}{LC}, \quad t>0}$$

$2\alpha = \frac{\overset{1/4}{R}}{\underset{1/8}{L}} \Rightarrow \alpha = 5$ $\omega_0^2 = \frac{1}{LC} = 16 \rightarrow \omega_0 = 4$

$\alpha > \omega_0 \rightarrow$ Overdamped response

$$\lambda_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$
$$= \{-2, -8\}$$

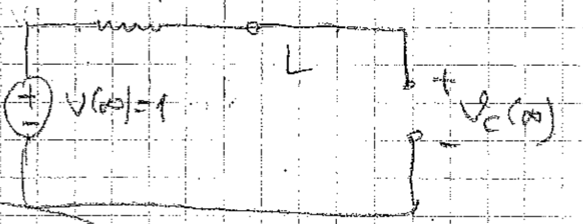
$\lambda^2 + 10\lambda + 16 = 0$
$(\lambda+8)(\lambda+2) = 0$

$v_c(t) = (A_1 e^{-2t} + A_2 e^{-8t}) + (\text{particular soln.})$

Then for step-response $v_s(t) = u(t)$ and $\left.\begin{matrix} v_c(0^-) = 0 \\ I_L(0^-) = 0 \end{matrix}\right\} \rightarrow$

$\rightarrow \left.\begin{matrix} v_c(0^+) = 0 \\ I_L(0^+) = 0 \end{matrix}\right\} \rightarrow v_c(\infty) = 1$

$t \rightarrow \infty$

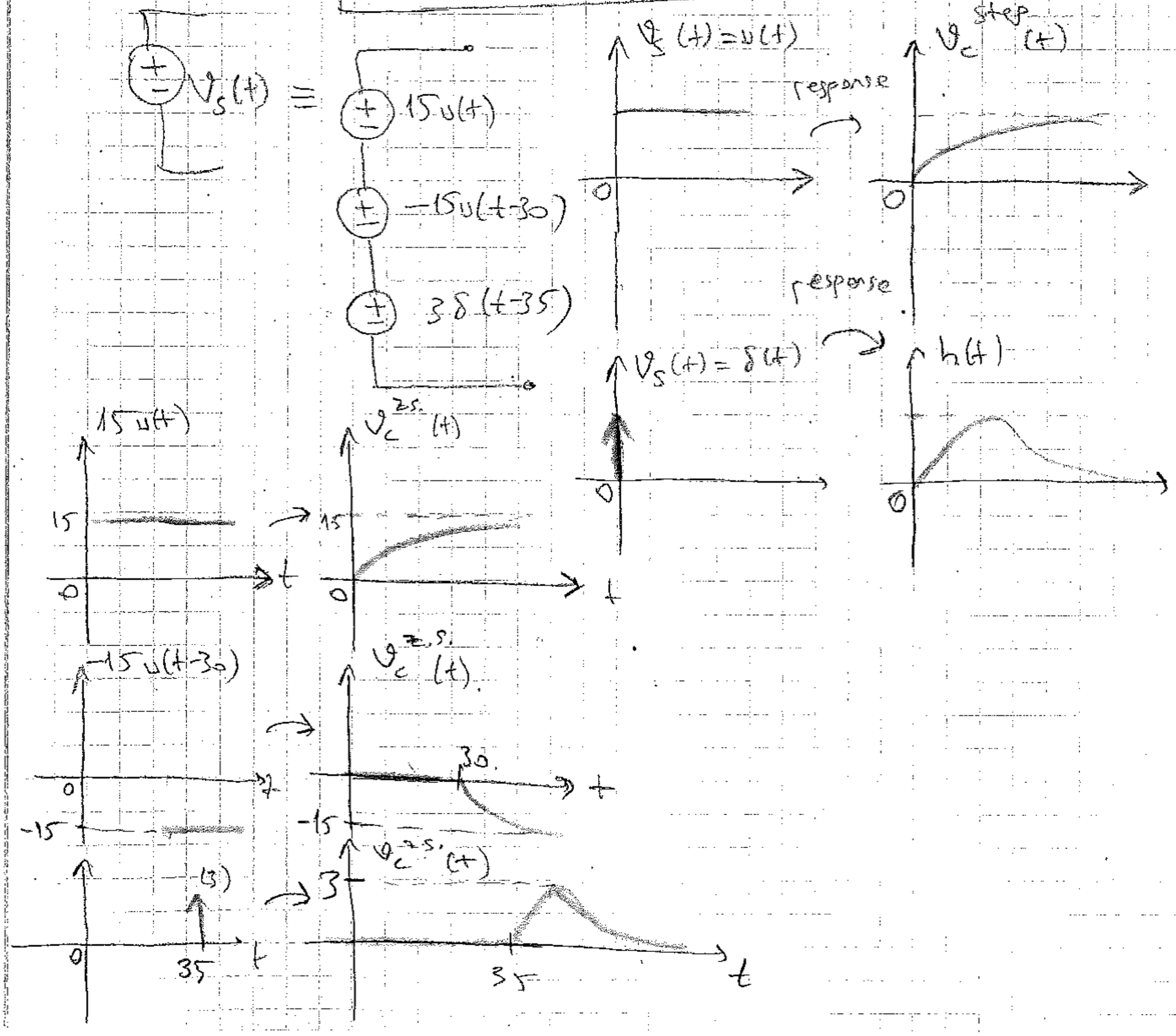


$$\rightarrow \boxed{v_c^{step}(t) = 1 - \frac{4}{3} e^{-2t} + \frac{1}{3} e^{-8t})u(t)}$$

Impulse response

$$\boxed{h(t) = \frac{d}{dt} v_c^{step}(t) = \frac{8}{3}\left(e^{-2t} - e^{-8t}\right)u(t)}$$

Impulse response

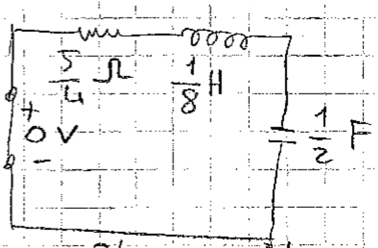


① $v_c^{zs}(t) = 15\, v_c^{step}(t) - 15\, v_c^{step}(t-30) + 3h(t-35)$

$$= 15\left(1 - \frac{4}{3}e^{-2t} + \frac{1}{3}e^{-8t}\right)u(t) - 15\left(1 - \frac{4}{3}e^{-2(t-30)} + \frac{1}{3}e^{-8(t-30)}\right)u(t-30)$$

$$+ 3\left(\frac{8}{3}\left(e^{-2(t-35)} - e^{-8(t-35)}\right)u(t-35)\right.$$

② Zero input



$\alpha = 5$

$\omega_0 = 4$

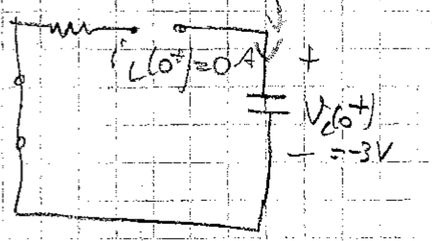
$\lambda_{1,2} = \{-2, -8\}$

$V_C(0^-) = -3V$

$I_L(0^-) = 0A$

$v_c^{z.i.}(t) = B_1 e^{-2t} + B_2 e^{-8t}, \ t > 0$

At $t = 0^+$ $C\dot{v}_c(0) = 0$



$v_c(0^+) = -3V$

$\dot{v}_c(0^+) = 0$

Finding B_1 and B_2

$\rightarrow B_1 + B_2 = -3$

$-2B_1 - 8B_2 = 0$

$B_1 = -4, \ B_2 = 1$

② $v_c^{z.i}(t) = \left(-4e^{-2t} + e^{-8t}\right)u(t)$

③ $\boxed{v_c^{compl.}(t) = v_c^{z.s.}(t) + v_c^{z.i.}(t)}$