EE202, Middle East Technical University, Dept. Electrical Engin 1

Nth Order Dynamic Circuits Lecture Notes of Kamil Nar (Spring 2010)

Generalized branch:

Graph theoretical node analysis

↓

we use generalized branches for voltage sources

① $\underline{\underline{A}} \cdot \underline{j} = 0$ KCL

② $\underline{J} = \underline{\underline{G}} \cdot \underline{v}$ Terminal eq

} $\underline{J}$ better be $\underline{i}$

③ $\underline{v} = \underline{\underline{A}}^T \underline{e}$ → node voltages (e_A, e_B) → Auxilary equation from graph

↓

Branch voltages

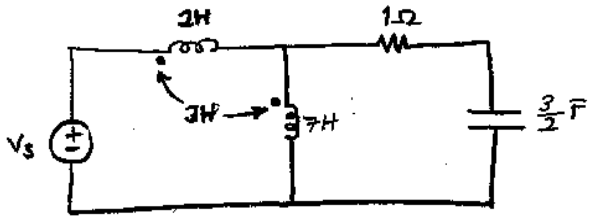
Inserting ③ into ②

$$\underline{J} = \underline{\underline{G}} \cdot \underline{\underline{A}}^T \cdot \underline{e}$$

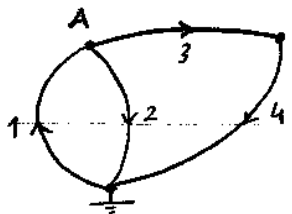
$$\underline{\underline{A}} \cdot \underline{J} = \underbrace{\underline{\underline{A}} \cdot \underline{\underline{G}} \cdot \underline{\underline{A}}^T}_{\text{known matrix}} \cdot \underline{e} = 0$$

$\underline{e}$ ↓ unknown vector

Ex



Graph theoretical node analysis:



$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$J_3 = \frac{V_3}{1\Omega} = V_3$$

$$J_4 = \frac{3}{2} \frac{d}{dt} V_4(t)$$

$$\begin{bmatrix} V_{2H}(t) \\ V_{7H}(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} d/dt \; i_{2H}(t) \\ d/dt \; i_{7H}(t) \end{bmatrix}$$

$$J_1 = i_{2H}(t)$$

$$J_2 = i_{7H}(t)$$

$$\begin{bmatrix} D\, i_{2H}'(t) \\ D\, i_{7H}(t) \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} V_{2H}(t) \\ V_{7H}(t) \end{bmatrix}$$

$$\begin{bmatrix} i_{2H}(t) - i_{2H}'(0) \\ i_{7H}(t) - i_{7H}(0) \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} \int_0^t V_{2H}(\tau)\,d\tau \\ \int_0^t V_{7H}(\tau)\,d\tau \end{bmatrix}$$

$$J_1 = i_{2H}'(0) + \frac{7}{5} D^{-1} V_{2H}(t) - \frac{3}{5} D^{-1} V_{7H}(t)$$

$$J_2 = i_{7H}(0) - \frac{3}{5} D^{-1} V_{2H}(t) + \frac{2}{5} D^{-1} V_{7H}(t)$$

$$V_{2H} = V_1 + V_S$$

$$V_{7H} = V_2$$

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{5}D^{-1} & -\frac{3}{5}D^{-1} & 0 & 0 \\ -3/5\, D^{-1} & 2/5\, D^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{2}D \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} i_{2H}'(0) \\ i_{7H}(0) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{5}D^{-1} \\ -3/5\, D^{-1} \\ 0 \\ 0 \end{bmatrix} V_S$$

$$\underline{J} = \underline{\underline{G}} \cdot \underline{\vartheta} + \underline{i}(0) + \underline{i_S}$$

$$\underline{\underline{A}} \cdot \underline{J} = \underline{\underline{A}} \cdot \underline{\underline{G}} \cdot \underline{\vartheta} + \underline{\underline{A}} \cdot \underline{i}(0) + \underline{\underline{A}} \cdot \underline{i_S} = 0$$

$$\underline{\underline{A}} \cdot \underline{\underline{G}} \cdot \underline{\underline{A}}^t \cdot \underline{e} + \underline{\underline{A}} \cdot \underline{i}(0) + \underline{\underline{A}} \cdot \underline{i_S} = 0$$

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\left(\underline{\underline{A}}\ \underline{\underline{G}}\ \underline{\underline{A}}^T\right)\begin{bmatrix} e_A \\ e_B \end{bmatrix} = -\begin{bmatrix} i_{7H}(0) - i_{2H}'(0) - 2 D^{-1} V_S(t) \\ 0 \end{bmatrix}$$

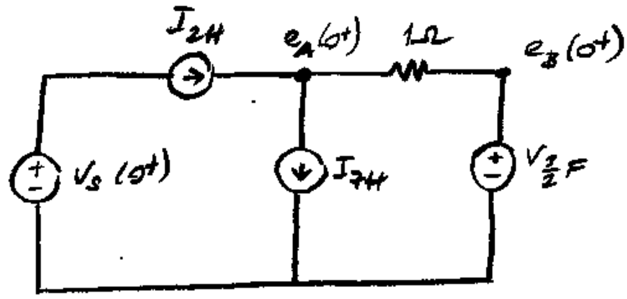
$$\begin{bmatrix} 3D^{-1} + 1 & -1 \\ -1 & 1 + \frac{3}{2}D \end{bmatrix}\begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} i_{2H}(0) - i_{7H}(0) + 2D^{-1} V_S(t) \\ 0 \end{bmatrix}$$

↳ integro-differential equation

Initial conditions for the solution of differential equation:

$i_{2H}'(0^-) = i_{2H}' \quad ; \quad i_{7H}'(0^-) = i_{7H}' \quad ; \quad V_{\frac{3}{2}F}(0^-) = V_{\frac{3}{2}F}$

At $\underline{t = 0^+}$



$$e_B(0^+) = v_{\frac{3}{2}F}(0^-)$$

$$e_A(0^+) = e_B(0^+) + i_{2H}(0^-) - i_{7H}(0^-)$$

KCL at e_B:

$$\frac{e_B - e_A}{1} + \frac{3}{2}\frac{d}{dt} e_B(t) = 0 \quad ✓$$

KCL at e_A:

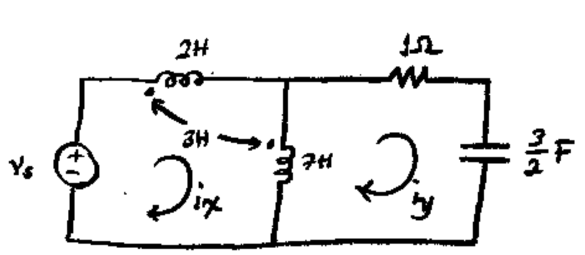
$$\frac{e_A - e_B}{1} + i_{7H}(t) - i_{2H}(t) = 0 \; \ldots \; (*)$$

$$\begin{bmatrix} i_{2H}(t) \\ i_{7H}(t) \end{bmatrix} = \begin{bmatrix} 7/5\, D^{-1} & -3/5\, D^{-1} \\ -3/5\, D^{-1} & 2/5\, D^{-1} \end{bmatrix} \begin{bmatrix} V_{2H}(t) \\ V_{7H}(t) \end{bmatrix} + \begin{bmatrix} i_{2H}(0^-) \\ i_{7H}(0^-) \end{bmatrix}$$

$$i_{7H}(t) - i_{2H}(t) = -2D^{-1}\underbrace{V_{2H}(t)}_{V_s(t) - e_A(t)} + D^{-1}\underbrace{V_{7H}(t)}_{e_A(t)} + i_{7H}(0^-) - i_{2H}(0^-)$$

$$= 3D^{-1} e_A - 2D^{-1} V_s(t) - i_{2H}(0) + i_{7H}(0) \rightarrow \text{into } (*)$$

<u>Ex</u>: (Mesh Analysis)



$$\begin{bmatrix} V_{2H} \\ V_{7H} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} \dot{i}_{2H} \\ \dot{i}_{7H} \end{bmatrix}$$

KVL around i_x:

$$-V_s + V_{2H}(t) + V_{7H}(t) = 0 \qquad i_{2H} = i_x$$

$$i_{7H} = i_x - i_y$$

$$-V_s + 5\dot{i}_x + 10\dot{i}_x - 10\dot{i}_y = 0$$

$$15\,D i_x - 10\,D i_y = V_s$$

KVL around i_y:

$$-V_{7H}(t) + 1\cdot i_y(t) + \underbrace{V_C(t)}_{V_C(0^-) + \frac{1}{C}\int_0^t \underbrace{i_c(\tau)}_{i_y(\tau)}\,d\tau} = 0$$

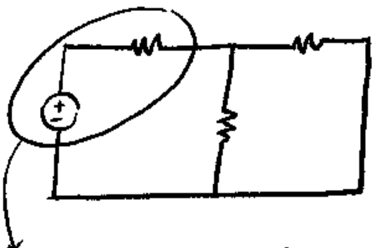
$$-10\,D i_x + 7\,D i_y + i_y + V_C(0^-) + \frac{2}{3}D^{-1} i_y(t) = 0$$

$$\begin{bmatrix} 15D & -10D \\ -10D & 7D + 1 + \frac{2}{3}D^{-1} \end{bmatrix} \begin{bmatrix} i_x(t) \\ i_y(t) \end{bmatrix} = \begin{bmatrix} V_s(t) \\ -V_c(0^-) \end{bmatrix}$$

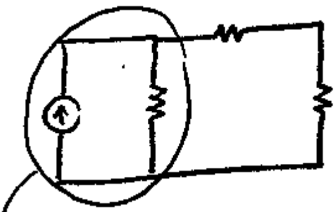
$$i_x(0^+) = i_{2H}(0^-)$$

$$i_y(0^+) = i_{2H}(0^-) - i_{7H}(0^-)$$

Generalized Branch: (when graph theoretical node / mesh is written)



Generalized branch for node analysis



Generalized branch for mesh analysis.

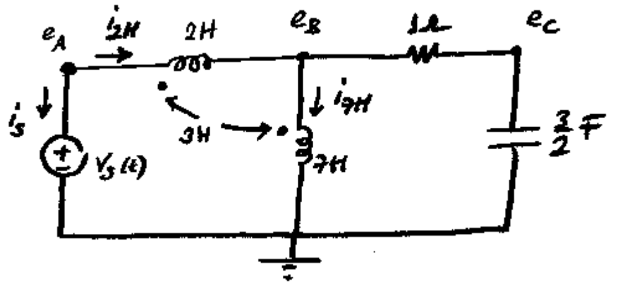
Modified Node Analysis (MNA)

In MNA, we write equations such that we end up with a differential equation instead of a integro-differential eqn. To do that:

① Assign <u>auxilary</u> variables (other than node voltage variables) for <u>inductors</u>, <u>voltage sources</u>

② write the KCL equations as in node analysis, use auxilary variables where necessary

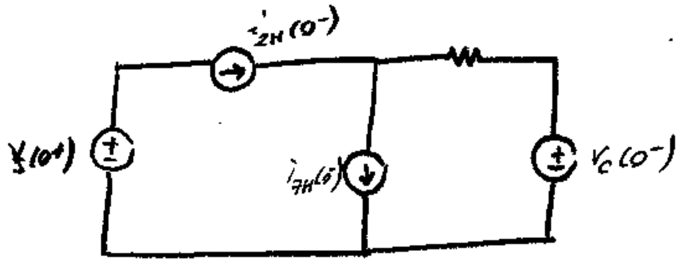
③ write additional equations for auxilary variables



$$
\begin{array}{l}
\text{KCL @ } e_A: \\
\phantom{\text{KCL @ }} e_B: \\
\phantom{\text{KCL @ }} e_C: \\
\\
\\
\\
\end{array}
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 \\
0 & -1 & 1+\frac{3}{2}D & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -2D & -3D \\
0 & 1 & 0 & 0 & -3D & -7D
\end{bmatrix}
\begin{bmatrix} e_A \\ e_B \\ e_C \\ i_s \\ i_{2H} \\ i_{7H} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ V_s \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} V_{2H} \\ V_{7H} \end{bmatrix} = \begin{bmatrix} e_A - e_B \\ e_B \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} D\, i_{2H} \\ D\, i_{7H} \end{bmatrix}
$$

<u>At $t = 0^+$</u>



$e_A(0^+) = V_s(0^+)$

$e_B(0^+) = V_c(0^-) + I_{2H}(0^-) - I_{7H}(0^-)$

$e_c(0^+) = V_c(0^-)$

$i_{2H}(0^+) = i_{2H}(0^-)$

$i_{7H}(0^+) = i_{7H}(0^-)$

$i_s(0^+) = -i_{2H}(0^-)$

State Equations

$$\underline{\dot{x}}(t) = \underline{\underline{A}} \cdot \underline{x}(t) + \underline{\underline{B}} \cdot \underline{u}(t)$$

Ex:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} V_s(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} i_s(t)}_{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} V_s(t) \\ i_s(t) \end{bmatrix}}$$

1st order matrix differential equation!

Solution is by Laplace transform, or eigen-decomposition of $\underline{\underline{A}}$, or through state-transition matrices, series expansion.

State variables: { Capacitor voltages, Inductor current }

Steps: ① Include all voltage sources in tree.
all current sources in co-tree

② Put maximum number of possible capacitors in tree.
minimum number of possible inductors

③ If there is a transformer, put only one - but one - branch of transformer in tree.

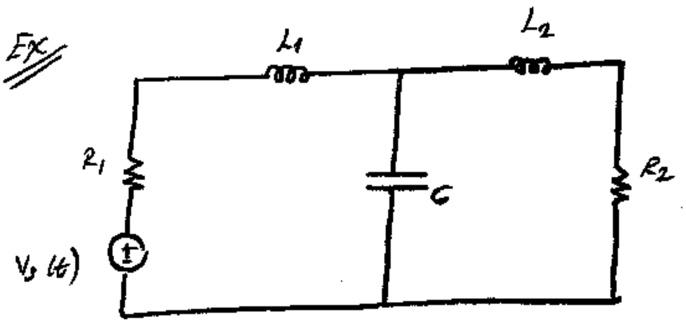
writing the equations:

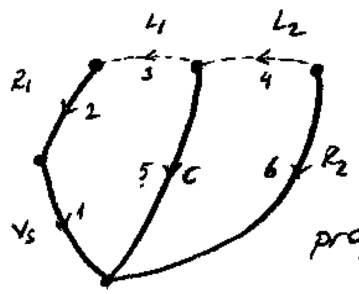
state variables: All capacitor voltages in the tree and all inductor currents in co-tree are state variables.

1) write fundamental-loop (KVL) for each inductor current in the co-tree.

2) write fundamental cut-set (KCL) for each capacitor voltage in the tree

Ex





proper tree

state variables = $\{ v_C(t), i_{L_1}(t) + i_{L_2}(t) \}$

Order of ... = # of state ... = 3

For L_1, KVL:

$$V_3 + V_2 + V_4 - V_5 = 0 \quad \text{(Fundamental Loop for } L_1 \text{ branch)}$$

$$L \cdot \dot{i}_{L_1} + R_1 \cdot i_{R_1} + V_S - \vartheta_C = 0$$

↳ not a state var. ⟶ express in terms of state var.

⇒ write fund. cut-set for i_{R_1} branch ☆

$$i_{R_1} = i_{L_1}$$

$$\dot{i}_{L_1}(t) = -\frac{R_1}{L_1} i_{L_1}(t) + \frac{1}{L_1} \vartheta_c(t) - \frac{\vartheta_s(t)}{L_1}$$

For L_2, KVL:

$$V_{L_2} + \vartheta_C - \vartheta_{R_2} = 0$$

$$L_2 \cdot \dot{i}_{L_2} + \vartheta_C - R_2 \cdot \underset{\substack{\downarrow \\ -i_{L_2}}}{i_{R_2}} = 0$$

$-i_{L_2}$ from fundamental cut-set

$$\dot{i}_{L_2}(t) = -\frac{R_2}{L_2} i_{L_2}(t) - \frac{1}{L_2} \vartheta_c(t)$$

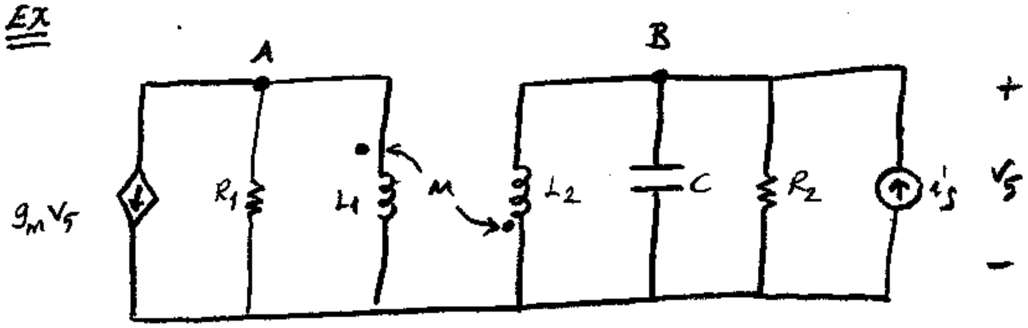
For C, KCL:

$$i_C - i_{L_2} + i_{L_1} = 0$$

$$\dot{V}_c = -\frac{1}{C} i_{L_1} + \frac{1}{C} i_{L_2}$$

$$\begin{bmatrix} \dot{\vartheta}_c(t) \\ \dot{i}_{L_1}(t) \\ \dot{i}_{L_2}(t) \end{bmatrix} = \begin{bmatrix} 0 & -1/C & 1/C \\ 1/L_1 & -R_1/L_1 & 0 \\ -1/L_2 & 0 & -R_2/L_2 \end{bmatrix} \begin{bmatrix} \vartheta_c(t) \\ i_{L_1}(t) \\ i_{L_2}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1/L_1 \\ 0 \end{bmatrix} \vartheta_s(t)$$

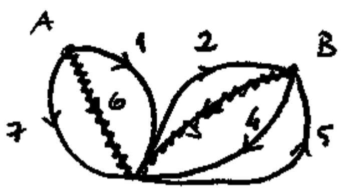
EX



$V_C(0^-) = \vartheta_0$

$I_{L_1}(0^-) = I_0^{L_1}$

$I_{L_2}(0^-) = I_0^{L_2}$



Current sources (dependent ones, too) are not included in tree.

State Eqn:

Fund. cut-set of capacitor branch:

$$i_3 + i_4 - i_5 - i_2 = 0$$

$$C\cdot \dot{v}_C + \frac{v_4}{R_2} - i_s - i_{L_2} = 0$$

$\xrightarrow{}$ Fund. loop $\longrightarrow$ $v_4 = v_C$

$$\rightarrow \boxed{\dot{v}_C(t) = \frac{1}{C}\, i_{L_2}(t) - \frac{1}{R_2\cdot C}\, v_C(t) + \frac{i_s(t)}{C}}$$

$$\begin{bmatrix} v_{L_1}(t) \\ v_{L_2}(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} d/dt\ i_{L_1}(t) \\ d/dt\ i_{L_2}(t) \end{bmatrix}$$

$\hookrightarrow$ things we are looking for!

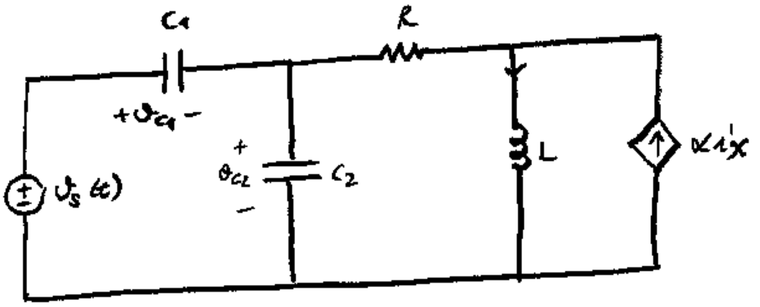
$$\rightarrow \begin{bmatrix} d/dt\ i_{L_1}(t) \\ d/dt\ i_{L_2}(t) \end{bmatrix} = \frac{1}{L_1 L_2 - M^2} \begin{bmatrix} L_2 & -M \\ -M & L_1 \end{bmatrix} \begin{bmatrix} V_{L_1}(t) \\ V_{L_2}(t) \end{bmatrix}$$

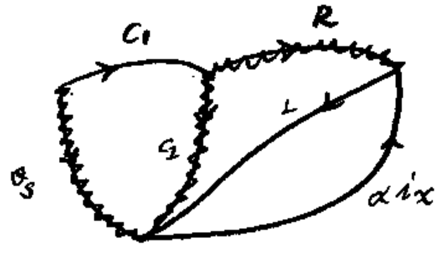
$$V_{L_1}(t) \xrightarrow{\text{fund. loop}} V_{R_1} \xrightarrow{\text{term. eqn.}} i_{R_1}\cdot R_1 \xrightarrow{\text{fund. cutset}} R_1\left(-g_m v_5 - i_{L_1}\right)$$

$$\xrightarrow{\text{fund. loop}} R_1\left(-g_m v_C - i_{L_1}\right)$$

$$V_{L_2}(t) \xrightarrow{\text{fund loop}} -v_C$$

Ex





State variables: $\{ v_{C_2} , i_L \}$ ★

$v_{C_2} \swarrow$ capacitors on tree

$i_L \searrow$ Inductors on co-tree

→ Fund. cut-set of C_2 branch:

$$-i_{C_1} + i_{C_2} + i_L - \alpha\, i_x = 0$$

$$-C_1 \cdot \dot{v}_{C_1} + C_2\, \dot{v}_{C_2} + i_L - \alpha\, C_2 \cdot \dot{v}_{C_2} = 0$$

$$-C_1 (\dot{v}_S - \dot{v}_{C_2}) + C_2 (1-\alpha)\, \dot{v}_{C_2} + i_L = 0$$

$$\boxed{\dot{v}_{C_2}(t) = \frac{-1}{C_1 + C_2(1-\alpha)}\, i_L + \frac{C_1}{C_1 + C_2(1-\alpha)}\, \dot{v}_S(t)}$$

→ Fund. Loop for L branch:

$$v_L = -v_R + v_{C_2}$$

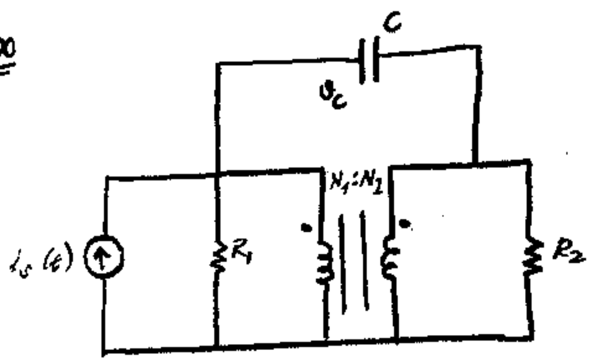
$$L \cdot \dot{i}_L \qquad i_R \cdot R \overset{F.C.}{=} (i_L - \alpha\, i_x) R = (i_L - \alpha\, C_2\, \dot{v}_{C_2}) R$$

from 1st state eqn.

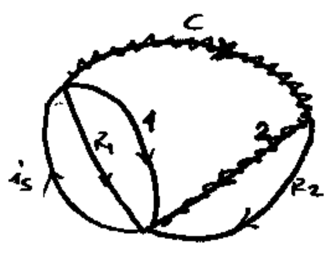
$$\boxed{L \cdot \dot{i}_L = -R \cdot i_L + R \cdot C_2 \cdot \alpha \left(\frac{-i_L}{C_1 + C_2(1-\alpha)} + \frac{C_1}{C_1 + C_2(1-\alpha)}\, \dot{v}_S(t) \right) + v_{C_2}}$$

Note: 3 dynamic elements but 2 state variables!

<u>EX</u>



$v_C(0^-) = v_0$



$$\frac{v_1}{v_2} = \frac{N_1}{N_2} \quad ; \quad \frac{i_1}{i_2} = \frac{-N_2}{N_1}$$

$i_C + i_1 + i_R - i_S = 0$

$C\cdot\dot{v}_C - \frac{N_2}{N_1} i_2 + \frac{v_{R_1}}{R_1} - i_S = 0$

F.L. $v_C + v_2 = v_C + v_1 \cdot \frac{N_2}{N_1} = v_1 \longrightarrow v_1 = \frac{v_C}{1 - N_2/N_1} = \frac{N_1 v_C}{N_1 - N_2}$

$$v_2 = \frac{N_2 v_C}{N_1 - N_2}$$

F.C.

$i_2 = -i_{R_2} - i_1 - i_{R_1} + i_S = -\frac{v_{R_2}}{R_2} + \frac{N_2}{N_1} i_2 - \frac{v_{R_1}}{R_1} + i_S$ ...

2nd try

$C\cdot\dot{v}_C = i_S - i_{R_1} - i_1$

State var = $\{ v_C \}$

$v_1 = \frac{v_C \cdot N_1}{N_1 - N_2}$

Fund. Cut-set for 2:

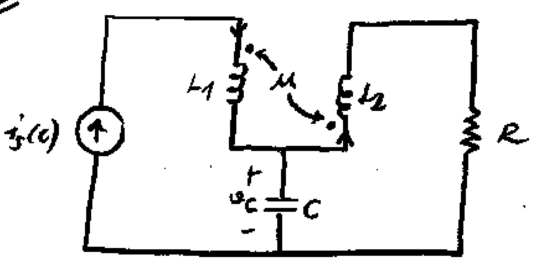
$i_2 = i_S - i_{R_1} - i_1 - i_{R_2}$

$i_2 = i_S - \frac{v_1}{R_1} - i_1 - \frac{v_2}{R_2}$

$-\frac{N_1}{N_2} i_1 = i_S - \frac{v_C}{R_1} \cdot \frac{N_1}{N_1 - N_2} - i_1 - \frac{v_C N_2}{R_2 \cdot (N_1 - N_2)}$

$$i_1 = \frac{i_S - \frac{v_C}{N_1 - N_2}\left(\frac{N_1}{R_1} + \frac{N_2}{R_2}\right)}{1 - N_1/N_2}$$

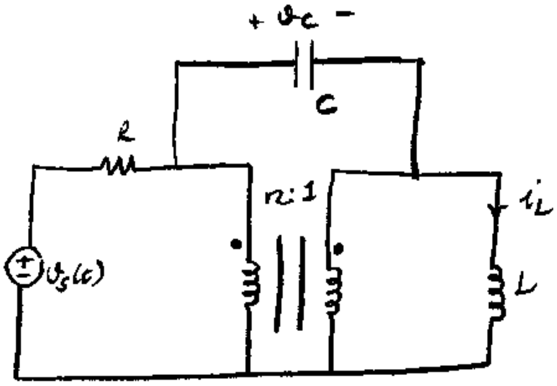
EX



Solution:

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_{L_2} \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L_2 & -R/L_2 \end{bmatrix} \begin{bmatrix} v_C \\ i_{L_2} \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} i_S + \begin{bmatrix} 0 \\ M/L_2 \end{bmatrix} \dot{i}_S$$

Ex



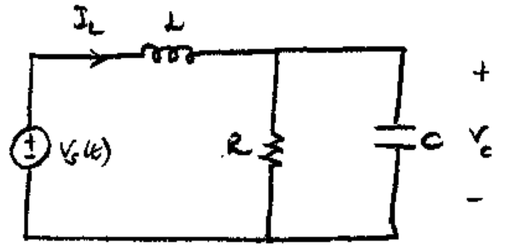
$$\begin{bmatrix} \dot{\vartheta}_c \\ \dot{I}_L \end{bmatrix} = \begin{bmatrix} \dfrac{-n^2}{(n-1)^2 RC} & \dfrac{-1}{(n-1)C} \\ \dfrac{1}{(n-1)L} & 0 \end{bmatrix} \begin{bmatrix} \vartheta_c \\ I_L \end{bmatrix} + \begin{bmatrix} \dfrac{1}{(n-1)RC} + \dfrac{1}{RC} \\ 0 \end{bmatrix} \vartheta_s(t)$$

Time - Invarience

when input is shifted in time, if output is also shifted system is <u>time-invariant</u>

Application : $\underset{\substack{\text{multiple} \\ R,\ \frac{1}{R}}}{\underset{\downarrow}{R}},\ \underbrace{L,\ C}_{d/dt,\ \int_{-\infty}^{t}}$

Solution of Nth order Dif eqn:



$\vartheta_c(0^-) = \vartheta_0$

$I_L(0^-) = I_0$

Node analysis:

$$C \cdot \dot{\vartheta}_c + \frac{\vartheta_c}{R} - I_L = 0$$

$$I_L(0^-) + \frac{1}{L} \int_0^t [\vartheta_s(\tau) - \vartheta_c(\tau)]\, d\tau$$

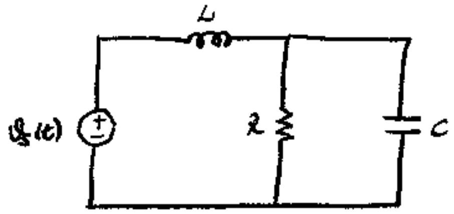
d/dt

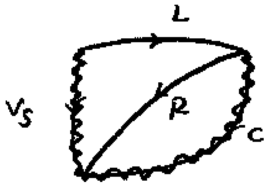
$$C \cdot \ddot{\vartheta}_c + \frac{\dot{\vartheta}_c}{R} + \frac{\vartheta_c - \vartheta_s(t)}{L} = 0 \longrightarrow \left(D^2 + \frac{1}{RC} D + \frac{1}{LC}\right) \vartheta_c(t) = \frac{\vartheta_s(t)}{LC}$$

$\vartheta_c(0^-) = \vartheta_0$

$\dot{\vartheta}_c(0^-) = \dfrac{i_c(0^-)}{C} = \left[I_0 - \dfrac{\vartheta_0}{R}\right] \cdot \dfrac{1}{C}$

State Eqn:





Cut-set for cap. branch: $C\dot{v}_c = -\frac{v_c}{R} + I_L$

Fund. loop for ind. branch: $L \cdot \dot{I}_L = v_s(t) - v_c$

$$\begin{bmatrix} \dot{v}_c(t) \\ \dot{I}_L(t) \end{bmatrix} = \begin{bmatrix} -1/RC & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_c(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v_s(t)$$

1 st order matrix diff. eqn = 2nd order scalar diff. eqn

Take another derivative of $\dot{v}_c(t)$ equation:

$$\ddot{v}_c(t) = -\frac{1}{RC}\dot{v}_c(t) + \frac{1}{C}\dot{I}_L(t)$$

$$D^2 v_c(t) = -\frac{1}{RC} D v_c - \frac{1}{LC} v_c + \frac{1}{LC} v_s(t)$$

$$\left(D^2 + \frac{1}{RC} D + \frac{1}{LC}\right) v_c(t) = \frac{1}{LC} v_s(t)$$ $\longrightarrow$ same eqn. with the one gotten by node analysis

Using state eqn., initial conditions are already given!
No need for 0^+ analysis.

Solution of 2nd order scalar Diff. Eqn

$R = 1/3$; $C = 1$; $L = 1/2$

$$(D^2 + 3D + 2)\, v_c(t) = 0$$

$$v_c(0) = v_0$$

$$\dot{v}_c(0) = I_0 - 3v_0$$

$\lambda_{1,2} = \{-1, -2\} \longrightarrow$ modes of the system

$v_c(t) = \alpha \cdot e^{-t} + \beta\, e^{-2t}$ $\longrightarrow$ zero-input response: superposition of 2 modes

$\alpha \cdot e^{-t}$ $\swarrow$ a mode of the circuit

$\beta\, e^{-2t}$ $\downarrow$ 2nd mode of the circuit.

How do you excite only 1 mode?

$\vartheta_c(0) = \vartheta_0$

$\dot{\vartheta}_c(0) = I_0 - 3\vartheta_0$

$$\vartheta_c(t)\Big|_{t=0} = \alpha e^{-t} + \beta \cdot e^{-2t}\Big|_{t=0} = \alpha + \beta = \vartheta_0$$

$$\dot{\vartheta}_c(t)\Big|_{t=0} = -\alpha \cdot e^{-t} - 2\beta e^{-2t}\Big|_{t=0} = -\alpha - 2\beta = I_0 - 3\vartheta_0$$

Two single modes $\alpha \neq 0$, $\beta = 0 \longrightarrow \vartheta_0 = \alpha$

$I_0 - 3\vartheta_0 = -\alpha \longrightarrow I_0 = 2\alpha$

$\vartheta_0 = 1V$

$I_0 = 2A$

$\vartheta_c(t) = e^{-t}$ ($\alpha = 1$, $\beta = 0$)

$\alpha = 0$, $\beta \neq 0$ $\quad \left.\begin{array}{l} \vartheta_0 = \beta \\ I_0 = \beta \end{array}\right\} \quad \vartheta_c(t) = \beta \cdot e^{-2t}$

Natural frequencies: Roots of characteistic equation (λ_1, λ_2)

Natural response: When we have zero input response $\longrightarrow$ natural response

Mode: Individual components of natural response

Solution by State Eqn

$$\begin{bmatrix} \dot{\vartheta}_c(t) \\ \dot{I}_L(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} \vartheta_c(t) \\ I_L(t) \end{bmatrix} \qquad \begin{array}{l} \vartheta_c(0) = \vartheta_0 \\ I_L(0) = I_0 \end{array}$$

$$\underline{\dot{x}}(t) = \underline{A} \cdot \underline{x}(t) \quad ; \quad \underline{x}(0) = \underline{x_0}$$

Let $\underline{x}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t}$

$$\lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \cancel{e^{\lambda t}} = \underline{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \cancel{e^{\lambda t}}$$

$$(\lambda I - A) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \det(\lambda I - A) = 0$$

$\det(\lambda I - A)$ is called characteristic polynomial

roots of $|\lambda I - A| = 0$ are the eigenvalues of $\underline{A}$ matrix.

$$|\lambda I - A| = \begin{vmatrix} \lambda+3 & -1 \\ 2 & \lambda \end{vmatrix} = \lambda(\lambda+3)+2 = \lambda^2 + 3\lambda + 2 \longrightarrow \text{charac eqn.}$$

Then natural frequencies are the roots of

$$\lambda^2 + 3\lambda + 2 = 0 \qquad \lambda = \{-1, -2\}$$

$$\underline{x}(t) = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{-t}, \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} e^{-3t} \right\}$$

$$\underline{x}(t) = \underline{c} \cdot e^{-t} + \underline{d} \cdot e^{-2t}$$

$$\dot{\underline{x}}(t) = \underline{\underline{A}} \cdot \underline{x}(t)$$

$$\dot{\underline{x}}(t) = \lambda_1 \cdot e^{\lambda_1 t} \cdot \underline{c} + \lambda_2 \cdot e^{\lambda_2 t} \underline{d} = \underline{\underline{A}} \left(e^{\lambda_1 t} \cdot \underline{c} + e^{\lambda_2 t} \underline{d} \right)$$

To excite $e^{\lambda_1 t}$ mode, $\underline{d} = \underline{0}$

$$\underline{c} \cdot \lambda_1 \cdot \cancel{e^{\lambda_1 t}} = \underline{\underline{A}} \cdot \underline{c} \cdot \cancel{e^{\lambda_1 t}}$$

$$\underline{\underline{A}} \cdot \underline{c} = \lambda_1 \cdot \underline{c}$$

To excite a single mode or mode with natural frequency, λ_1 $v_c(0^-)$, $I_L(0^-)$ should be the eigenvector of A corresponding to λ_1.

$\underline{\underline{A}}$ matrix of state eqn. is fundamental for analysis; since its eigenvalues are the natural frequencies, the eigenvectors of $\underline{\underline{A}}$ are the required initial conditions to excite a mode.

To excite $\lambda_1 = -1$ natural freq:

$$\begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \underline{e_1} = \lambda_1 \cdot \underline{e_1}$$

$$(\lambda_1 \underline{\underline{I}} - \underline{\underline{A}}) \, \underline{e_1} = \underline{0} \qquad \begin{bmatrix} \lambda+3 & -1 \\ 2 & \lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$$

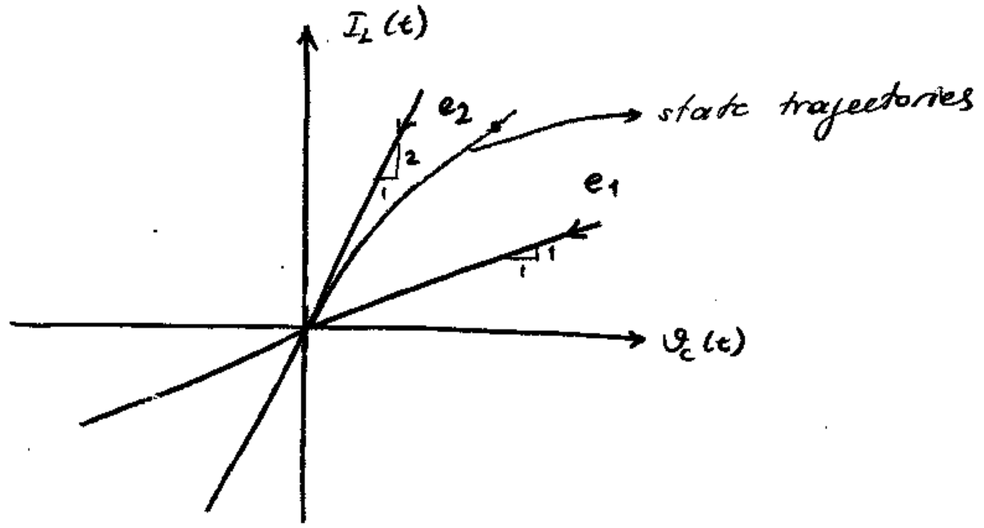
$$\lambda = -1 \longrightarrow \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} k \qquad \begin{bmatrix} v_c(0^-) \\ I_L(0^-) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} k \longrightarrow \text{excites } e^{-t} \text{ mode}$$

$\lambda = -2$ $\quad \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

eigen vector $= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} v_c(0^-) \\ I_L(0^-) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} k \longrightarrow$ excites e^{-2t} mode.

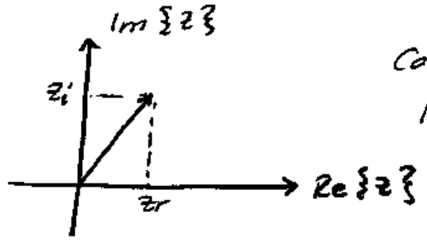


On Complex Exponential Function

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad x \in \mathbb{R}$$

$z \in \mathbb{C} \rightarrow$ complex field

$z = z_r + j \cdot z_i$, $\quad z_r, z_i \in \mathbb{R}$, $\quad j = \sqrt{-1}$ $\qquad \underline{z} = \begin{bmatrix} z_r \\ z_i \end{bmatrix}$



Complex plane

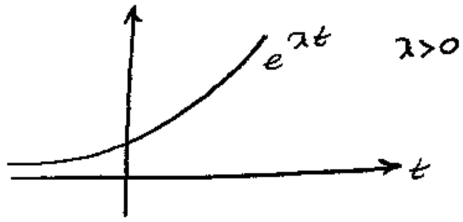
$$f(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$f(z)$: $\quad z$: purely real and positive $(z=2)$
negative $(z=-2)$

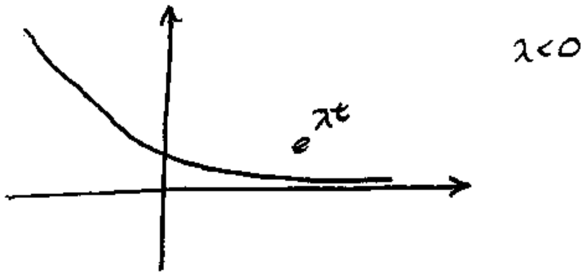
imaginary $(z = 2j)$

complex valued $(z = 1 + 2j)$

1) Purely real, positive valued



2) Purely real, negative valued



3) Purely imaginary

$f(t) = e^{J\omega t} \qquad \omega \in \mathbb{R},\ t \in \mathbb{R}$

$$e^{J\phi} = \sum_{k=0}^{\infty} \frac{(J\phi)^k}{k!} = \sum_{n=0}^{\infty} \frac{(J\phi)^{2n}}{2n!} + \sum_{n=0}^{\infty} \frac{(J\phi)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \phi^{2n}}{2n!} + J \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \phi^{2n+1}}{(2n+1)!}$$

$$= \cos\phi + J \cdot \sin\phi$$

$$e^{J\phi} = \cos\phi + J\cdot\sin\phi \qquad \text{Euler's formula}$$

$$e^{JA} \cdot e^{JB} = e^{J(A+B)}$$

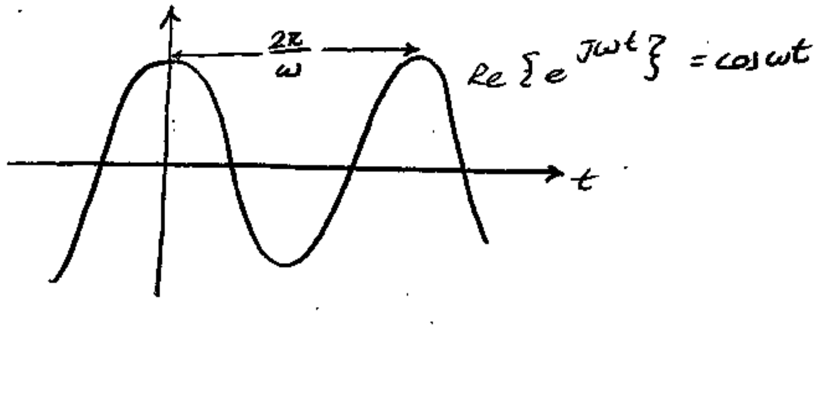
$$(\cos A + J\sin A)(\cos B + J\sin B) = \cos(A+B) + J\sin(A+B)$$

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

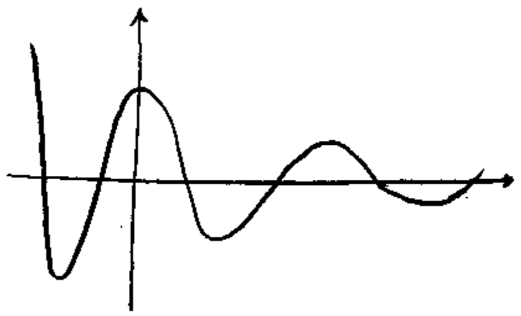
HW $\displaystyle\sum_{k=0}^{N-1} \cos(\omega k) \Rightarrow \cos(\omega N)$

Hint: $\cos(\omega k) = Re\{e^{J\omega k}\}$



4) $\lambda \in \mathbb{C}$

$f(t) = e^{\lambda t} = e^{\lambda_r \cdot t} \cdot e^{j \cdot \lambda_i t}$



$\lambda_r < 0$

Stability of LTI systems

Stability refers to keeping current and voltage of each branch bounded (finite) through circuit components.

1) zero-input
2) stability concept with inputs

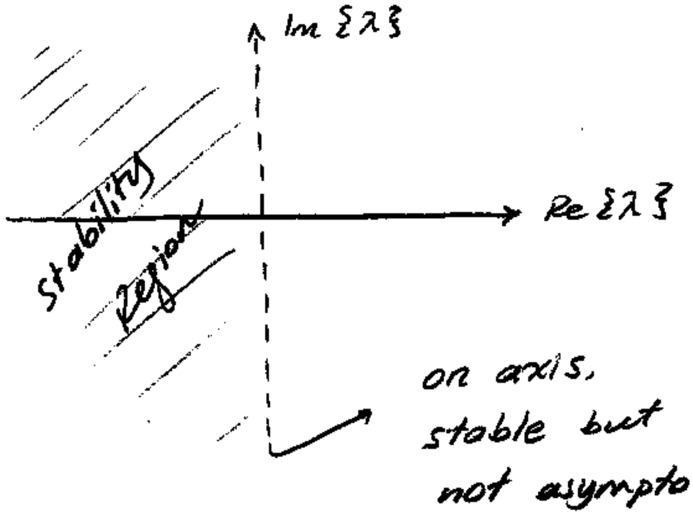
Given any initial condition, if each state goes to zero as $t \to \infty$; then such a system is called <u>asymptotically stable</u>

<u>Note:</u> There are more than one stability definition, such as, exponentially stable, asymptotically stable, Lyapunov stability etc.

For LTI systems, all stability definitions converge to the same condition.

LTI stable system $\longleftrightarrow$ natural freq. have negative valued real parts.

natural freq: roots of char. eq. (eigenvalues of $\underline{A}$)



λ : natural freq. of a system

Solution of Dynamic System with Inputs

Particular Solution of Diff. Eqn.

$(D^2+3D+2)\,x(t) = f(t)$

↳ forcing term (external input)

$f(t) = e^{st}$ (We limit our interest to exponential inputs. Later we will use more powerful techniques covering more function types)

Ex: $(D^2+3D+2)\,x(t) = e^{st}$

$x_p = A e^{st}$

$A(s^2+3s+2)\,e^{st} = e^{st}$

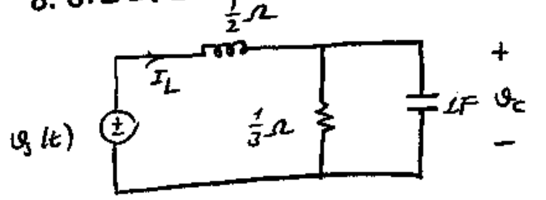
$A = \dfrac{1}{s^2+3s+2}$, provided that $s^2+3s+2 \neq 0$ ($s \notin$ natural freq.)

Important Remark:

For stable systems, as $t\to\infty$, the zero-input solution goes to zero and we are left with zero-state solution
then if the input is in the form of e^{st}, for LTI systems, the particular solution is also a complex exponential with the same exponent.

State eqns are favourite tool for analysis.
MNA, NA, MA require more number of eqns and initial conditions of the equations have to be found and integro differential

8.3.2010



$\vartheta_c(0^-)=\vartheta_0$, $I_L(0^-)=I_0$

$(D^2+3D+2)\,\vartheta_c(t) = 2\,\vartheta_s(t)$

$\vartheta^{complete}(t) = \vartheta^h(t) + \vartheta^p(t)$

ϑ^h: system determined, natural freq.

ϑ^p: input dependent

Homogeneous: $(D^2+3D+2)\,\vartheta_c(t) = 0 \rightarrow \lambda_{1,2} = \{-1,-2\}$ natural freq.

Particular solution :

Let $\vartheta_s(t) = e^{-5t} \longrightarrow$ case ①

Assume $\vartheta_p(t) = A \cdot e^{-5t}$

$$(s^2 + 3s + 2) A \cdot e^{st} = 2 e^{st} \longrightarrow A = \frac{2}{s^2+3s+2}, \quad s \notin \{-1,-2\}$$

For $\vartheta_s(t) = e^{-5t}$, $\quad A = \frac{2}{12} = \frac{1}{6}$

$$\vartheta^{complete}(t) = \underbrace{c_1 \cdot e^{-t} + c_2 \cdot e^{-2t}}_{\substack{\text{homogeneous} \\ \text{(natural resp.)}}} + \underbrace{\frac{1}{6} \cdot e^{-5t}}_{\text{particular}}$$

steady state : part of complete response as $t \longrightarrow \infty$

transient : the remaining part in the complete response apart from steady state.

No steady state $\longrightarrow$ $\vartheta^{ss}(t) = 0$!

Case ② : $\vartheta_s(t) = 1V = e^{st}\big|_{s=0}$

$\vartheta_c^P(t) = A$

$(D^2 + 3D + 2)\, \vartheta_c^P(t) = 2 \qquad A = 1 \longrightarrow \vartheta_c^P(t) = 1V$

$\vartheta^{ss}(t) = 1 \longrightarrow$ steady state solution.

<u>Method</u> : For exponential inputs, to find particular solution, we make the guess of $A \cdot e^{st}$ then find A.

Case ③ : $\vartheta_s(t) = \cos(5t)$

$(D^2 + 3D + 2)\, \vartheta_c^P(t) = 2 \cdot \cos(5t)$

$\vartheta_c^P(t) = A \cdot \cos(5t) + B \sin(5t)$

$\vartheta_s(t) = \cos 5t = Re\{e^{J5t}\}$

$(D^2 + 3D + 2)\, \vartheta_c^P(t) = 2\, Re\{e^{J5t}\}$

$\vartheta_c^P(t) = Re\{A_c \cdot e^{J5t}\}, \quad A_c \in \mathbb{C}$

$\frac{d}{dt}\vartheta_c^P(t) = Re\{\frac{d}{dt} A_c e^{J5t}\} = Re\{J5 \cdot A_c \cdot e^{J5t}\}$

$\frac{d^2}{dt^2}\vartheta_c^P(t) = Re\{-25 \cdot A_c \cdot e^{J5t}\}$

Do the substitution

$$D^2 \vartheta_c P(t) + 3D \cdot \vartheta_c P(t) + 2\vartheta_c P(t) = Re\{2e^{j5t}\}$$

$$Re\{-25 A_c e^{j5t}\} + Re\{j15 A_c \cdot e^{j5t}\} + Re\{2A_c e^{j5t}\} = Re\{2e^{j5t}\}$$

$$Re\{A_c(-23+j15)e^{j5t}\} = Re\{2e^{j5t}\}$$

$$A_c = \frac{2}{-23+j15} = \frac{2(-23-j15)}{23^2+15^2}$$

$$\vartheta_c P(t) = Re\left\{\frac{-46-j15}{23^2+15^2} e^{j5t}\right\}$$

$$= -\frac{2}{23^2+15^2} \cdot Re\{(23+j15)(\cos 5t + j\sin 5t)\}$$

$$= -\frac{2}{23^2+15^2}[23\cos(5t) - 15\sin(5t)]$$

$$\cos(5t) = Re\{e^{j5t}\}$$

$$\sin(5t) = \cos(5t - 90°)$$

Exponential functions are the eigenfunction of an LTI dynamic system.

Stability

$$[D^2 - (\lambda_1+\lambda_2)D + \lambda_1\lambda_2]\vartheta_c(t) = \vartheta_s(t)$$

λ_1, λ_2 : natural freq.

1) Asymptotically stable ($Re\{\lambda_1\}<0$ & $Re\{\lambda_2\}<0$)

$$\vartheta_s(t) = e^{st}$$

$s=0$ (DC)	$s=1$	$s=j2$ (AC)
↓ $\vartheta_c(t)$ = natural resp. + A	$\vartheta_c(t)$ = nat. resp. + $A \cdot e^t$	$\vartheta_c(t)$ = nat. resp. + $A_c \cdot e^{j2t}$
BIBO stability seems to be OK.		BIBO is OK for real (imaginary) part of the solution

BIBO Stability :

All bounded inputs $\longrightarrow$ Bounded outputs

Bounded func : $|x(t)| < M$, $\exists M$

② Stable $\longrightarrow$ $Re\{\lambda_1\} \leq 0$, $Re\{\lambda_2\} \leq 0$

$$v_s(t) = e^{st}$$

$s = 0$ (DC)

$$v^{comp.}(t) = c_1 \cdot e^{0t} + c_2 \cdot e^{\lambda_2 t}$$

+ particular

$\downarrow$

$A + Bt$

NOT BIBO STABLE

$s = -5$

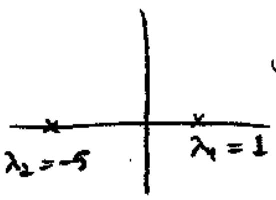
$$v^{comp}(t) = c_1 + c_2 e^{-5t} + A e^{-5t} + B t \cdot e^{-5t}$$

seems OK for BIBO stability.

$s = -3$

$$c_1 + c_2 e^{-5t} + A \cdot e^{-3t}$$

③ Unstable $\longrightarrow$ $Re\{\lambda_1\} > 0$ <u>or</u> $Re\{\lambda_2\} > 0$



stability plane

Ex/

$$\begin{bmatrix} D+1 & D+2 & 0 \\ 3 & D-1 & 2 \\ 1 & 5 & D \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \\ 0 \end{bmatrix} v_s(t)$$

$$\begin{vmatrix} s+1 & s+2 & 0 \\ 3 & s-1 & 2 \\ 1 & 5 & s \end{vmatrix} \longrightarrow \text{gives char. eqn.}$$

PHASORS

$\xrightarrow[A\cdot e^{st} \text{ input}]{}$ LTI $\xrightarrow[B\cdot e^{st} \text{ particular soln}]{}$

we know that exponential family of signals (Ae^{st}) at the input of a N^{th} order dynamic system produces an output (particular soln, soln in the forcing term, i.e input) in the form $B\cdot e^{st}$. So exponential family, the output is determined by finding B.

Then the coefficient of the exponent is called the phasor. In general A, B are complex numbers.

Specific for A.C. inputs, that is $f(t) = M\cdot\cos(\omega t+\phi)$; we have the following phasor definition

$$f(t) = M\cdot\cos(\omega t+\phi) = Re\{M\cdot e^{J(\omega t+\phi)}\} \quad [M,\phi,\omega \in \mathbb{R}]$$

$$= Re\{\underbrace{M\cdot e^{J\phi}}_{A}\cdot e^{J\omega t}\}$$

$$= Re\{A\cdot e^{J\omega t}\}$$

↳ complex valued; phasor, coefficient of $e^{J\omega t}$

Then,

* $2\cos(\omega t+\frac{\pi}{6}) \xrightleftharpoons[\text{form}]{\text{phasor}} 2e^{J\frac{\pi}{6}}$ $\quad (2\angle 30^\circ)$

  $[= Re\{2e^{J\frac{\pi}{6}}\cdot e^{J\omega t}\}]$

* $(1\angle 45^\circ) \xrightarrow{\text{time}} Re\{\underbrace{1\angle 45^\circ}_{e^{J\pi/4}}\, e^{J2t}\}$

  $\omega = 2$

  $= \cos(2t+\frac{\pi}{4})$

Ex $\cos(\underbrace{4t+30^\circ}_{A}) + \cos(\underbrace{4t+60^\circ}_{B}) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= Re\{e^{J30^\circ} e^{J4t}\} + Re\{e^{J60^\circ} e^{J4t}\}$

$= Re\{e^{J4t}(e^{J30^\circ} + e^{J60^\circ})\}$

$= Re\left\{e^{J4t}\left(\begin{bmatrix}\cos 30^\circ \\ J\sin 30^\circ\end{bmatrix} + \begin{bmatrix}\cos 60^\circ \\ J\sin 60^\circ\end{bmatrix}\right)\right\}$

$= Re\left\{e^{J4t}\left(\frac{\sqrt{3}+1}{2} + J\frac{\sqrt{3}+1}{2}\right)\right\} = Re\left\{e^{J4t}\underbrace{(1+J)}_{\sqrt{2}\cdot e^{J\frac{\pi}{4}}}\left(\frac{\sqrt{3}+1}{2}\right)\right\}$

polar coor. representation

$= Re\left\{\left(\frac{1+\sqrt{3}}{2}\right)\cdot\sqrt{2}\cdot e^{J(4t+\frac{\pi}{4})}\right\} = \frac{1+\sqrt{3}}{\sqrt{2}}\cdot\cos\left(4t+\frac{\pi}{4}\right)$

In the previous example, we have repeatedly used $Re\{e^{J4t}(\dots)\}$ at every step. We do not write this at every step, but prefer to write the phasor instead.

That is,

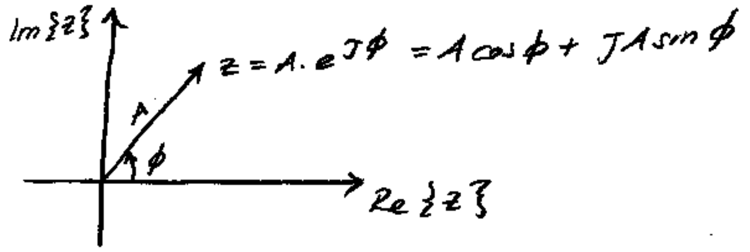
$$\cos(4t+30^\circ) + \cos(4t+60^\circ) \xrightarrow{\text{phasor}} \overset{w=4}{1\angle 30^\circ + 1\angle 60^\circ}$$

$= \left(\frac{\sqrt{3}+J}{2}\right) + \left(\frac{1+J\sqrt{3}}{2}\right)$ ← resulting phasor

$= \frac{1+\sqrt{3}}{\sqrt{2}} e^{J\pi/4}$

$= \frac{1+\sqrt{3}}{2}\cos(4t+45^\circ)$ time domain

$A\angle\phi = A\cdot e^{J\phi} = A(\cos\phi + J\sin\phi)$



Note:

NA → some of natural frequencies at zero can be missing due to Δ, Δ^{-1} cancellation

MNA → used to find natural freq. (has only Δ's)

State Eqn → used for natural freq. (has only Δ's)

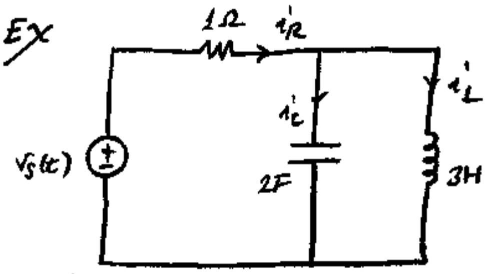
natural freq. can be complex number.

If λ is complex valued, its conjugate is also a natural freq.

Ex
$$\frac{d}{dt} A\cos(\omega t+\phi) = -A\omega\sin(\omega t+\phi)$$
$$= A\omega \cdot \cos(\omega t+\phi+90^\circ)$$

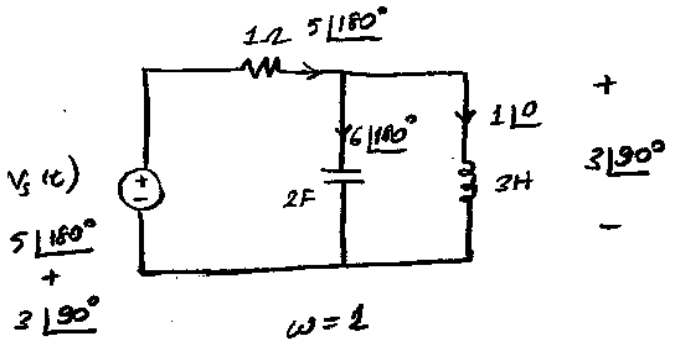
$$\boxed{A\angle\phi \xrightarrow[d/dt]{} A\omega \angle\phi+90^\circ}$$

Ex



$$i_L(t) = \cos(t)$$
$$V_L(t) = V_C(t) = -3\sin(t)$$
$$i_C(t) = -6\cos(t)$$
$$i_R = i_C + i_L = -5\cos(t)$$

$$V_s(t) = R\cdot i_R + V_L(t) = -5\cos(t) - 3\sin(t)$$



$\omega = 1$

$$V_s(t) = 5\angle 180^\circ + 3\angle 90^\circ$$
$$= -5 + j3$$
$$= \sqrt{25+9}\, e^{j\left(\tan^{-1}\frac{3}{-5}\right)}$$

$$V_s(t) = \sqrt{34}\cdot\cos\left(t - \tan^{-1}\frac{3}{5}\right)$$
$$= \sqrt{34}\cdot \operatorname{Re}\left\{ e^{j\left(\omega t - \tan^{-1}\frac{3}{5}\right)} \right\}$$

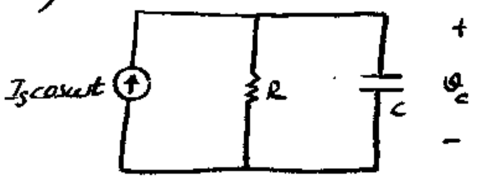
Phasors allow us

1) Add cosines (with the same frequency)

2) Differentiate cosines

So adding differentiated and scaled many cosine terms is possible with phasors.

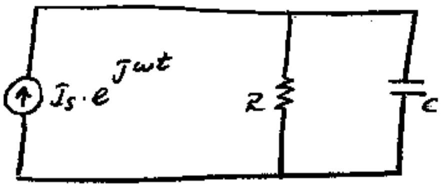
Ex



$$\left(D + \frac{1}{RC}\right) v_c(t) = \frac{i_s(t)}{C} = \frac{I_s \cos\omega t}{C}$$

$$v_c(0^-) = V_0$$

$$\text{①}\ v_c^{\,complete}(t) = c_1 \cdot e^{-t/RC} + A\cos\omega t + B\sin\omega t$$



$$\left(D + \frac{1}{RC}\right) v_c(t) = \frac{I_s \cdot e^{j\omega t}}{C}$$

$$\text{Ⅱ}\ v_c^{\,complete}(t) = d_1 \cdot e^{-t/RC} + D \cdot e^{j\omega t}$$

can be complex number

$$\text{①}\ V_c^{\,complete}(t) = Re\left\{\text{Ⅱ}\ V_c^{\,complete}(t)\right\}$$

$I_s \cdot e^{j\omega t} \longrightarrow$ complex excitation

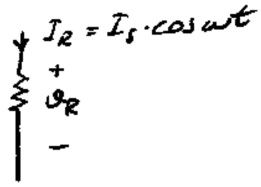
$$\text{Ⅰ}\ V_c^{\,complete}(t) = d_1 \cdot e^{-t/RC} + \frac{I_s \cdot R}{1+(\omega RC)^2}\cos\omega t + \frac{I_s\,\omega R^2 C}{1+(\omega RC)^2}\sin\omega t$$

$$d_1 + \frac{I_s \cdot R}{1+(\omega RC)^2} = v_0 \quad \longleftarrow d_1 \text{ is found.}$$

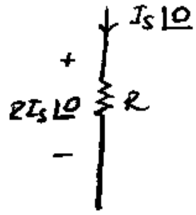
If we are only interested in particular solution for A.C. excitation; we have some simplifications.

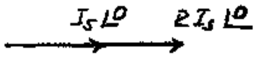
Phasor Circuit Analysis

R:

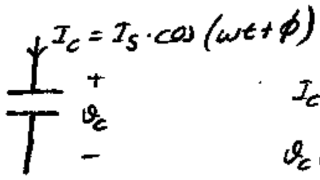


	Time-domain	Phasor
	$I_R(t) = I_s \cdot \cos\omega t$	$i_R = I_s \angle 0$
	$v_R(t) = R \cdot I_s \cdot \cos\omega t$	$v_R = R I_s \angle 0$





C:



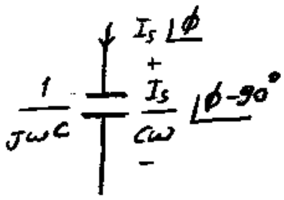
$I_C(t) = I_s \cdot \cos(\omega t + \phi)$

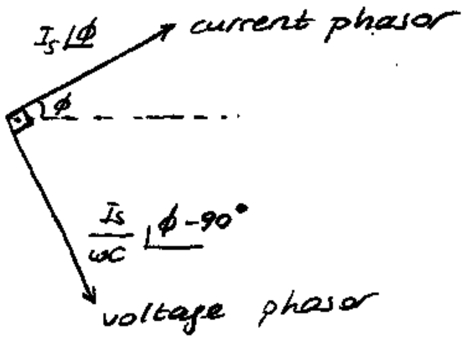
$v_C(t) = \dfrac{I_s}{C\omega} \sin(\omega t + \phi)$

$I_C = I_s \angle \phi$

$v_C = \dfrac{I_s}{C\omega} \angle \phi - 90^\circ$

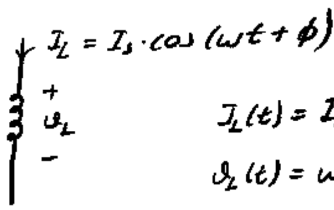
$= \dfrac{1}{j\omega C} I_s \angle \phi$





Capacitor voltage (phasor) **lags** the current (phasor) by 90°.
capacitor current **leads the voltage by** 90°.

L:



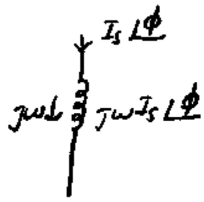
$I_L(t) = I_s \cos(\omega t + \phi)$

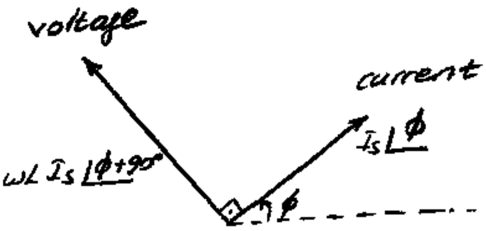
$v_L(t) = \omega L\, I_s \cdot \cos(\omega t + \phi + \frac{\pi}{2})$

$I_L = I_s \angle \phi$

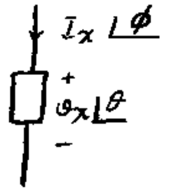
$v_L = \omega L\, I_s \angle \phi + 90^\circ$

$v_L = j\omega L\, I_s \angle \phi$





Impedance and Admittance



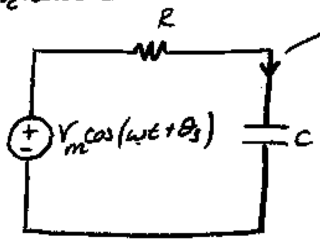
Impedance : $Z = \dfrac{\text{Voltage phasor}}{\text{current phasor}} = \dfrac{v_x}{I_x} \angle \theta-\phi \quad (\Omega)$

Admittance : $Y = \dfrac{\text{Current phasor}}{\text{Voltage phasor}} \quad (\mho)$

$$Z = \underbrace{Re\{Z\}}_{\text{Resistance}} + j \cdot \underbrace{Im\{Z\}}_{\text{Reactance}}$$

$$Y = \underbrace{Re\{Y\}}_{\text{conductance}} + j\, \underbrace{Im\{Y\}}_{\text{susceptance}}$$

Ex $v_c \cdot \omega RC \cdot \cos(\omega t + \theta_c + 90°)$



$\omega C \cdot v_c \cdot \cos(\omega t + \theta_c + 90°)$

+

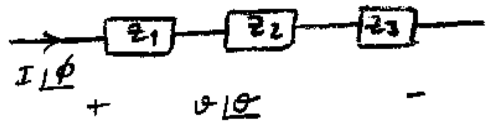
$v_c \cos(\omega t + \theta_c)$

−

$$v_m \cos(\omega t + \theta_s) = v_R(t) + v_c(t)$$

Series and Parallel Combination

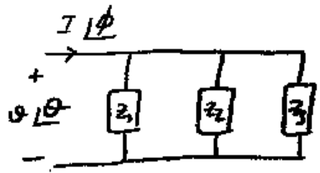
Series :



$$v_1 + v_2 + v_3 = I\angle\phi \cdot z_1 + I\angle\phi \cdot z_2 + I\angle\phi \cdot z_3$$

$$v\angle\theta = I\angle\phi \underbrace{(z_1 + z_2 + z_3)}_{z \text{ combined}}$$

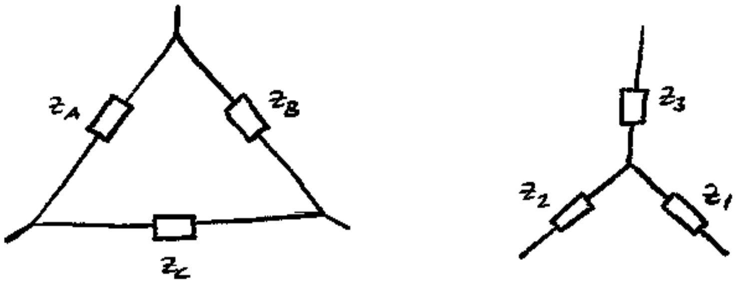
Parallel :



$$I\angle\phi = \sum_{k=1}^{3} I_{z_1} \angle \phi_{z_1}$$

$$= \sum_{k=1}^{3} \frac{v\angle\theta}{z_k} = v\angle\theta \cdot \sum_{k=1}^{3} \frac{1}{z_k}$$

$$\frac{v\angle\theta}{I\angle\phi} = \left(\sum_{k=1}^{N} \frac{1}{z_k} \right)^{-1} \rightarrow z_{\text{combined}}$$



$$z_1 = \frac{z_B\, z_C}{z_A + z_B + z_C} \qquad z_A = \frac{z_1 z_2 + z_1 z_3 + z_2 z_3}{z_1}$$

$$z_2 = \frac{z_A\, z_C}{z_A + z_B + z_C} \qquad z_B = \frac{z_1 z_2 + z_1 z_3 + z_2 z_3}{z_2}$$

$$z_3 = \frac{z_A \cdot z_B}{z_A + z_B + z_C} \qquad z_C = \frac{z_1 z_2 + z_1 z_3 + z_2 z_3}{z_3}$$

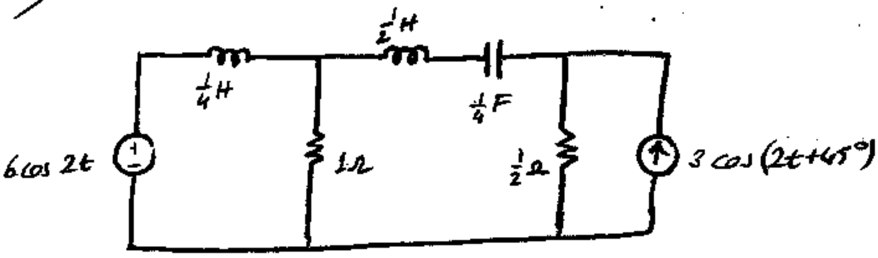
General AC Circuit Analysis

AC steady state analysis (particular soln. due to AC excitation)

1) Node
2) Mesh
3) Thevenin - Norton
4) other simplification methods

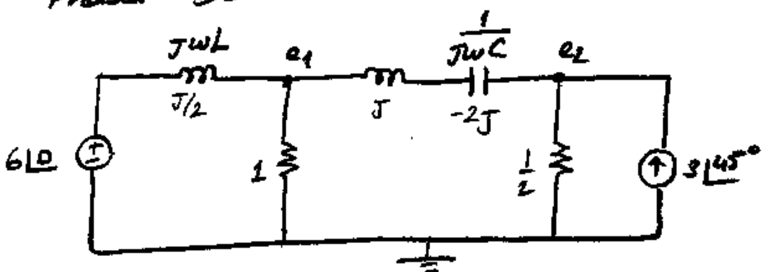
} AC analysis.

Ex Node Analysis



Find AC steady-state branch current & voltages.

Phasor Domain



$\omega = 2$

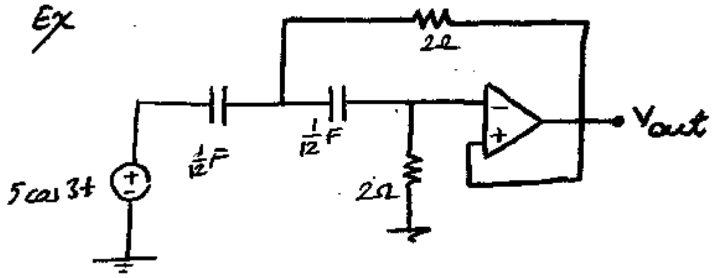
KCL at e_1: $\dfrac{e_1}{1} + \dfrac{e_1 - 6\angle 0}{j/2} + \dfrac{e_1 - e_2}{-j} = 0$

KCL at e_2: $\dfrac{e_2}{1/2} + \dfrac{e_2 - e_1}{-j} - 3\angle 45^\circ = 0$

$$\begin{bmatrix} 1-J & -J \\ -J & 2+J \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -12J \\ 3\angle 45^\circ \end{bmatrix} \longrightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} ? \\ 3.63\angle 14^\circ \end{bmatrix}$$

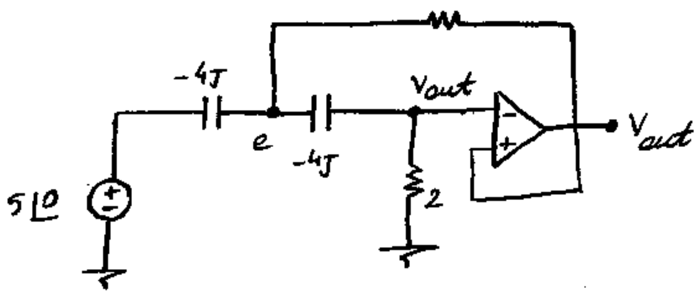
$$e_2(t) = 3{,}63 \cos(2t + 14^\circ)$$

Ex



Assume op-amp is in linear region. Find Vout.

Phasor domain



$\omega = 3$

KCL at e: $\dfrac{e-5}{-4J} + \dfrac{e-V_{out}}{2} + \dfrac{e-V_{out}}{-4J} = 0$ … ①

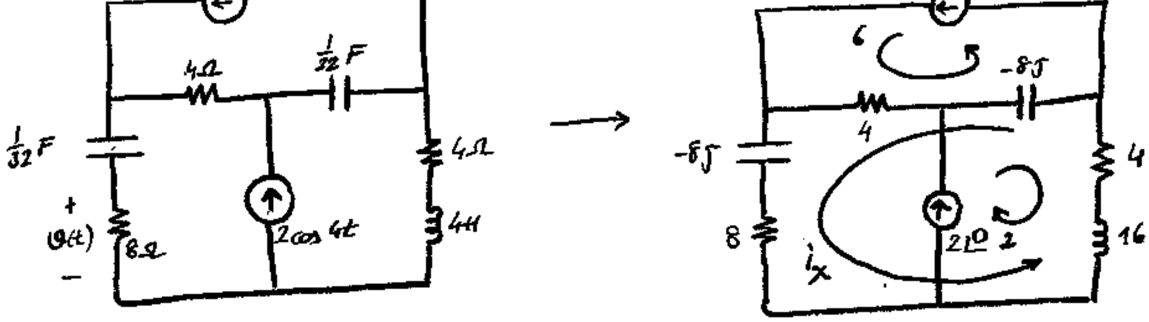
KCL at V_-: $e = v_{out} + \dfrac{v_{out}}{2}(-4J) = v_{out}(1-2J)$

① $\longrightarrow$ $e(2+2J) + v_{out}(-2-J) = 5J$

$v_{out}\left[(1-2J)(2+2J) - 2 - J\right] = 5J$

$v_{out} = \dfrac{5J}{4-3J} = \dfrac{5J}{5\angle -37^\circ} = 1\angle 127^\circ \longrightarrow v_{out} = \cos(3t + 127^\circ)$ V

Ex Mesh Analysis



$\omega = 4$

KVL around i_x loop:

$$4(i_x-6) + (8-8j)i_x + (4+16j)(i_x-2) - 8j(i_x-2-6) = 0$$

$$i_x = \frac{24+8(4j+1)-64j}{4+8-8j+16j+4-8j} = \frac{8(4-4j)}{16} = \frac{1}{2}\,5\angle -45°$$

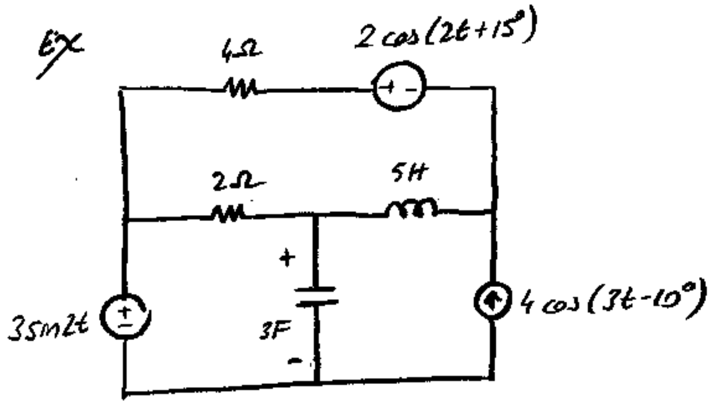
$$v = i_x \cdot 8 = 20\angle -45°$$

$$v(t) = 20\cos(4t-45°)\ \text{V}$$

Sources with different Frequencies (ω)

↳ angular frequency (rad/s)

Ex

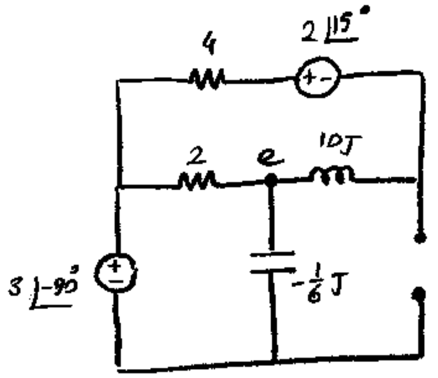


$v_{3F}(t) = ?$

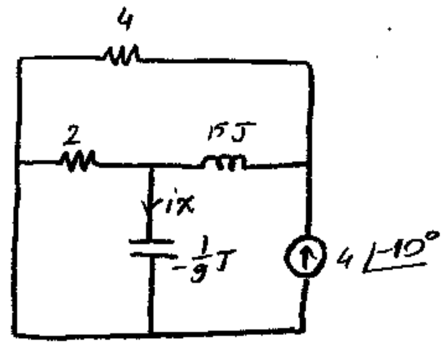
$$\sin(t) = \cos(t-90°)$$

$$1\angle -90°$$

Superpose:



$\omega = 2$



$\omega = 3$

$$\frac{e}{-j/6} + \frac{e-(3\angle -90 - 2\angle 15°)}{4+10j} + \frac{e-3\angle 90}{2} = 0$$

$$i_x = 4\angle -10° \cdot \frac{4}{4+(15j + 2\,\|\,(-j/9))} \cdot \frac{2}{2-9j}$$

$$e = v_{3F}^{(A)}\angle\theta^{(A)}$$

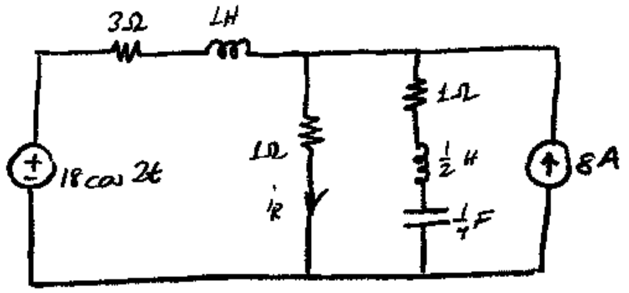
$$V_{3F} = -\frac{j}{9}\, i_x = v_{3F}^{(B)}\angle\theta^{(B)}$$

$$v_{3F}^{(A)}(t) = v_{3F}^{(A)}\cos(2t+\theta^{(A)})$$

$$v_{3F}^{(B)}(t) = v_{3F}^{(B)}\cos(3t+\theta^{(B)})$$

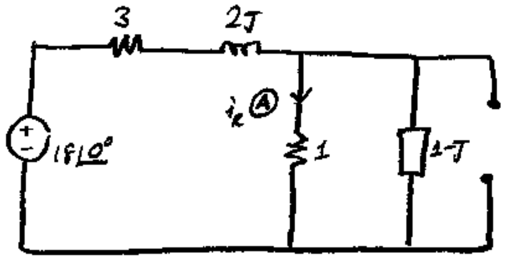
$$v_{3F}(t) = v_{3F}^{(A)}(t) + v_{3F}^{(B)}(t).$$

Ex



$i_R^{ss}(t) = ?$

$\omega = \{2, 0\}$

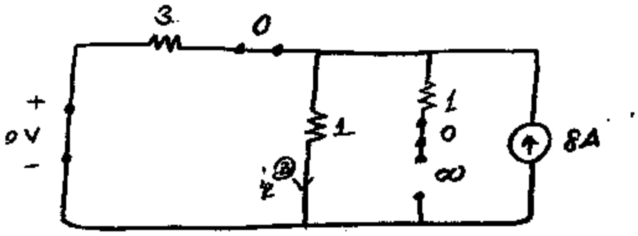


$\omega = 2$

$$i_R Ⓐ = \frac{18}{3+2J} \cdot \frac{1}{\frac{1}{1} + \frac{1}{1-J} + \frac{1}{3+2J}} = 2(1-J)$$

$$= 2\sqrt{2} \angle -45^\circ$$

$$i_R Ⓐ(t) = 2\sqrt{2} \cos(2t - 45^\circ)$$



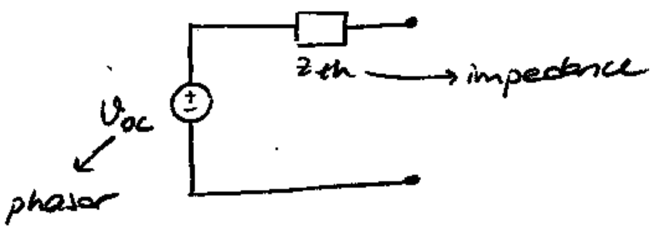
$\omega = 0$

$$i_R Ⓑ = 8 \cdot \frac{3}{3+1} = 6A$$

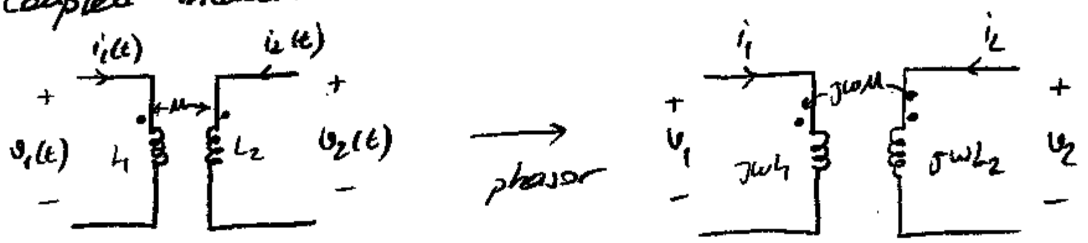
$$\Rightarrow i_R(t) = 6 + 2\sqrt{2} \cos(2t - 45^\circ) \ A$$

Thevenin - Norton

Exactly as in resistive circuits with phasor domain!



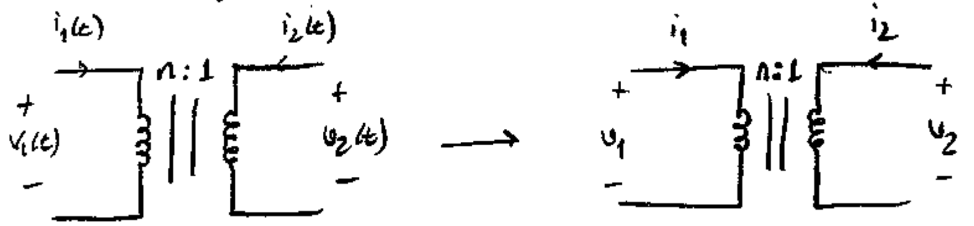
Coupled Inductor



$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} D\, i_1(t) \\ D\, i_2(t) \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} J\omega L_1 & J\omega M \\ J\omega M & J\omega L_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Ideal Transformer



$$\frac{V_1}{V_2} = n \quad , \quad \frac{i_1}{i_2} = \frac{-1}{n}$$

phasor.

Ex (Prev. Unit)

$$(D^2+4)\, v_c(t) = \cos 2t = Re\{e^{j2t}\}$$

$$(D^2+4)\, v_c(t) = 0$$

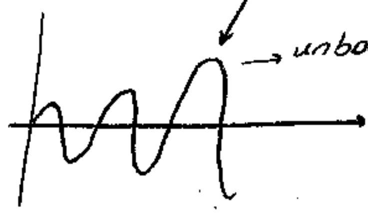
$$\lambda^2 + 4 = 0 \longrightarrow \lambda_{1,2} = \{\mp 2j\}$$

$$v_c^{comp}(t) = c_1 \cdot e^{\lambda_1 t} + c_2 \cdot e^{\lambda_2 t} + A t \cdot e^{j2t}$$

particular soln.

$$\cos(2t) + \dots \quad (\dots) t \cos(2t+\phi)$$

unbounded



AC Power Analysis

RMS or Effective Values

$x(t)$: Periodic function $[\, \exists T,\ T \neq 0 \rightarrow x(t-T) = x(t) \ \forall t \,]$

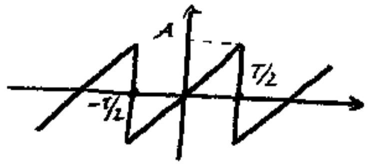
Root mean square $\Rightarrow$ $$RMS = \sqrt{\frac{1}{T} \underbrace{\int_{t_0}^{t_0+T}}_{\text{integrate over a full period}} [x(t)]^2\, dt}$$

$$RMS = \sqrt{\frac{1}{T} \int_0^T [x(t)]^2 dt}$$

① $x(t) = A \cdot \cos(\omega t + \phi)$

$$RMS \longrightarrow \sqrt{\frac{1}{T}\int_0^T A^2 \cos^2(\omega t+\phi)\, dt} = \sqrt{\frac{A^2}{T}\int_0^T \frac{1+\cos(2\omega t + 2\phi)}{2}} = \frac{A}{\sqrt{2}}$$

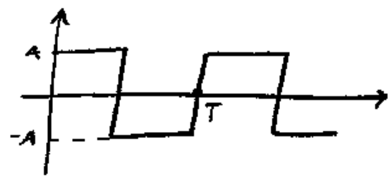
② $x(t)$: triangular (saw-tooth)



$$\text{rms} = \sqrt{\frac{1}{T}\int_{-T/2}^{T/2}\left(\frac{2A}{T}t\right)^2 dt}$$

$$= \sqrt{\frac{2\cdot 4A^2}{T^3}\int_0^{T/2} t^2 dt} = \boxed{\frac{A}{\sqrt{3}}}$$

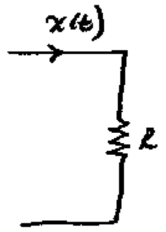
③ Rectangular



$\text{rms}\{x(t)\} = A.$

(effective

<u>why RMS?</u>



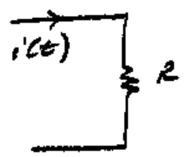
$p(t) = R\,[x(t)]^2$

$$E(\text{absorbed by } R) = \int_0^T p(\tau)d\tau = \int_0^T R\,[x(t)]^2 dt$$

$$= R\cdot T\cdot \frac{1}{T}\int_0^T x^2(t)dt$$

$$= RT\cdot(x_{rms})^2 = (R\cdot X_{rms}^2)\,T \longrightarrow \begin{array}{l}\text{energy}\\ \text{calculation}\end{array}$$

<u>Average Power.</u>

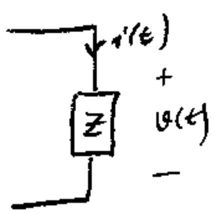


$$P_{AV} = \frac{1}{T}\int_0^T p(t)dt \quad \text{or} \quad P_{AV} = \lim_{t\to\infty}\frac{1}{T}\int_0^T p(t)dt$$

$$= R\cdot\frac{1}{T}\int i^2(t)dt = \boxed{(i_{rms})^2\cdot R}$$

E consumed in 5 periods $\longrightarrow P_{AV}\cdot 5T$

<u>Average and Instantenous Power</u>



$$\left.\begin{array}{ll} v(t) = V_m\cdot\cos(\omega t+\theta_v) \longrightarrow & V = V_m\angle\theta_v \\ i(t) = I_m\cdot\cos(\omega t+\theta_i) & I = I_m\angle\theta_i \end{array}\right\} Z = \frac{V_m}{I_m}\angle\theta_v-\theta_i$$

$$p(t) = v(t)\,i(t) = V_m\cdot I_m\cdot\cos(\omega t+\theta_v)\cos(\omega t+\theta_i)$$

$$= \frac{V_m I_m}{2}\left[\cos\underbrace{(\theta_v-\theta_i)}_{\theta_z} + \cos(2\omega t + \underbrace{\theta_v+\theta_i}_{2\theta_i+\theta_z})\right]$$

$$p(t) = \frac{V_m I_m}{2}\left[\cos\theta_z + \cos(2\omega t + 2\theta_i)\cos(\theta_z) - \sin(2\omega t + 2\theta_i)\sin(\theta_z)\right]$$

$$= \frac{V_m I_m}{2}\cos(\theta_z)\left\{1 + \cos(2\omega t + 2\theta_i)\right\} - \frac{V_m I_m}{2}\sin(\theta_z)\sin(2\omega t + 2\theta_i)$$

$$P_{AV} = \frac{1}{T}\int_0^T p(t)\,dt = \frac{V_m I_m}{2}\cos(\theta_z)$$

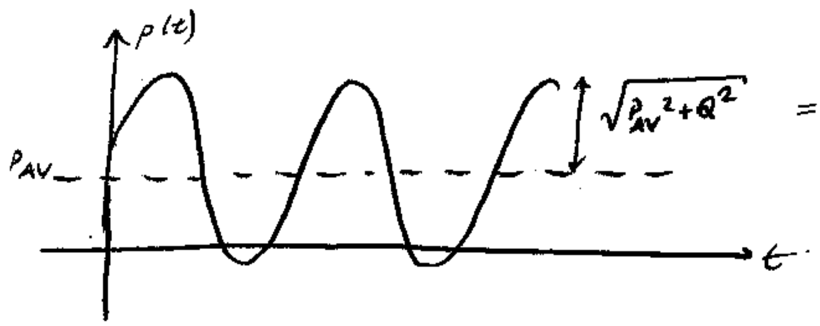
$$= \boxed{V_{RMS} \cdot I_{RMS} \cdot \cos(\theta_z)}$$

$$p(t) = P_{AV}\left\{1 + \cos(2(\omega t + \theta_i))\right\} - Q \cdot \sin(2(\omega t + \theta_i))$$

$$Q = \frac{V_m I_m}{2}\sin(\theta_z)$$

$$p(t) = P_{AV} + P_{AV}\cdot\cos(2\omega t + 2\theta_i) - Q\sin(2\omega t + 2\theta_i)$$

$$= P_{AV} + \sqrt{P_{AV}^2 + Q^2}\cos\left(2\omega t + 2\theta_i + \tan^{-1}\frac{Q}{P_{AV}}\right)$$



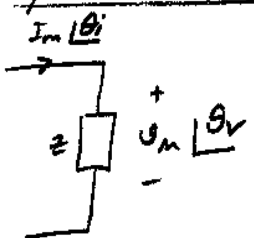
$$\sqrt{P_{AV}^2 + Q^2} = \frac{V_m I_m}{2} = V_{RMS} \cdot I_{RMS}$$

$$\sqrt{P_{AV}^2 + Q^2} = \sqrt{\left[\frac{V_m I_m}{2}\cos(\theta_z)\right]^2 + \left[\frac{V_m I_m}{2}\sin(\theta_z)\right]^2} = \frac{V_m I_m}{2}$$

$V_{RMS} \cdot I_{RMS}$: apparent power

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2 + B^2}\cos\left(\omega t - \tan^{-1}\frac{B}{A}\right)$$

Special Cases



$$z = \frac{V_m}{I_m}\angle\overbrace{\theta_v - \theta_i}^{\theta_z}$$

1) $z = R \longrightarrow \theta_z = 0$

$$P_{AV} = \frac{V_m I_m}{2} = V_{RMS} I_{RMS}$$

$$p(t) = P_{AV}\left\{1 + \cos(2\omega t + 2\theta_i)\right\}$$

② z : inductor

$\theta_z = 90°$

$P_{AV} = 0.$ $\qquad\qquad p(t) = -\frac{V_m I_m}{2} \sin(2\omega t + 2\theta_i)$

$Q = \frac{V_m I_m}{2}$

Q : Reactive Power (unit VAR)
↳ volt - amper reactive

$Q = \frac{V_m I_m}{2} \sin(\theta_z)$

25.3.2010

$p(t) = P_{AV} \{ 1 + \cos(2\omega t + 2\theta_i) \} - Q \cdot \sin(2\omega t + 2\theta_i)$

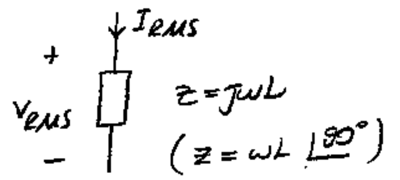
$P_{AV} = V_{RMS} \cdot I_{RMS} \cdot \cos(\theta_z)$

$Q = V_{RMS} \; I_{RMS} \cdot \sin(\theta_z)$
$\quad\downarrow$
$\quad V_{eff}$

$\frac{1}{T}\int_0^T p(t)dt = P_{AV}$ in watts (time avarage of $p(t)$)

Work done in T_s secs. $= P_{AV} \cdot T_s \rightarrow$ Joule's

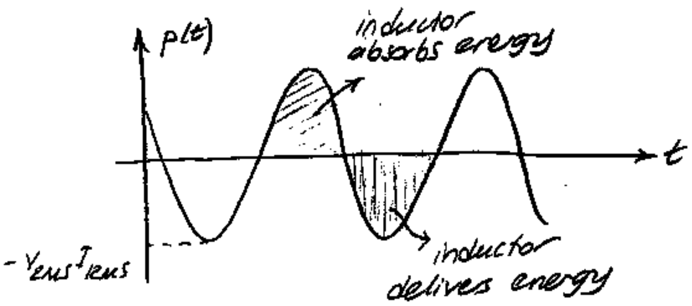
Let's check:



$p(t) = -V_{RMS} \cdot I_{RMS} \sin(2\omega t + 2\theta_i)$

$P_{AV} = V_{RMS} \cdot I_{RMS} \cos(90°) = 0$

$Q = V_{RMS} \cdot I_{RMS} \sin(90°) = V_{RMS} I_{RMS}$



Inductor does not do any work!

$\langle p(t) \rangle = \frac{1}{T}\int_0^T p(t)dt = 0$

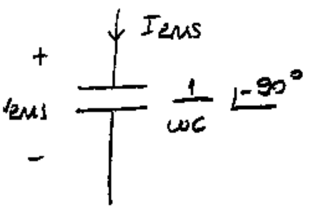
Avarage Stored Energy:

① capacitor: $E_C(t) = \frac{1}{2} C V_c^2(t)$

$E_C^{AVG} = \frac{1}{T}\int_0^T (E_c(t))dt = \boxed{\frac{1}{2} C \cdot (V_c^{RMS})^2}$

② Inductor:

$$\boxed{E_L^{AVG} = \frac{1}{2} \cdot L \cdot (I_L^{RMS})^2}$$



$$Q = V_{RMS} \cdot I_{RMS} \sin(\theta_z) = -V_{RMS} \cdot I_{RMS}$$

$$P_{AV} = 0$$

$$= -V_{RMS} \cdot \frac{V_{RMS}}{|Z|}$$

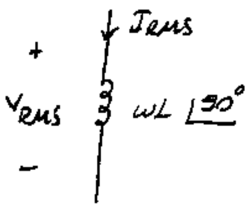
$$= -\omega C \, V_{RMS}^2$$

$$\frac{V_{phasor}}{I_{phasor}} = Z = \frac{1}{j\omega C}$$

$$= -2\omega \cdot \left(\frac{1}{2} C \, V_{RMS}^2\right)$$

$$\frac{V_{RMS}}{I_{RMS}} = |Z| = \frac{1}{\omega C}$$

$$\boxed{= -2\omega \cdot E_C^{AVG}}$$



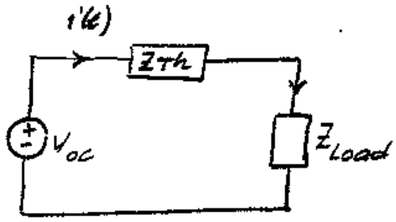
$$Q = V_{RMS} \cdot I_{RMS} \cdot \sin(90^\circ) = V_{RMS} \, I_{RMS}$$

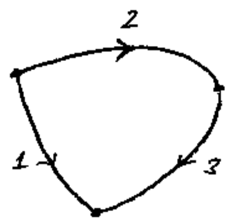
$$= \omega L \, I_{RMS}^2$$

$$= 2\omega \left(\frac{1}{2} \cdot L \, I_{RMS}^2\right)$$

$$\boxed{= 2\omega \, E_L^{AVG}}$$

Conservation of P_{AVG} and Q:





$$i_k(t) = I_m^k \cos(\omega t + \theta_i^k)$$

$$V_k(t) = V_m^k \cdot \cos(\omega t + \theta_v^k)$$

Tellegen's Theorem:

$$\sum_{k=1}^{3} P_k(t) = 0$$

$$P_2(t) + P_3(t) = -P_1(t) = P_S(t) \rightarrow \text{power supplied by } V_{oc}$$

$$P_{AVS}\{1 + \cos(2\omega t + 2\theta_{is})\} - Q_S \sin(2\omega t + 2\theta_{is})$$

$$P_{AV2}\{1 + \cos(2\omega t + 2\theta_{i2})\} - Q_2 \sin(2\omega t + 2\theta_{i2})$$

$$P_{AV3}\{1 + \cos(2\omega t + 2\theta_{i3})\} - Q_3 \sin(2\omega t + 2\theta_{i3})$$

Due to validity for all t

$\sum P_{AV}$ absorbed $= \sum P_{AV}$ supplied

$\sum Q$ absorbed $= \sum Q$ supplied

Complex Power

$$S = P + jQ = V_{RMS} I_{RMS} \cos(\theta_z) + j V_{RMS} I_{RMS} \sin(\theta_z)$$

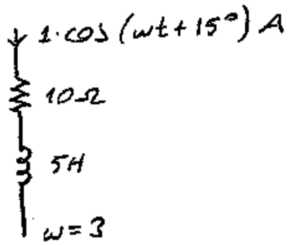
S ← complex power

<u>Note:</u>

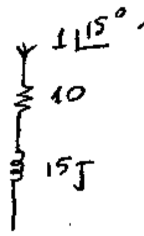
$$p(t) = P_{AV} + P_{AV}\cos(2\omega t + 2\theta_i) - Q\sin(2\omega t + 2\theta_i)$$

$$= P_{AV} + \mathrm{Re}\left\{ (P_{AV} + jQ)\, e^{j2(\omega t + \theta_i)} \right\}$$

Ex



Find P_{AV}, Q, S for the component.



$I_{RMS} = 1/\sqrt{2}$

$$V = (1\angle 15^\circ)(10 + j15) = (1\angle 15^\circ)\left(5\sqrt{13}\,\angle \tan^{-1} 3/2\right)$$

$$= 5\sqrt{13}\,\angle 15^\circ + \tan^{-1} 3/2$$

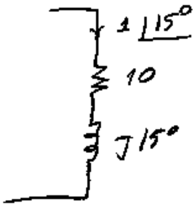
$$V_{RMS} = \frac{5\sqrt{13}}{\sqrt{2}}$$

$$P_{AV} = I_{RMS} V_{RMS} \cos(\theta_z) = \frac{5\sqrt{13}}{2} \cdot \frac{10}{5\sqrt{13}} = 5 \text{ Watts}$$

$$Q = I_{RMS} \cdot V_{RMS} \cdot \sin(\theta_z) = \frac{5\sqrt{13}}{2} \cdot \frac{15}{5\sqrt{13}} = \frac{15}{2} \text{ VAR}$$

<u>2nd Method</u>

P_{AV} of $(10 + j15)$ is only due to 10Ω



$I_{RMS} = 1/\sqrt{2}$

$$P_{AV}^{10\Omega} = V_{RMS} \cdot I_{RMS} \cdot \cos(\theta_R)$$

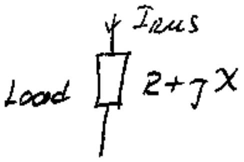
0 for a resistor.

$$= I_{RMS}^2 R = \frac{1}{2} \cdot 10 = 5 \text{ Watts}.$$

$$Q^{j15} = V_{RMS} \cdot I_{RMS} \sin(\theta_L)$$

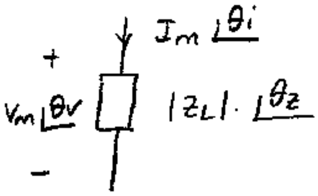
$$= |Z_{j15}| \cdot I_{RMS} I_{RMS} = 15 \cdot I_{RMS}^2 = \frac{15}{2} \text{ VAR}.$$

So in general



$$P_{Load} = I_{RMS}^2 R$$
$$Q_{Load} = I_{RMS}^2 X$$

<u>3rd Method</u>



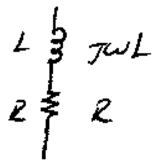
$$S = \frac{1}{2} \cdot V \cdot I^*$$

phasors

$$S = V_{RMS} \cdot I_{RMS} \left(\cos(\theta_z) + j \sin(\theta_z) \right)$$
$$= \frac{|V|}{\sqrt{2}} \cdot \frac{|I|}{\sqrt{2}} \, e^{j\theta_z} \qquad \theta_z \to \theta_v - \theta_i$$
$$= \frac{1}{2} \left(|V| \cdot e^{j\theta_v} \right) \left(|I| \cdot e^{-j\theta_i} \right)$$

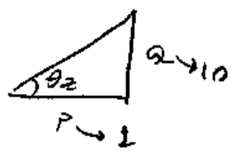
$$S_{Load} = \frac{1}{2} V \cdot I^*$$

<u>Note:</u> Inductive Load



$$S = P + jQ$$
$$\left. \begin{aligned} P &= I_{RMS}^2 \cdot R \\ Q_{ind} &= I_{RMS}^2 \cdot \omega L \end{aligned} \right\} \begin{aligned} &P > 0 \\ &\underline{\underline{Q > 0}} \end{aligned}$$

Ex

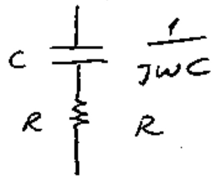


$S = 1 + j10$

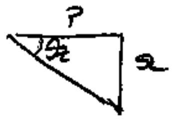
power factor : $\cos(\theta_z)$

lagging (current lags)

Capacitive load



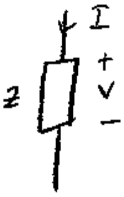
$$S = P + jQ$$
$$\left. \begin{aligned} P &= I_{RMS}^2 \cdot R \\ Q_{cap} &= -I_{RMS}^2 \cdot \frac{1}{\omega C} \end{aligned} \right\} \begin{aligned} &P > 0 \\ &\underline{\underline{Q < 0}} \end{aligned}$$



leading (current leads)

29.3.2010

Review: AC steady-state Power



$$S = \frac{1}{2} V \cdot I^* = \frac{1}{2} |I|^2 Z = I_{RMS}^2 \cdot Z$$
$$= I_{RMS}^2 \left(R + jX \right)$$

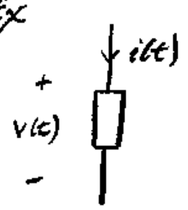
$I_{RMS}^2 \to I_{eff}^2$

$$\angle S = \angle Z$$

$pf = \cos(\angle Z)$ lagging - leading

$$S = P + jQ$$
$$P = I_{RMS}^2 R$$
$$Q = I_{RMS}^2 X$$

Ex



$v(t) = 3\cos(4t + 30°)$

$i(t) = 6\cos(4t - 20°)$

1) Find real power (watts) and Reactive power (VAR) of the component.

$S = \frac{1}{2} V \cdot I^* = \frac{1}{2} \cdot 3\angle 30° \,(6\angle -20°)^* = 9\angle 50° = 5.7 + j\,6.8$

$P = 5.7$ watts

$Q = 6.8$ VAR

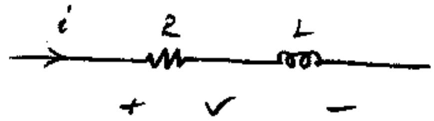
2) Find pf of the component.

$p.f. = \cos(\measuredangle S) = \cos(50°) = 0.64$ (lagging) (inductive) (current lags voltage)

3) Assuming that the component is constructed from R, L, C, find suitable R, L, C to realize the component

$Z = \frac{V}{I} = \frac{3\angle 30°}{6\angle -20°} = \frac{1}{2}\angle 50° = 0{,}32 + j\,0.38$

$= R + j\omega L \longrightarrow L = \frac{0{,}38}{4} = 0{,}095\,H$



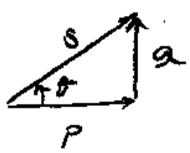
Ex

$v = 100\,V$ (RMS)

$P = 5\,kW$

$pf = 0.8$ lagging

a) Find VAR (Reactive power)



$pf = 0{,}8 \longrightarrow \cos\theta = 0{,}8$

$\theta = 37{,}3°$

$Q = P \cdot \tan(37.3°) = 5000 \cdot \frac{3}{4} = 3750$ VAR

$S = P + jQ \longrightarrow S = 5000 + j\,3750$

↳ lagging

b) Apparent Power:

Apparent Power = $V_{RMS} \cdot I_{RMS}$

$S = (5000 + j\,3750) = I_{RMS}^2\,(R + jX) = \frac{1}{2}|I|^2 Z = \frac{1}{2}\left|\frac{V}{Z}\right|^2 Z = \frac{1}{2}\frac{|V|^2}{Z^*}$

$= \boxed{\frac{V_{RMS}^2}{Z^*}}$

$$S = (5 + j\,3{,}75)k = \frac{V_{RMS}^2}{Z^*}$$

$$Z^* = \frac{1}{0{,}5 + 0{,}375j}$$

$$Z = \frac{1}{0{,}5 - j\,0{,}375} = \frac{0{,}5 + j\,375}{(0{,}5)^2 + (0{,}375)^2}$$

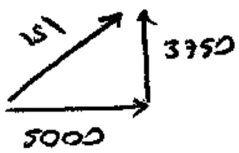
$$\text{Apparent power} = V_{RMS} \cdot I_{RMS} \qquad \frac{V}{Z} = I \longrightarrow I_{RMS} = \frac{V_{RMS}}{|Z|}$$

$$= 100 \cdot \frac{100}{|Z|} = 10.000\sqrt{(0.5)^2 + (0.375)^2}$$

Apparent Power $= |S|$

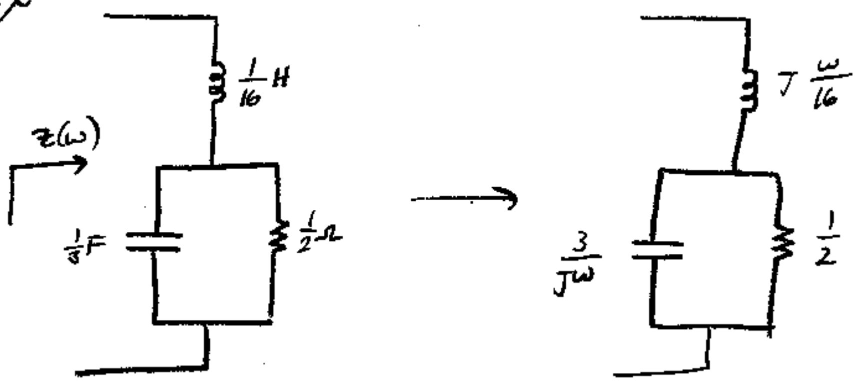
$$S = V_{RMS}\, I_{RMS}\, (\cos\theta_z + j \sin\theta_z)$$

$$|S| = V_{RMS}\, I_{RMS}$$



$$|S| = \frac{5000}{0.8} = 6250 \text{ VA} \longleftarrow$$ Unit of S is VA (volt-Amper)

Ex



$$Z(\omega) = j\frac{\omega}{16} + \frac{\frac{3}{j\omega}\cdot\frac{1}{2}}{\frac{3}{j\omega} + \frac{1}{2}} = j\frac{\omega}{16} + \frac{3}{6 + j\omega} = \frac{j\omega}{16} + \frac{3(6 - j\omega)}{36 + \omega^2}$$

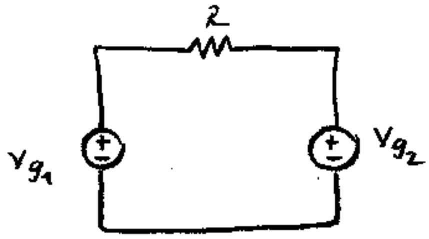
$$= \frac{j\omega^3 - 12 j\omega + 288}{16(36 + \omega^2)} = \frac{288 + j\omega(\omega^2 - 12)}{16(36 + \omega^2)}$$

① $\omega = \sqrt{12} \longrightarrow Z(\omega)\Big|_{\omega = \sqrt{12}} = \dfrac{288}{16 \cdot 48} \longrightarrow$ purely real

② $\omega > \sqrt{12} \longrightarrow$ inductive load

③ $\omega < \sqrt{12} \longrightarrow$ capacitive load

Superposition in AC Power



$V_{g_1} = V_1 \cos(\omega_1 t + \theta_1)$

$V_{g_2} = V_2 \cdot \cos(\omega_2 t + \theta_2)$

$\boxed{\omega_1 \neq \omega_2\ !}$

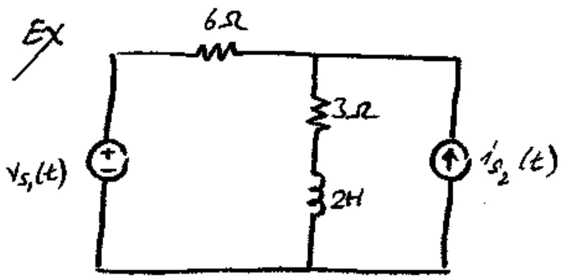
$i_R(t) = I_1 \cos(\omega_1 t + \phi_1) + I_2 \cos(\omega_2 t + \phi_2)$ A (by superposition principle)

$$P_{AV} = \frac{1}{T}\int_0^T (i_R(t))^2 R\,dt \longrightarrow P_{AV} = \lim_{T\to\infty} \frac{1}{T}\int_0^T (i_R(t))^2 \cdot R\,dt$$

$$i_R^2(t) = I_1^2 \cos^2(\omega_1 t + \phi_1) + I_2^2 \cos^2(\omega_2 t + \phi_2) + 2 I_1 I_2 \cos(\omega_1 t + \phi_1)\cos(\omega_2 t + \phi_2)$$

$$P_{AV} = (I_1^{RMS})^2 R + (I_2^{RMS})^2 \cdot R + \frac{2R}{T} I_1 I_2 \underbrace{\cancel{\int_0^T \cos(\omega_1 t + \phi_1)\cos(\omega_2 t + \phi_2)\,dt}}_{\frac{1}{2}\left(\cancel{\cos((\omega_1+\omega_2)t + \phi_1 + \phi_2)} + \cancel{\cos((\omega_1-\omega_2)t + \phi_1 - \phi_2)}\right)} \to 0$$

Ex

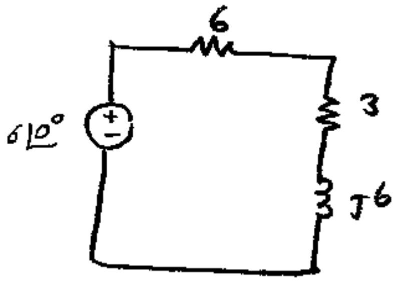


$v_{s_1}(t) = 6\cos(3t)$ V

$i_{s_2}(t) = 2\cos(4t + 30°)$ A

$P_{3\Omega}^{AVG} = ?$

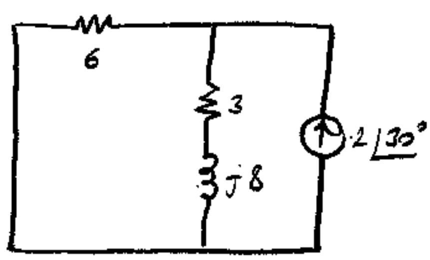
$\xrightarrow{\omega = 3}$



$$i_{3\Omega} = \frac{6\angle 0°}{9 + 6J} = \frac{2}{3 + J2} = \frac{2}{\sqrt{13}} \angle -\tan^{-1}\frac{2}{3} = 0.555\angle -33.69°$$

$$I_{3\Omega,RMS}^{\omega=3} = \frac{2}{\sqrt{13}} \cdot \frac{1}{\sqrt{2}} = \sqrt{\frac{2}{13}}$$

$\xrightarrow{\omega = 4}$



$$i_{3\Omega} = 2\angle 30° \cdot \frac{6}{9 + J8} = \frac{12\angle 30°}{\sqrt{145}\angle \tan^{-1} 8/9} = 0{,}997\angle -11.634°$$

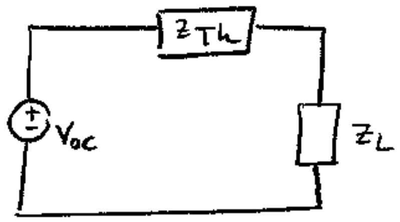
$$I_{3\Omega,RMS}^{\omega=4} = \frac{6\sqrt{2}}{\sqrt{145}}$$

$$i_{3\Omega} = 0{,}555\cos(3t - 33.69) + 0{,}997\cos(4t - 11{,}634°)\ \text{A}$$

$$P_{AV} = P_{AV}^{(\omega=3)} + P_{AV}^{(\omega=4)} = \left(I_{RMS}^{\omega=3}\right)^2 \cdot 3 + \left(I_{RMS}^{\omega=4}\right)^2 \cdot 3 = 0.462 + 1.491$$

$$= 1.153 \text{ Watt}$$

Maximum Power Transfer



V_{oc}, Z_{Th} : fixed

What should the value of Z_L be such that P_{AVG} of the load (Z_L) is maximized?

$$\boxed{Z_L = Z_{Th}^*}$$

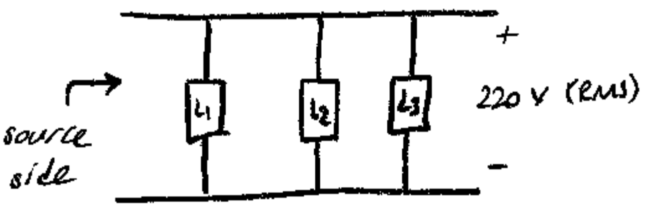
is the maximum transfer solution

Proof $P_{Load} = I_{RMS}^2 \cdot R_L$

$$P_{Load}(R_L, X_L) = \frac{1}{2}\left|\frac{V_{oc}}{(R_{Th}+R_L)+j(X_{Th}+X_L)}\right|^2 \cdot R_L$$

$$\left.\begin{aligned} \frac{\partial P_{Load}}{\partial R_L} &= 0 \\ \frac{\partial P_{Load}}{\partial X_L} &= 0 \end{aligned}\right\} \text{ Solve together } \quad \left.\begin{aligned} X_L &= -X_{Th} \\ R_L &= R_{Th} \end{aligned}\right\} Z_L = Z_{Th}^*$$

Ex



Load 1 : 16 kW , 18 kVAR

Load 2 : 10 k VA at 0.6 lead

Load 3 : 8 kW at unity p.f

a) Find p.f. on the source side

b) Find the impedance seen by the source

a) $S_1 = 16 + j18$ kVA

$S_2 = 6 - j8$ kVA

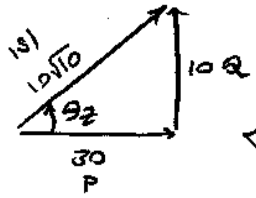
$S_3 = 8$ kVA

$S_{total} = 30 + j10$ kVA

$S_{source}^{supplied} = 30 + j10$ kVA

$p.f. = \cos(\theta_Z)$

$S_{total} = I_{RMS}^2 \, Z_{combined} \rightarrow \angle S = \angle Z$



$p.f = \cos(\theta_z) = \frac{30}{10\sqrt{10}} = \frac{3\sqrt{10}}{10}$ lagging

always state leading or lagging with pf.

b) $S = I_{RMS}^2 Z_{comb.}$; $|S| = V_{RMS} \cdot I_{RMS}$ $\quad (S = \frac{1}{2} V \cdot I^*)$

$30+J$ $\quad$ $220\,V$

$$I_{RMS} = \frac{10\sqrt{10} \times 10^3}{220} = 143{,}74 \text{ A (RMS)}$$

$$Z_{comb} = \frac{S}{I_{RMS}^2} = \frac{(30+J10)\times 10^3}{(143{,}74)^2} = 1.45 + J0.48$$

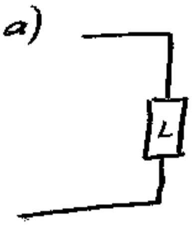
Power Factor Compensation

Ex/ A mill consumes 100 kW from 220 V RMS line, at pf 0.85 lagging.

a) Find the current (RMS) supplied by the source

b) Find the current (RMS) " if pf were 0.95 lagging

a)



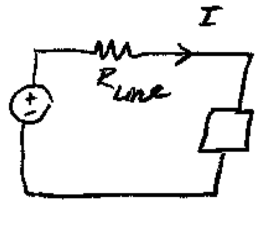
$S_L = 100 + J\,100 \tan(31{,}7°)$ $\qquad$ $|S_L| = V_{RMS} \cdot I_{RMS}$

$|S_L| = \frac{100}{0{,}85} = 117{,}66$ kVA. $\qquad$ $I_{RMS} = \frac{|S_L|}{V_{RMS}} = 534.8$ A.

b) $S_L = 100 + J \tan(18°)$

$|S_L| = \frac{100}{0{,}95} = 105.3$ kVA $\rightarrow I_{RMS} = \frac{|S_L|}{220\,V} = 478$ A.

c) If there is a line connecting generator to the load and $R_{line} = 0.1$; Find the power loss over the line for $pf = 0{,}85$, $pf = 0.95$.



i) $pf = 0{,}85$ $\quad P_{line} = (0.1) \cdot I_{RMS}^2 = 0.1 \cdot (534.8)^2$

$= 28.6$ kW

ii) $pf = 0.95$ $\quad P_{line} = 0.1 \cdot (478)^2 = 22.9$ kW

d) Define efficiency:

$$\text{efficiency} = \frac{\text{Real power supplied to load}}{\text{Real power generated by source}}$$

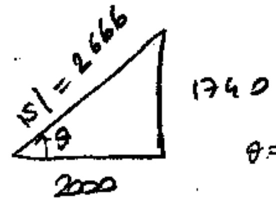
$pf = 0.85 \longrightarrow \quad eff = \dfrac{100}{100+28.6} = 77\ \%$

$pf = 0.95 \longrightarrow \quad eff = \dfrac{100}{100+22} = 82\ \%$

Ex Load requires 2kW at 0.75 pf lagging at 220 Vrms. Calculate the reactive power supplied by the compensating capacitor to make pf 0.9 and find the impedance and assuming 50 Hz, 220 V(RMS) system find the capacitor value in terms of Farads.

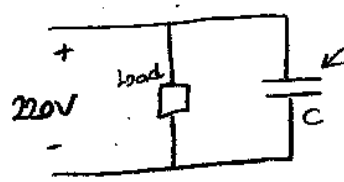
Before:

$S_{before} = 2000 + j1764$

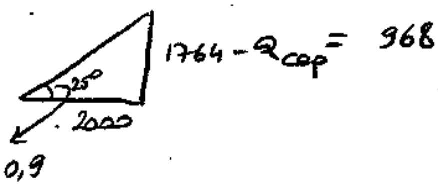


$\theta = \arccos 0.75 = 41°$

We assume that Load & compensating element always operate at 220V (RMS)

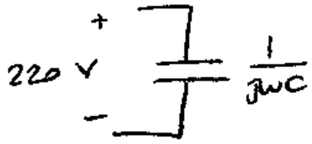


$S_{cap} = -j\,Q_{cap}$



$Q_{cap} = 796$

$S_{cap} = -j\,796$ VAR



$I_{rms} = \dfrac{|S_{cap}|}{220} = 3.61$ A (rms)

$S_{cap} = I_{rms}^2 \cdot Z_{cap} = -j\,796$

$Z_{cap} = \dfrac{-j796}{(3.61)^2} = \boxed{-j\,61.07}$

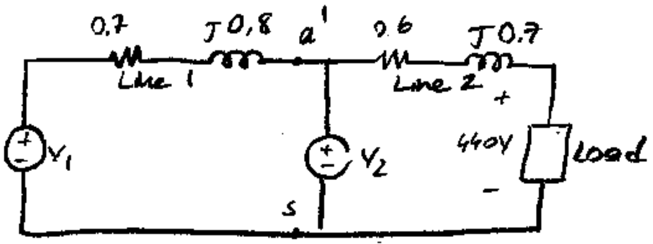
$-j\dfrac{1}{C\omega} = -j\,61.07 \longrightarrow \quad C = \dfrac{1}{100\pi\, 61.07} = 52\ \mu F$

Volt-Ampere Method

We make use of conservation of complex power in the Volt-Ampere method.

$$\sum_k S_k^{supplied} = \sum_k S_k^{absorbed}$$

Ex Two generators supply 10 kW load at 0.8 pf lagging. The generator-2 supplies 5 kW at 0.6 pf lagging. Find the voltages of V_1 & V_2 (RMS), the apparent power of generator 1 and pf of generator 1.



$$S_{G_2}^{supplied} = 5000 + J\,6666 \text{ VA}$$

$$S_L = 10000 + J\,7500 \text{ VA}$$

$$I_{RMS}^{Load} = \frac{|S_{Load}|}{V_{RMS}^{Load}} = \frac{10000/0.8}{440} = 28.4 \text{ A RMS}$$

$$S_{Line-2} = (I_{Load}^{RMS})^2 \cdot (0.6 + J\,0.7) = 484 + J\,565$$

$$S_{Load} + S_{Line-2} = 10\,484 + J\,8065 \text{ VA}$$

$$|S_{Load+Line-2}| = I_{Load}^{RMS} \cdot V_2^{RMS}$$

$$(0.484^2 + 8065^2)^{1/2} = 28.4\, V_2^{RMS} \longrightarrow \boxed{V_2^{RMS} = 466 \text{ V RMS}}$$

$$S_{\substack{right\\ of\ a-a'}} = S_{Load+Line-2} - S_{G_2}^{sup} = 5,484 + J\,1400$$

$$|S_{\substack{right\\ of\ a-a'}}| = I_{Line\,1}^{RMS} \cdot \underset{V_2^{RMS}}{V_{aa'}}$$

$$(5,484^2 + 1400^2)^{1/2} = I_{Line1}^{RMS} \cdot 466$$

$$I_{Line\,1}^{RMS} = 12.14 \text{ A}$$

$$S_{Line-1} = (I_{Line-1})^2 \times (0.7 + J\,0.8) = 103 + J\,118$$

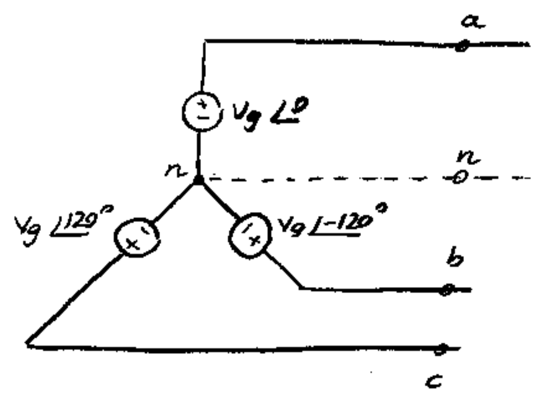
$$S_{G_1}^{sup} = S_{Line-1} + S_{\substack{right\ of\\ a-a'}} = 5587 + J\,1516$$

$$|S_{G_1}^{sup}| = V_1^{RMS} \cdot I_{Line-1}^{RMS} \Rightarrow V_1^{RMS} = \frac{(5587^2 + 1516^2)^{1/2}}{12.14} = \boxed{477\text{V}}$$

$$pf = \cos^{-1} \frac{5587}{(5587^2 + 1516^2)^{1/2}} = \boxed{0.965}$$

3-Phase Balanced Circuits

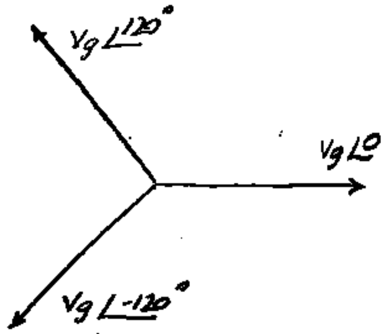
Y- Connection

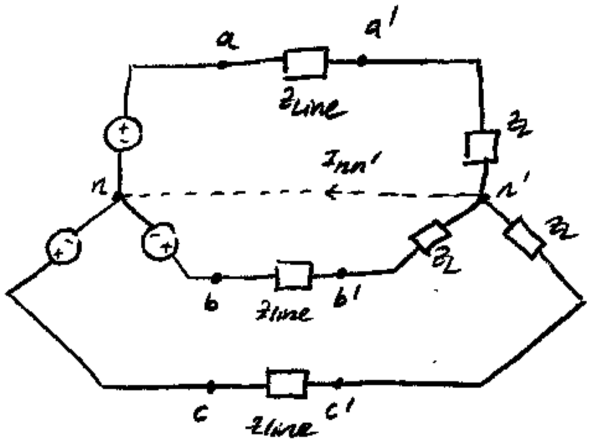


$V_{an} = V_g \angle 0°$

$V_{bn} = V_g \angle -120°$

$V_{cn} = V_g \angle 120°$

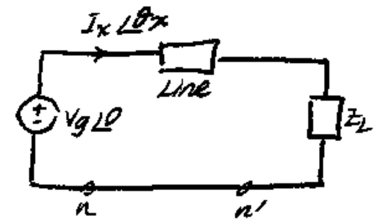


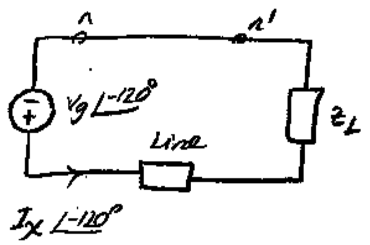


Balanced 3-φ systems.

Note:

Solving only phase a is equivalent to solving all phases since



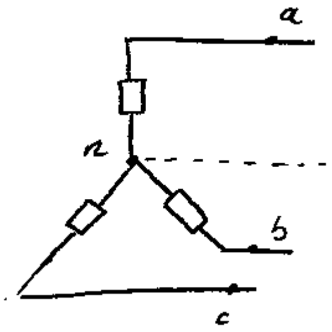


Then

$$I_{nn'} = I_x \angle \theta_x + I_x \angle \theta_x - 120° + I_x \angle \theta_x + 120°$$

Phase-Line Current / Voltages

Y-Connection:



V_{an}, V_{bn}, V_{cn} : Phase voltages

V_{ab}, V_{ac}, V_{bc} : Line voltages (line to line)

$$V_{ab} = V_a - V_b = V_{an} + V_{nb}$$

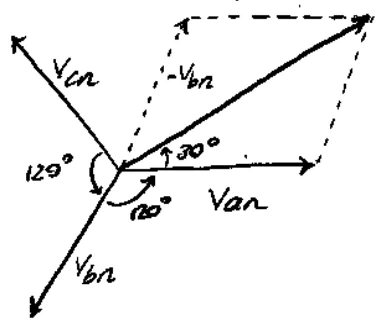
I_{an} : phase current
$I_{aa'}$: Line current

$$I_{aa'} = I_{an}$$

Phase voltage $\neq$ Line voltage
Phase current = Line current

} For Y-connection

Line-voltage, Phase-voltage relation for Y-connection



$$V_{ab} = \sqrt{3}\, V_g \angle 30^\circ$$

$$V_{ac} = \sqrt{3}\, V_g \angle -30^\circ$$

$$V_{bc} = \sqrt{3}\, V_g \angle -90^\circ$$

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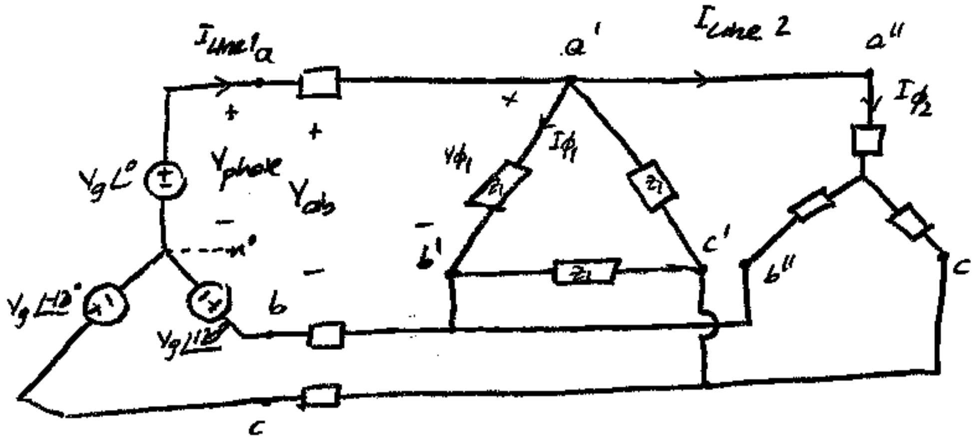
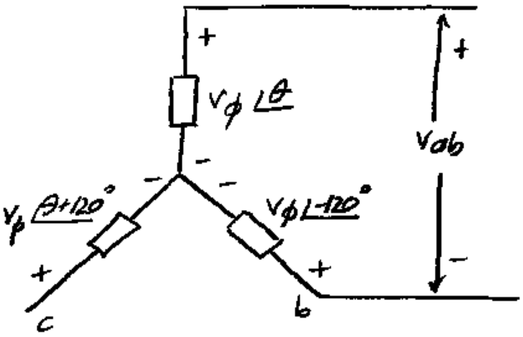


V_ϕ : phase voltage

I_ϕ : phase current
(current in a single phase of the component)

V_{line} : Line voltage (V_{ab}, V_{ac}, V_{bc})

I_{line} : Current passing thru the line (Z line)

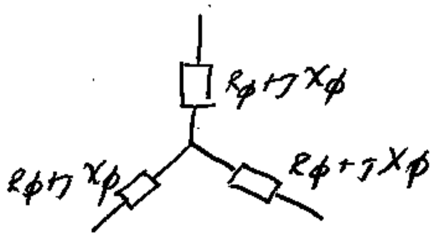


$$V_{ab} = V_\phi \angle\theta - V_\phi \angle\theta-120°$$
$$= V_\phi \angle\theta \left(1 - \angle -120°\right)$$
$$= V_\phi \angle\theta \left(1 + \angle 60°\right)$$
$$= V_\phi \angle\theta \left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$
$$= \sqrt{3} V_\phi \angle\theta \underbrace{\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right)}_{1\angle 30°} = \sqrt{3} V_\phi \angle\theta+30°$$

(Line voltage in RMS) = $\sqrt{3}$ × (Phase voltage in RMS)

(Line current in RMS) = (Phase current in RMS)

Power Relations for Y-Connection



$P_{Tot} = 3 \cdot P_\phi \longrightarrow$ per phase

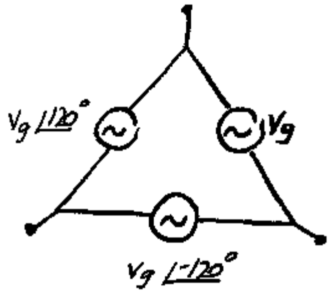
$P_\phi = |I_{\phi\text{-rms}}|^2 R_\phi$

$S_{Tot} = 3 S_\phi$

$S_\phi = |I_{\phi,rms}|^2 (\overbrace{R_\phi + jX_\phi}^{Z_\phi})$

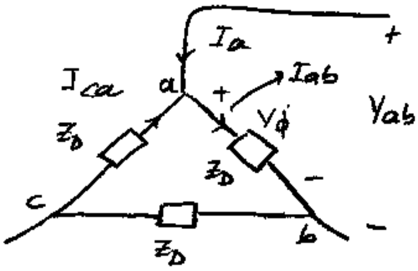
$$P_{Tot} = Re\{S_{Tot}\} = 3\,Re\{S_\phi\} = 3\,Re\{|I_{\phi,rms}|^2 |Z_\phi| e^{j\angle Z_\phi}\}$$
$$= 3\,Re\{I_{\phi,rms} \cdot V_{\phi,rms} \cdot e^{j\angle Z_\phi}\}$$
$$= 3 \cdot \underbrace{I_{\phi rms}}_{I_{Line}} \cdot \underbrace{V_{\phi rms}}_{\frac{V_{Line}}{\sqrt{3}}} \cdot \cos(\angle Z_\phi)$$
$$\underbrace{= \sqrt{3} \cdot V_{Line,rms} \cdot I_{Line,rms} \cdot \cos(\angle Z_\phi)}_{P_{Total} \text{ for Y-Load}}$$

Δ- Connection



No neutral line

Generators are Y connected in general, not Δ connected.



For Δ-Load

$V_{line,\,rms} = V_{\phi,\,rms}$

$I_{line,\,rms} = \sqrt{3}\cdot I_{\phi,\,rms}$

$$I_a = I_{ab} - I_{ca} = \frac{V_{ab}}{Z_\Delta} - \frac{V_{ca}}{Z_\Delta} = \frac{V_\phi \angle 0^\circ}{Z_\Delta} - \frac{V_\phi \angle -120^\circ}{Z_\Delta}$$

$$= I_\phi - I_\phi \angle -120^\circ$$

$$\underbrace{I_a}_{\text{Line current}} = \sqrt{3}\, I_\phi \angle 30^\circ$$

Power Calculation for Δ-Load

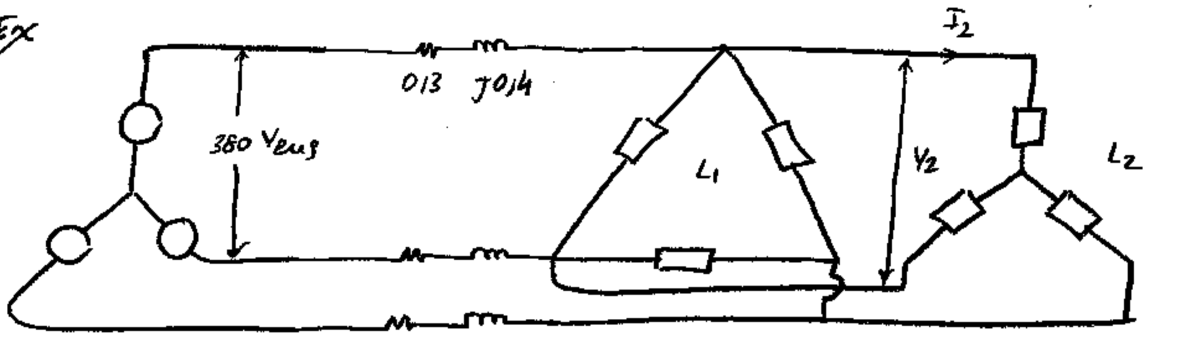
$$P_{Tot} = 3P_\phi = 3\,\mathrm{Re}\left\{ V^\Delta_{\phi\text{-}rms} \cdot I^\Delta_{\phi\text{-}rms} \cdot e^{j\angle Z_\Delta} \right\}$$

$$= 3\,\mathrm{Re}\left\{ V^\Delta_{line,\,rms} \cdot \frac{I^\Delta_{line,\,rms}}{\sqrt{3}} \cdot e^{j\angle Z_\Delta} \right\}$$

$$= \sqrt{3}\cdot V^\Delta_{line\text{-}rms} \cdot I^\Delta_{line\text{-}rms} \cdot \cos(\angle Z_\Delta)$$

same formula for both Δ and Y Loads!

Ex



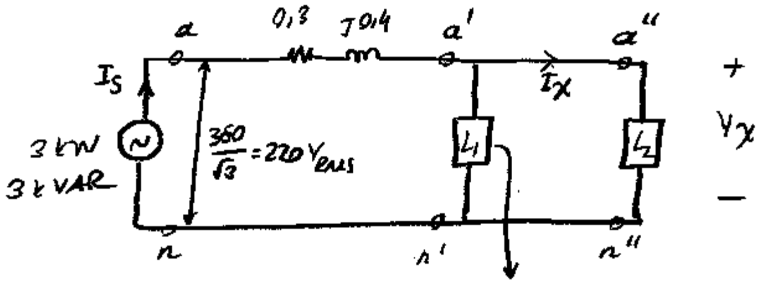
Generator produces 380V-rms line to line voltage

produces 9 kW and 9 k-VAR

Load 1 6kW at 0.6 p.f. Lagging

Find V_2, I_2 and power absorbed by $Load_2$.

Soln 1: Using single phase equivalent circuit.



converted to an Y-Load
by Δ→Y transformation

$S_G = 3 + j3$ kVA $\qquad |S_G| = 3\sqrt{2}$ kVA $= 220 \cdot I_{S\text{-}rms}$

$I_{S\text{-}rms} = 19.3$ A

$S_{Line} = I_{S,rms}^2\ (0,3 + j0.4) = 112 + j149$

$S_{L_1} + S_{L_2} = 3000 + j3000 - (112 + j149)$

$= 2888 + j2851$

$|S_{L_1+L_2}| = |V_{x\,rms}|\,|I_{S\text{-}rms}|$

$4{,}058$ kVA $= V_{x\text{-}rms} \cdot 19{,}3 \longrightarrow V_{x\text{-}rms} = 210{,}3$

$V_2 = \sqrt{3} \cdot V_x = 364\ V_{rms}$

$S_{L_2} = S_{L_1+L_2} - S_{L_1} = 2888 + j2851 - \left(\dfrac{6000 + j6000}{3}\right)$

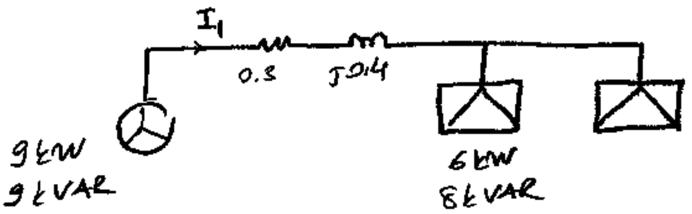
$= 888 + j167$

$|S_{L_2}| = V_{x\text{-}rms} \cdot I_{x\text{-}rms} \longrightarrow I_{x\text{-}rms} = 4{,}31\ A_{rms} = I_2$

Power absorbed by Load 2

$S_{L_2}^{tot} = 3S_{L_2} = 2664 + j561$

Soln 2: Single Line diagram



For both Loads

$P_{Load} = \sqrt{3}\ V_{Line} \cdot I_{Line} \cdot \cos(\measuredangle Z_{Load})$

$Q_{Load} = \sqrt{3}\ V_{Line} \cdot I_{Line} \cdot \sin(\measuredangle Z_{Load})$

$S_{Load} = P_{Load} + jQ_{Load}$

3φ total power quantities

① Find $I_1 \longrightarrow |S_{gen}| = \sqrt{3} \cdot \underbrace{V_{Line}^{gen}}_{380} \cdot I_{Line}^{gen.}$

$9000\sqrt{2}$ $\qquad I_{Line} = 19.3\ A$

② Line Losses $\quad S_{Line} = 3(19,3)^2(0,3 + j\,0,4) = 336 + j\,447$

③ Power left to loads

$$S_{L_1+L_2} = 8664 + j\,8553$$

④ $|S_{L_1+L_2}| = \sqrt{3}\,|V_{L_1+L_2}^{rms}| \cdot I_{Line}$

$$V_{L_1+L_2}^{rms} = 364\ V_{rms}$$

⑤ $S_{L_2} = 8664 - 6000 + j(8553 - 8000) = 2664 + j\,561\ VA.$

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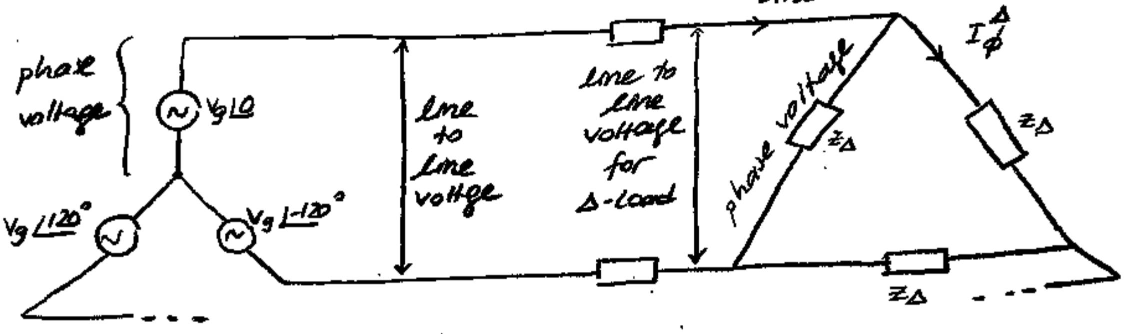
3-ϕ Systems

Y-Connection $\qquad \Delta$-Connection



phase voltage: V_g (RMS)

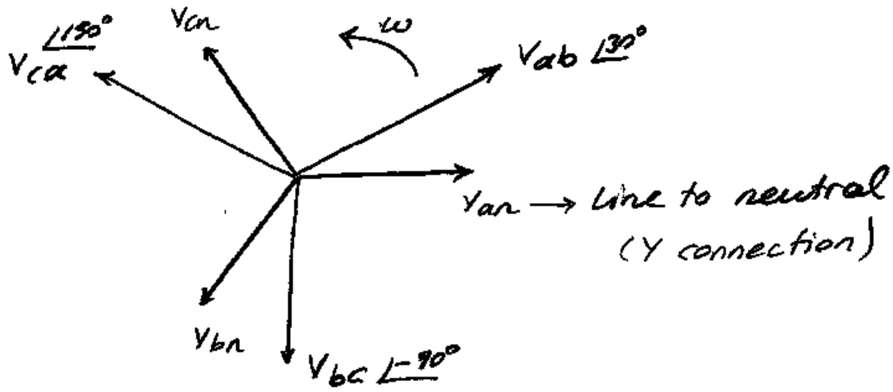
Line to Line voltage $= \sqrt{3}\,V_g$

$V_{phase} = V_{Line}$

$\sqrt{3} \cdot I_\phi^{\Delta} = I_{Line}^{\Delta}$

$$P_{Load}^{Tot,Y(\Delta)} = \sqrt{3} \cdot V_{Load,rms}^{Y(\Delta)} \cdot I_{Load,rms}^{Y(\Delta)} \cdot \underbrace{\cos(\angle\theta_Z)}_{\text{p.f of load.}}$$

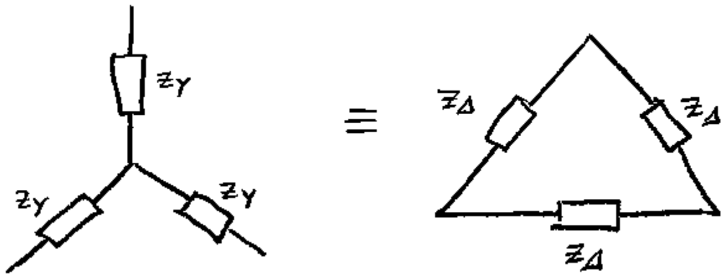
$$Q_{Load}^{Tot,Y(\Delta)} = \ \ldots \qquad \cdot \sin(\angle\theta_Z)$$



a b c (positive sequence) (a at 0°, then b at 0°, c at 0°)

a c b (negative sequence)

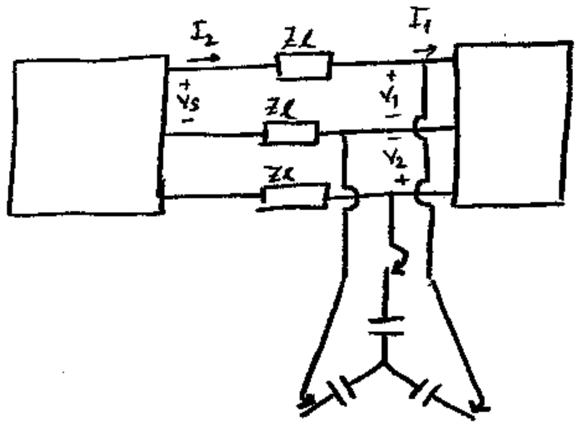
Δ-Y



$Z_\Delta = 3 Z_Y$

ZPS IV

Prob ⑥



A balanced 3-φ

Load : 400 kW, $pf = \frac{1}{\sqrt{2}}$ lagging

$V_1 = 400\angle 30°$ $V_2 = 400\angle{-30°}$ V_{rms} → RMS phasor

$Z_l = 0{,}07 + j\,0.16$

a) Switches are open. Find I_1, $V_{s,eff}$, the complex power supplied by source, and efficiency.

3φ Problems:

① By default, given powers are 3φ

② By default, given voltages are line-to-line.

a) $P_{Tot} = 400\,kW = \sqrt{3}\, V_{Line} \cdot I_{Line} \cdot \cos(\angle\theta_Z) \rightarrow I_1 = 816{,}5\,A\,(RMS)$

$S_{Line}^{Tot} = 3\, I_{1,RMS}^2 \cdot (Z_l) = 140 + j\,320\ kVA$

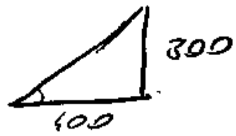
$S_{sup}^{tot} = 540 + j(320 + 400) = 540 + j\,720\ kVA$

$Eff = \frac{400}{540} = 74\,\%$

b) Switches are closed, and pf of the compensated load becomes 0,8 lagging. Find the susceptance of the capacitors in the bank, I_1, I_2, $V_{s,eff}$, Complex power supplied by the load.

Before Comp. | After comp.

$p = 400$ kW $\quad pf = 0.8$



$$|S_{Tot}| = \sqrt{3} \cdot V_{Line} \cdot I_{Line}$$

$$500 \text{ kVA} = \sqrt{3} \cdot 400 \cdot I_{Line} \longrightarrow I_{Line} = I_2 = 721.6 \text{ A}$$

$$S_{Line}^{tot} = 3 \cdot (I_{2rms})^2 \cdot Z_2 = 109 + j\,250 \text{ kVA}$$

$$S_{tot}^{sup} = 509 + j\,550 \text{ kVA}$$

$$Eff = \frac{400}{509} = 78\,\%$$

$$V_{Line}^{sup} = \frac{|S_{tot}^{sup}|}{\sqrt{3}\,|\underbrace{I_{Line}^{sup}}_{I_2 = 721.6}|} = 600 \text{ V}_{rms}$$

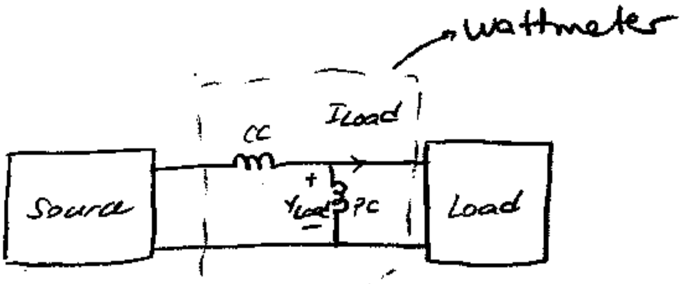
$$S_{cap\text{-}bank} = -100\,j \text{ kVAR}$$

$$\underbrace{S_{\phi}^{cap}}_{\text{per phase}} = -j\,\frac{100}{3} \text{ kVAR}$$

$$-j \cdot \frac{100}{3} = \frac{|V_{RMS}^{cap}|^2}{Z_{cap}^{*}} \longrightarrow \frac{1}{Z_{cap}^{*}} = \frac{-j\frac{100}{3} \text{ kVA}}{\left(\frac{400}{\sqrt{3}}\right)^2}$$

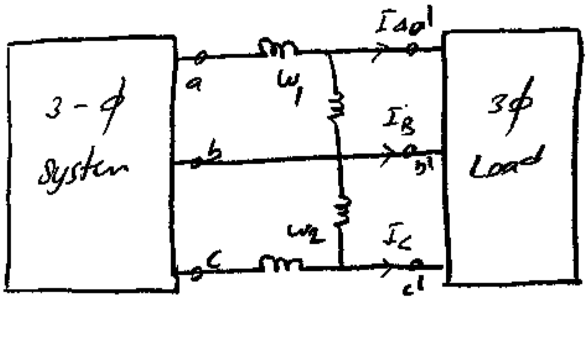
$$Z_{cap} = -j\,\frac{16}{10}$$

3-ϕ Power measurement



CC : current coil (low impedance)

PC : Potential coil (High impedance)



Measurement of W_1 : $Re\{V_{a'b'} \cdot I_A^*\}$

" W_2 : $Re\{V_{c'b'} \cdot I_C^*\}$

$V_{a'b'} = V_{line} \angle 30^\circ$

$V_{b'c'} = V_{line} \angle -90^\circ$

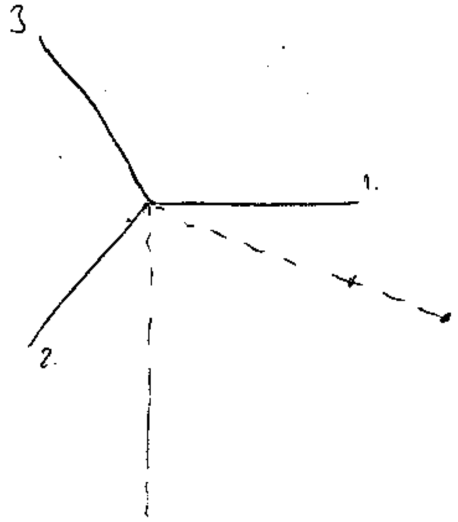
$I_A = I_{line} \angle -\theta$

$I_C = I_{line} \angle -\theta + 120^\circ$

$W_1 = Re\{V_{line} \angle 30^\circ \cdot I_{line} \angle \theta\} = V_{line} \cdot I_{line} \cos(\theta + 30^\circ)$

$W_2 = Re\{-V_{line} \angle -90^\circ \; I_{line} \angle \theta - 120^\circ\} = V_{line} \cdot I_{line} \cdot \cos(\theta - 30^\circ)$

$W_1 + W_2 = V_{line} \cdot I_{line} \; (2\cos\theta \cdot \cos(30^\circ)) = \underbrace{\sqrt{3} \cdot V_{line} \cdot I_{line} \cdot \cos\theta}_{\text{total power absorbed by the load.}}$



$V_{line} \angle -30 \cdot I_{line} \angle \theta$

$V_{line} \, I_{line} \cos(\theta - 30^\circ)$

$V_{line} \cdot \angle -90^\circ \cdot I_{line} \angle \theta - 120$

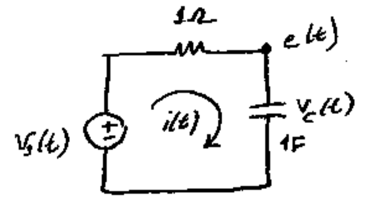
$V_{line} \, I_{line} \cos(\theta + 150^\circ)$

$2\cos(\theta : 60^\circ)\cos(90^\circ)$

Z

a) LTI dyn. circuit

i) formulation ii) solution



$(D+1)\cdot e(t) = V_S(t)$

$e(0^-) = V_0$

planar circuit: $(1+D^{-1})\, i(t) = V_S(t) - V_0$

$V_c(0^-) = V_0$

t-domain

$d(s) = s+1 = 0 \longrightarrow s = -1$

$V_{ch} = a\cdot e^{-t}$

$V_S(t) = 0$: $\left.\begin{array}{l} V_{czi}(t) = a\cdot e^{-t} \\ V_{czi}(0^-) = V_0 \end{array}\right\} a = V_0$

$V_0 = 0$; $V_{czs}(t)$

$V_{czs}(0^-) = 0$

$V_c(t) = V_{czi}(t) + V_{czs}(t)$

$V_S(t) = \delta(t)$

$\dot{h}(t) = -h(t) + \delta(t)$

$\uparrow$

$V_{czs}(t)$ $\quad h(0^-) = 0$

$h(t) = \underbrace{V_{cn}(t)}_{a\cdot e^{-t}}\cdot u(t)$ $\qquad h(0^+) = 1 \rightarrow a = 1 \longrightarrow h(t) = e^{-t}\cdot u(t)$

Note:

$h(t) = a\cdot e^{-t} u(t)$

$\dot{h}(t) = -a\cdot e^{-t} u(t) + \underbrace{a\cdot e^{-t}\,\delta(t)}_{a\cdot\delta(t)}$

$-a\cdot e^{-t} u(t) + a\cdot\delta(t) = -a\cdot e^{-t} u(t) + \delta(t)$

$a = 1$

$V_{czs}(t) = h(t) * u_S(t)$

$= \int_{0^-}^{t} h(t-t')\, u_S(t')\, dt'$

$= e^{-t}\int_{0^-}^{t} e^{t'} u_S(t')\, dt'$, $t \geq 0$

$V_S(t) = u(t)$: $\quad V_{czs}(t) = 1 - e^{-t}$ V , $t \geq 0$

$t\,u(t)$: $\quad t - 1 + e^{-t}$ V , $t \geq 0$

$e^{t} u(t)$: $\quad \frac{1}{2}(e^{t} - e^{-t})$ V , $t \geq 0$

$e^{-t} u(t)$: $\quad t\cdot e^{-t}$ V , $t \geq 0$

for $\cos(t)\cdot u(t)$: $\quad \int_0^t e^{t'}\cos(t')\,dt' = \mathrm{Re}\left\{\int_0^t e^{t'} e^{jt'}\, dt'\right\}$

$= \mathrm{Re}\left\{\frac{1}{1+j} e^{(1+j)t'}\Big|_0^t\right\} = \mathrm{Re}\left\{\underbrace{\frac{1}{1+j}}_{\frac{1}{\sqrt{2}}\angle -45°} e^{(1+j)t} - \underbrace{\frac{1}{1+j}}_{1-j}\right\}$

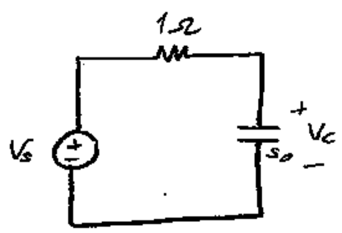
$= \frac{1}{\sqrt{2}} e^{t} (\cos t - 45°) - \frac{1}{2}$

$\cos(t) \cdot u(t)$: $\frac{1}{\sqrt{2}} (\cos t - 45°) - \frac{1}{2} e^{-t}$ V, $t \geq 0$

<u>H</u> $v_s(t) = V_s \cdot e^{s_0 t}$

let $v_{cp}(t) = v_c \cdot e^{s_0 t}$

$v_c = \frac{1}{s_0+1} V_s$



$V_c = \frac{\frac{1}{s_0}}{\frac{1}{s_0}+1} V_s = \frac{1}{s_0+1} V_s$

$v_{czs}(t) = \underbrace{a \cdot e^{-t}}_{v_{ch}(t)} + v_{cp}(t)$

$v_{czs}(0) = 0 = a + v_{cp}(0) \Rightarrow a = -v_{cp}(0)$

$v_s(t) = 1$, $t \geq 0$

$v_s = 0$, $s_0 = 0 \Rightarrow v_c = 1 \quad v_{cp}(t) = 1$

$v_{zs}(t) = \underset{\downarrow \atop -1}{a} \cdot e^{-t} + 1$

e^{jt}: $V_s = 1$, $s_0 = j$

$v_c = \frac{1}{j+1} = \frac{1}{\sqrt{2}} e^{-j45°}$

$v_{cp}(t) = Re \{ v_c \cdot e^{jt} \}$

$= \frac{1}{\sqrt{2}} \cos(t - 45°)$

$v_{czs}(t) = a e^{-t} + \frac{1}{\sqrt{2}} \cos(t - 45°)$

$a + \frac{1}{\sqrt{2}} \cos(-45°) = 0 \rightarrow a = -\frac{1}{2}$

$\dot{v}_c(t) = -v_c(t) + v_s(t)$

$v_c(0^-) = v_0$

<u>s-domain</u>

$\mathcal{L}\{ \dot{v}_c(t) \} = \mathcal{L}\{ -v_c(t) + v_s(t) \}$

$s V_c(s) - \underbrace{v_c(0^-)}_{v_0} = -V_c(s) + V_s(s)$

$(s+1) V_c(s) = v_0 + V_s(s) \longrightarrow$

$V_c(s) = \underbrace{\frac{1}{s+1} V_s(s)}_{V_{czs}(s)} + \underbrace{\frac{v_0}{s+1}}_{V_{czi}(s)}$

$v_c(t) = v_{czs}(t) + v_0 \cdot e^{-t}$

$v_s(t) = \delta(t) \Rightarrow V_s(s) = 1$

$\Rightarrow V_{czs}(s) = \frac{1}{s+1}$

$u(t) \rightarrow \frac{1}{s}$

$V_{czs}(s) = \frac{1}{s+1} \cdot \frac{1}{s} = \frac{1}{s} - \frac{1}{s+1}$

$\frac{1}{s} \cdot \frac{1}{s+1} = \frac{a}{s+1} + \frac{b}{s}$

$\frac{1}{s}\Big|_{s=-1} = a + b\,\frac{s+1}{s}\Big|_{s=-1}$

$a = -1, \quad b = 1$

$t\,u(t) \rightarrow \frac{1}{s^2}$

$V_{czs}(s) = \frac{1}{s+1} \cdot \frac{1}{s^2} = \frac{a}{s+1} + \frac{b}{s} + \frac{c}{s^2}$

$a = 1, \quad c = 1$

to find b, differentiate

$\frac{1}{s+1} = a \cdot \frac{s^2}{s+1} + bs + c \longrightarrow -\frac{1}{(s+1)^2}\Big|_{s=0} = a\,\frac{2s(s+1) - s^2}{(s+1)^2} + b + 0\Big|_{s=0}$

$b = -1$

$v_{czs}(t) = 1 \cdot e^{-t} - 1 + 1 \cdot t \quad V, \quad t \geq 0.$

$e^{t}u(t) \longrightarrow \frac{1}{s-1}$

$V_{czs}(s) = \frac{1}{(s+1)(s-1)} = \frac{1/2}{s-1} - \frac{1/2}{s+1}$

$v_{czs}(t) = -\frac{1}{2}e^{-t} + \frac{1}{2} \cdot e^{t} \quad V, \quad t \geq 0$

$e^{-t}u(t) \longrightarrow \frac{1}{s+1}$

$V_{czs}(s) = \frac{1}{(s+1)^2}$

$v_{czs}(t) = t \cdot e^{-t} \quad V, \quad t \geq 0$

$\cos t\, u(t) \longrightarrow \frac{s}{s^2+1}$

$V_{czs}(s) = \frac{s}{(s+1)(s^2+1)} = \frac{a}{s+1} + \frac{bs+c}{s^2+1}$

$a = \frac{-1}{1+1} = -\frac{1}{2}$

$-\frac{1}{2}(s^2+1) + bs^2 + bs + cs + c = s$

$(b - \frac{1}{2})s^2 + (b+c)s + (c - \frac{1}{2}) = s$

$b = \frac{1}{2} \quad c = \frac{1}{2}$

$v_{czs}(t) = -\frac{1}{2}e^{-t} + \underbrace{\frac{1}{2}\cos t + \frac{1}{2}\sin t}_{\frac{1}{\sqrt{2}}\cos(t - 45^\circ)} \quad V, \quad t \geq 0$

$\cos t\, u(t) \rightarrow \frac{s}{s^2+1}$

$V_{czs}(s) = \frac{a}{s+1} + \frac{b}{s-j} + \frac{c}{s+j}$

$a = -\frac{1}{2}$

$b = \frac{s}{(s+1)(s+j)}\Big|_{s=j} = \frac{j}{(j+1)\,2j} = \frac{1}{2} \cdot \frac{1}{j+1} = \frac{1-j}{4}$

$c = \frac{s}{(s+1)(s-j)}\Big|_{s=-j} = \frac{1+j}{4} = b^*$

$v_{czs}(t) = -\frac{1}{2} \cdot e^{-t} + \underbrace{b \cdot e^{jt} + b^* e^{-jt}}_{2\,\mathrm{Re}\{b \cdot e^{jt}\}} = 2\,\mathrm{Re}\left\{\frac{1}{2\sqrt{2}}\,e^{-j45^\circ} \cdot e^{jt}\right\} = \frac{1}{\sqrt{2}}\cos(t - 45^\circ)$

$$e^{jt}\cdot u(t) \longrightarrow \frac{1}{s-j}$$

$$V_{c_{zs}}(s) = \frac{1}{(s+1)(s-j)} = \frac{a}{s+1} + \frac{b}{s-j}$$

$$a = \frac{1}{-1-j} = -\frac{1-j}{2}$$

$$b = \frac{1}{j+1} = \frac{1}{\sqrt{2}} e^{-j45^\circ}$$

$$V_{c_{zs}}(t) = -\frac{1-j}{2} e^{-t} + \frac{1}{\sqrt{2}} e^{-j45^\circ} e^{jt} \quad V,\ t \geq 0$$

$$\cos(t)\cdot u(t) = \mathrm{Re}\left\{ V_{c_{zs}}(t) \right\} u(t)$$

$$v_R(t) = R \cdot i_R(t)$$

$$V_R(s) = R \cdot I_R(s)$$

$$v_L(t) = L \cdot D \cdot i_L(t)$$

$$V_L(s) = L\left(s I_L(s) - I_0\right)$$

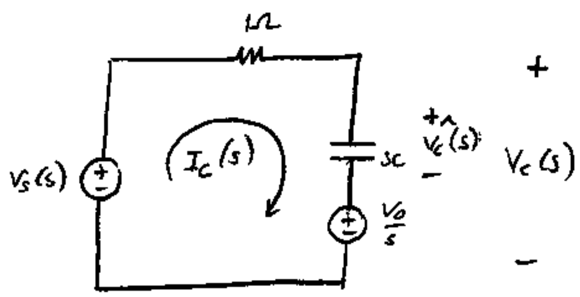
$$= sL I_L(s) - L I_0$$

$$i_c(t) = c \cdot D\, v_c(t)$$

$$I_c(s) = C\left(s V_c(s) - V_0\right)$$

$$= sC\, V_c(s) - C V_0$$

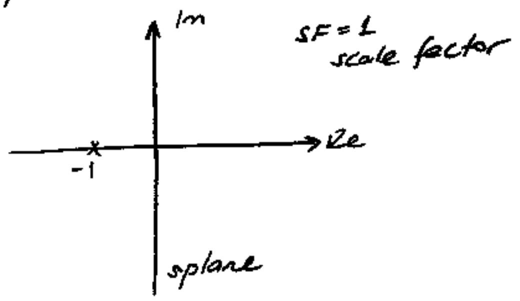
$$V_c(s) = \underbrace{\frac{1}{sC} I_c(s)}_{\hat{V_c}(s)} + \frac{V_0}{s}$$



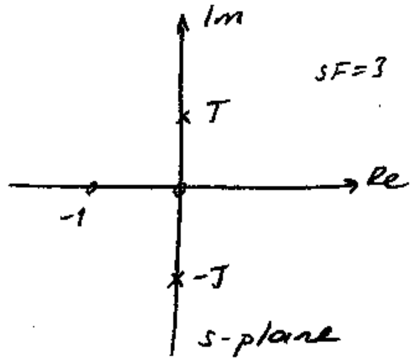
$$V_c(s) = \frac{V_0}{s} + \frac{\frac{1}{s}}{\frac{1}{s}+1}\left(V_s(s) - \frac{V_0}{s}\right)$$

$$H(s) = \frac{1}{s+1} \leftarrow \text{a system function}$$

pole / zero diagram



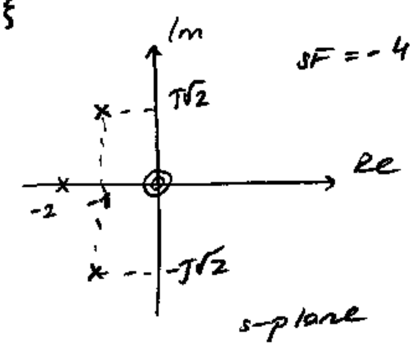
$$H(s) = \frac{3s(s+1)}{s^2+1}$$



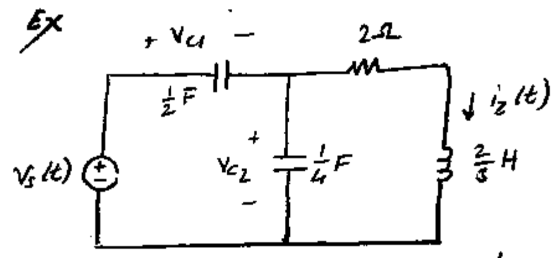
$$H(s) = \frac{P(s)}{Q(s)}$$ rational function.
real rational function

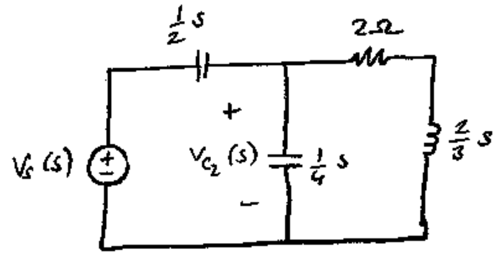
$$\deg H = \max\{\deg P, \deg Q\}$$

$$H(s) = \frac{-4s^2}{(s+2)(s^2+2s+3)}$$



Ex





at zero-initial state

$v_s(t) = 4\cos(2t - 15°)$

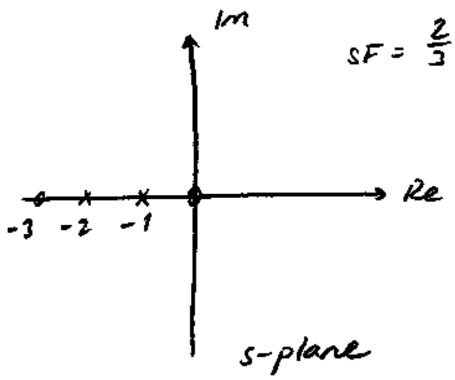
$$Z_0 = \frac{4}{s} \parallel \left(2 + \frac{2}{3}s\right)$$

$$V_{C_2}(s) = \frac{Z_0}{Z_0 + \frac{2}{s}} V_s(s)$$

$$Z_0 = \frac{\frac{8}{s} + \frac{8}{3}}{\frac{4}{s} + 2 + \frac{2}{3}s}$$

$$= \frac{8\left(\frac{s}{3} + 1\right)}{4 + 2s + \frac{2s^2}{3}} = \frac{4(s+3)}{s^2+3s+6}$$

$$V_{C_2}(s) = \frac{\frac{4(s+3)}{s^2+3s+6}}{\frac{4(s+3)}{s^2+3s+6} + \frac{2}{s}} V_s(s) = \underbrace{\frac{2}{3}\,\frac{s(s+3)}{(s+1)(s+2)}}_{H(s)} V_s(s)$$



$SF = \frac{2}{3}$

$$\frac{s(s+3)}{s^2+3s+2} = 1 - \frac{2}{s^2+3s+2}$$

$$H(s) = \frac{2}{3} - \frac{4}{3}\,\frac{1}{(s+1)(s+2)}$$

$$= \frac{2}{3} - \frac{4}{3}\,\frac{1}{s+1} + \frac{4}{3}\,\frac{1}{s+2}$$

$$h(t) = \frac{2}{3}\delta(t) - \frac{4}{3}e^{-t} + \frac{4}{3}e^{-2t}$$

19.04.2010

Brief Review of Laplace Transform and S-domain Analysis

$$(D^2 + 3D + 2)\,x(t) = f(t)$$

$$\Downarrow$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \\ \\ \end{bmatrix} f(t)$$

$$F(s) = \int_{-\infty}^{\infty} f(t)\,e^{-st}\,dt \quad , \; s \in R.O.C. \;\left(\text{for which Laplace transform integral converges}\right)$$

$\hookrightarrow s \in \mathbb{C}$

$\mathcal{L}\{f(t)\} = F(s)$

$$f(t) = \frac{1}{2\pi j} \int_{\delta - j\infty}^{\delta + j\infty} F(s) e^{st} ds \cdot (?) (\delta \in ROC)$$

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t) e^{-st} dt = \int_0^\infty e^{-st} dt = \frac{e^{-st}}{s} \Big|_{t=\infty}^{t=0} = \frac{1}{s}, \quad s > 0$$

$Re\{s\} > 0$

Two types of Laplace transforms:

① Unilateral

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

② Bilateral

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

↳ more of interest, since we have initial value problems.

$$\mathcal{L}\{\dot f(t)\} = \int_{0^-}^{\infty} \dot f(t) e^{-st} dt = \underbrace{e^{-st} f(t) \Big|_{t=0^-}^{\infty}}_{-f(0^-)} + \underbrace{\int f(t) s \, e^{-st} dt}_{sF(s)}$$

$$= sF(s) - f(0)$$

$$\mathcal{L}\{\ddot f(t)\} = s\mathcal{L}\{\dot f(t)\} - \dot f(0) = s^2 \cdot \mathcal{L}\{f(t)\} - sf(0) - \dot f(0)$$

$$(D^2 + 3D + 2) \, v_c(t) = f(t) \longrightarrow u(t)$$

$v_c(0^-) = V_0$

$\dot v_c(0^-) = \dot V_0$

Let there be a solution,

$$v_c(t) \rightleftharpoons V_c(s)$$

$$\dot v_c(t) \rightleftharpoons sV_c(s) - V_0$$

$$\ddot v_c(t) \rightleftharpoons s^2 V_c(s) - sV_0 - \dot V_0$$

$$\mathcal{L}\{\alpha f_1(t) + \beta f_2(t)\} = \alpha \mathcal{L}\{f_1(t)\} + \beta \mathcal{L}\{f_2(t)\}$$

$$(D^2 + 3D + 2) v_c(t) \rightleftharpoons V_c(s)[s^2 + 3s + 2] - V_0(s+3) - \dot V_0$$

$$= u(t) \qquad = \frac{1}{s}$$

$$V_c(s)[s^2 + 3s + 2] - V_0(s+3) - \dot V_0 = \frac{1}{s}$$

$$V_c(s) = \frac{s(s+3)V_0 + s\dot V_0 + 1}{s(s^2 + 3s + 2)}$$

$\mathcal{L}\{u(t)\} = \frac{1}{s}$

$\mathcal{L}\{\delta(t)\} = 1$

$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}$

$\mathcal{L}\{\cos(\alpha t)\} = \frac{s}{s^2+\alpha^2}$

$\mathcal{L}\{\sin(\alpha t)\} = \frac{\alpha}{s^2+\alpha^2}$

$$V_c(s) = \frac{s(s+3)V_0 + s\dot{V}_0 + 1}{s(s+1)(s+2)}$$

partial fraction expansion

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$s=0 \longrightarrow A = \frac{1}{2}$

$s=-1 \longrightarrow B = \frac{-2V_0 - \dot{V}_0 + 1}{-1}$

$s=-2 \longrightarrow C = \frac{-2V_0 - 2\dot{V}_0 + 1}{2}$

$\mathcal{L}^{-1}\{\ \}$ of RHS / LHS

$$v_c(t) = \frac{1}{2}u(t) + (2V_0 + \dot{V}_0 - 1)e^{-t} + (-V_0 - \dot{V}_0 + \tfrac{1}{2})e^{-2t}, \quad t \geq 0.$$

Ex
$$F(s) = \frac{s-2}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$\boxed{\mathcal{L}\{V_0\} = \frac{V_0}{s}}$$

$s=0 \longrightarrow A = -2$

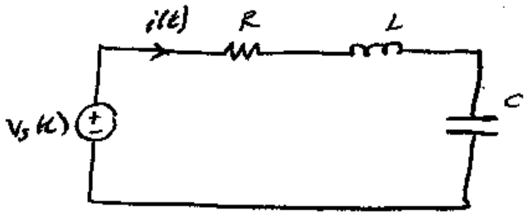
$s=-1 \longrightarrow C = 3$

$$F(s)\cdot(s+1)^2 = \frac{s-2}{s(s+1)^2}(s+1)^2 = \frac{A}{s}(s+1)^2 + B(s+1) + C$$

$$\frac{d}{ds}\Big(\quad \frac{2}{s^2} = A\left(1 - \frac{1}{s^2}\right) + B \longrightarrow s=-1 \longrightarrow B = 2.$$

Circuit Response Using s-Domain Methods

① Taking Laplace Transform of Time-Domain Relation:



$V_c(0^-) = V_0$

$I_L(0^-) = I_0$

KVL:
$$-V_s(t) + R\,i(t) + L\frac{d}{dt}i(t) + V_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i(\tau)d\tau = 0$$

$\big(\mathcal{L}\{\ \}$

$$-V_s(s) + R\cdot I(s) + L\big(sI(s) - I_0\big) + \frac{V_0}{s} + \frac{1}{C}\cdot\frac{I(s)}{s} = 0$$

$$I(s) = \frac{Cs\,V_s(s) + sLC\,I_0 - CV_0}{s^2LC + RCs + 1}$$

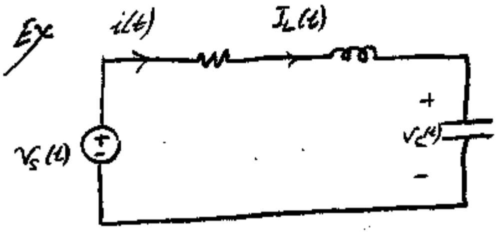
$$I(s) = \frac{\frac{s V_s(s)}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} + \frac{s I_0 - \frac{1}{L}V_0}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$i(t)$: complete solution $= i^{zs}(t) + i^{zi}(t)$

22.04.2010

Circuit Response using Laplace Transform

Ex



$V_c(0^-) = V_0$

$I_L(0^-) = I_0$

$$I(s) = \underbrace{\frac{s I_0 - V_0/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}}_{zi} + \underbrace{\frac{s V_s(s)/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}}_{zs}$$

① Let $L = 1H$; $R = 6\Omega$; $C = 0{,}04F$

$I_0 = 5A$; $V_0 = 1V$

$$I(s) = \frac{s V_s(s)}{s^2+6s+25} + \frac{5s-1}{s^2+6s+25} \quad (*)$$

Also let $V_s(t) = 12 \sin(5t) \quad t \geq 0$

$$\rightarrow V_s(s) = \mathcal{L}\{12 \sin(5t)\} = \frac{60}{s^2+25}$$

$$I(s) = \frac{60s}{(s^2+6s+25)(s^2+25)} + \frac{5s-1}{s^2+6s+25}$$

$$\hookrightarrow (s+3)^2 + 16 = (s+3+j4)(s+3-j4)$$

$$I^{zs}(s) = \frac{60s}{(s^2+6s+25)(s^2+25)}$$

$$= \frac{K_1}{s+3+j4} + \frac{K_1^*}{s+3-j4} + \frac{K_2}{s+j5} + \frac{K_2^*}{s-j5}$$

$$K_1 = \left.\frac{60s}{(s+3-j4)(s+j5)(s-j5)}\right|_{s=-3-j4} = -j1{,}25$$

$$K_2 = I^{zs}(s) \cdot (s+j5)\Big|_{s=-j5} = j$$

$$I^{zs}(s) = \frac{j1,25}{s+3-j4} + \frac{-j1,25}{s+3+j4} + \frac{-j}{s-j5} + \frac{j}{s+j5}$$

$$I^{zs}(s) = \frac{-10}{(s+3)^2+16} + \frac{10}{s^2+25}$$

$$\mathcal{L}^{-1}\{\}\quad i^{zs}(t) = \underbrace{-\frac{5}{2}\cdot e^{-3t}\sin(4t)}_{\text{transient part for zs solution}} + \underbrace{2\sin(5t)}_{\text{steady state part of zero state response}}$$

$$\mathcal{L}^{-1}\{\} = i^{zs}(t) = j1,25\, e^{(-3+j4)t} - j1,25\cdot e^{-(3+j4)t} - j e^{j5t} + j e^{-j5t}$$

$$= 2\,Re\{j1,25\, e^{-(3-j4)t}\} + 2Re\{-j e^{j5t}\}$$

$$= \frac{5}{2}e^{-3t} Re\{\underbrace{e^{j(4t+90°)}}_{\cos(4t+90°)}\} - 2\,Re\{\underbrace{e^{j(5t+90°)}}_{\cos(5t+90°)}\}$$

$$I^{zi}(s) = \frac{5s-1}{(s+3)^2+4^2} = \frac{k_1}{s+3+j4} + \frac{k_1^*}{s+3-j4}$$

$$k_1 = \frac{5s-1}{s+3-j4}\Bigg\downarrow_{s=-3-j4} = \frac{-16-j20}{-j8} = \frac{20-j16}{8} = \frac{5}{2} - j2$$

$$i^{zi}(t) = 2\,Re\{k_1\cdot e^{-(3+j4)t}\} = 2\cdot e^{-3t} Re\{(\tfrac{5}{2}-j2)(\cos 4t - j\sin 4t)\}$$

$$= 2e^{-3t}\left[\frac{5}{2}\cos(4t) - 2\sin(4t)\right]$$

$$= 2e^{-3t}\sqrt{\frac{25}{4}+4}\cdot\cos\left(4t - \tan^{-1}\frac{-2}{5/2}\right)$$

② Step response: (zero-state response for $u(t)$ input)

Same values for R, L, C etc.

$$I^{zs}(s) = \frac{s\,V_s(s)}{s^2+6s+25} \longleftarrow \mathcal{L}\{u(t)\} = 1/s = V_s(s)$$

$$I^{zs}(s) = \frac{1}{(s+3)^2+4^2} = \frac{1}{4}e^{-3t}\sin(4t)$$

③ Impulse response:

$$I^{zs}(s) = \frac{s\,V_s(s)}{s^2+6s+25} \longrightarrow \mathcal{L}\{\delta(t)\} = 1$$

$$= \frac{s}{(s+3)^2+4^2} = \frac{s+3}{(s+3)^2+4^2} + \frac{-3}{(s+3)^2+4^2} = e^{-3t}\cos(4t) - \frac{3}{4}e^{-3t}\sin(4t)$$

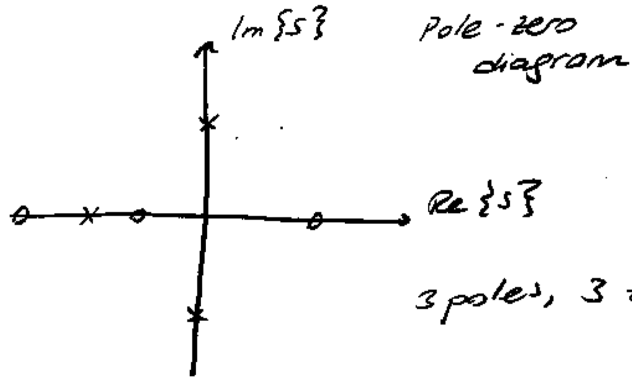
Poles and zeros in S-domain

$$I^{zs}(s) = K \cdot \frac{\prod_{k=1}^{Z} (s - z_k)}{\prod_{k=1}^{P} (s - p_k)}$$

a ratio of two polynomials in "s domain" $\mathcal{L}\{i^{zs}(t)\}$

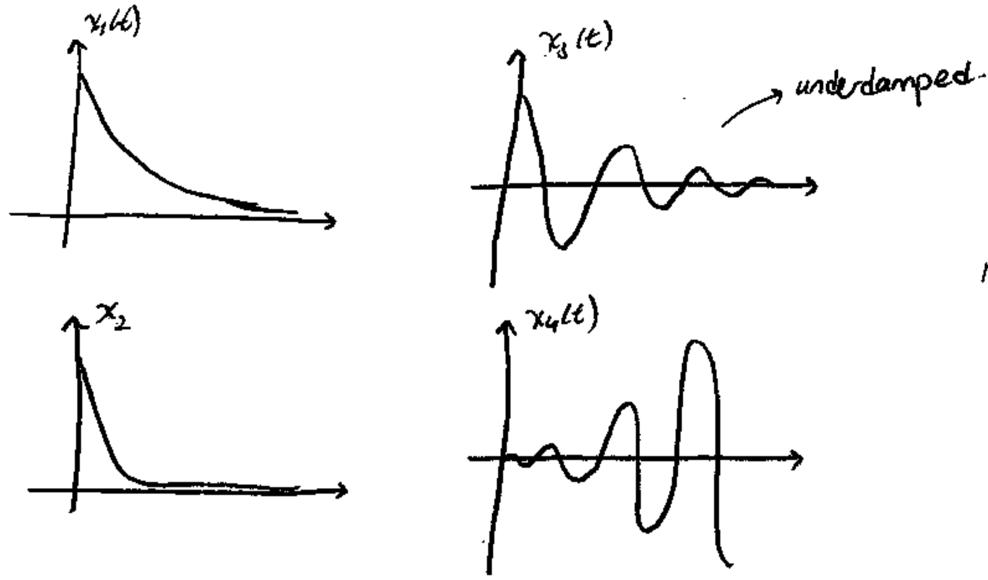
p_k : Poles of $I^{zs}(s)$ (singularities)

z_k : zeros of $I^{zs}(s)$ $\quad (I^{zs}(z_k) = 0)$

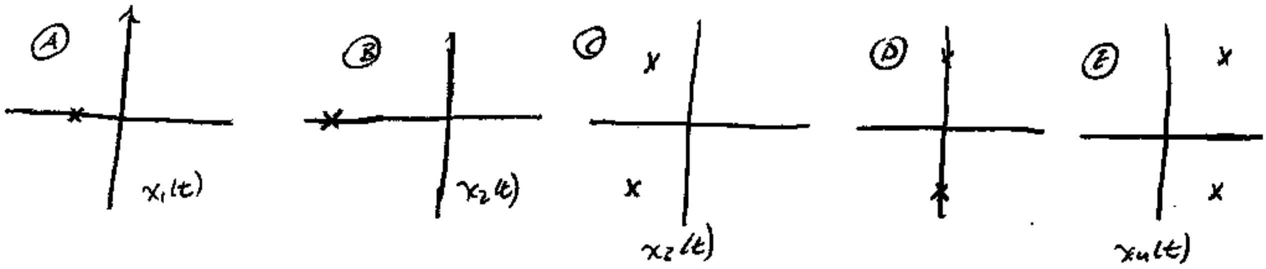


3 poles, 3 zeros!

Ex



monoment
monoment



distance to Im{s} determines how fast it decay and distance to Re{z} its frequency.

Initial and Final Value Theorems

Initial Value: $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$

Final Value $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

Ex Step response:

$i^{zs}(t) = \frac{1}{4} e^{-3t} \sin 4t \xrightarrow{\mathcal{L}\{\}} I^{zs}(s) = \frac{1}{s^2+6s+25}$

$i^{zs}(0^+) = 0$ $\quad \lim_{s \to \infty} s\, I^{zs}(s) = 0$; $\longrightarrow$ initial value theorem

$i^{zs}(\infty) = 0$ $\quad \lim_{s \to 0} s I^{zs}(s) = 0$; $\longrightarrow$ Final value theorem

Some important details

To use initial value theorem $s=\infty$ should be in R.O.C.

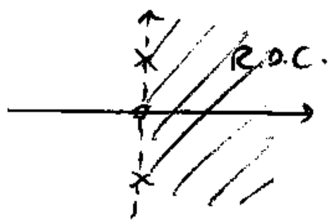
Final value theorem $s=0$

$\mathcal{L}\{\cos(\beta t)\} = \frac{s}{s^2+\beta^2} \xrightarrow[\text{theorem}]{\text{final value}} \lim_{s \to 0} s \frac{s}{s^2+\beta^2} = 0$!!

but $\lim_{t \to \infty} \cos\beta t$ does not exist.

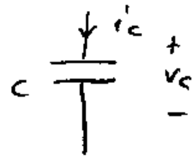
do not never forget R.O.C. when $\mathcal{L}$ is discussed.

$\mathcal{L}\{\cos\beta t\} = \int_0^\infty \cos\beta t \cdot e^{-st} dt$ has R.O.C.



Circuit Components in s-domain

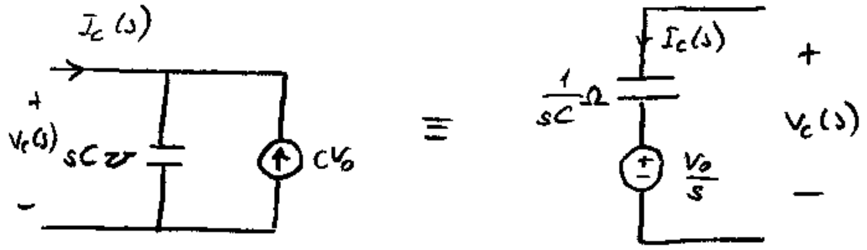
Capacitor: time domain s-domain

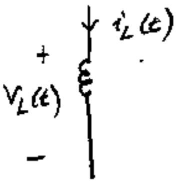


$i_c(t) = C \frac{dv_c(t)}{dt}$

$I_c(s) = Cs\, V_c(s) - C V_c(0^-)$

$V_c(s) = \frac{I_c(s)}{sC} + \frac{V_0}{s}$

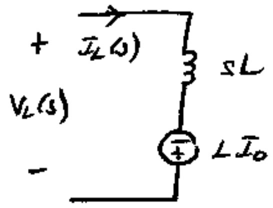




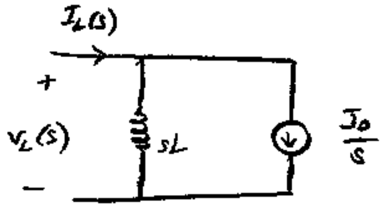
$$V_L(t) = L \cdot \frac{d\, i_L(t)}{dt}$$

$$i_L(0^-) = I_0$$

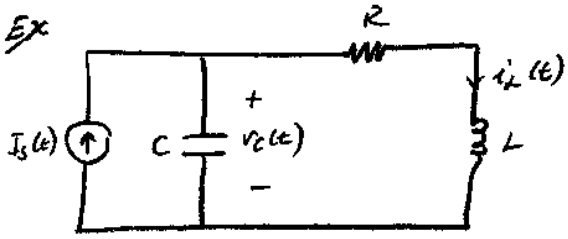
$$V_L(s) = Ls\, I_L(s) - L \cdot I_0$$



$$I_L(s) = \frac{V_L(s)}{sL} + \frac{I_0}{s}$$



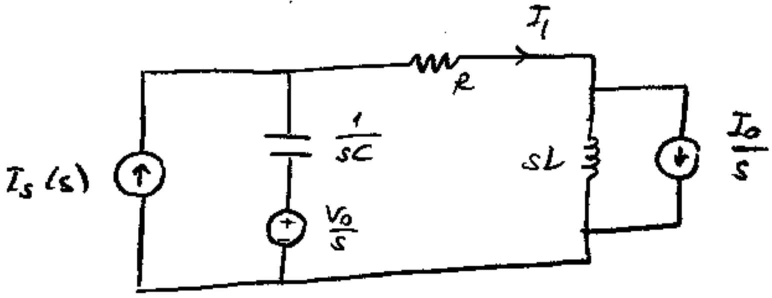
Ex



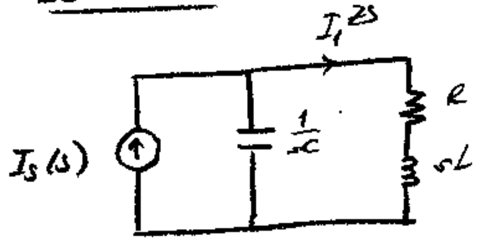
$$V_C(0^-) = V_0$$

$$I_L(0^-) = I_0$$

↓ s domain



zero-state:

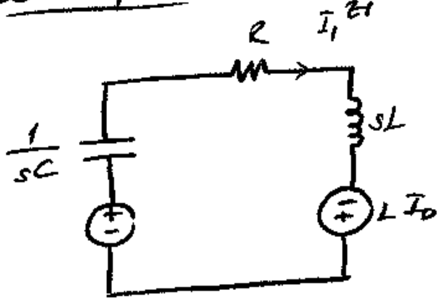


$$I_1^{zs}(s) = I_s(s)\,\frac{\frac{1}{sC}}{\frac{1}{sC} + sL + R}$$

$$= \frac{1}{LCs^2 + sRC + 1}\, I_s(s)$$

$$= \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\, I_s(s)$$

Zero-input



$$I_1^{zi}(s) = \frac{\frac{V_0}{s} + L I_0}{\frac{1}{sC} + R + sL} = \frac{\frac{V_0}{L} + s I_0}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Zero-state:

$$I_1(s) = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} I_s(s) \longrightarrow \left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) I_1(s) = \frac{I_s(s)}{LC}$$

$$\Big\downarrow \mathcal{L}^{-1}\{\ \}$$

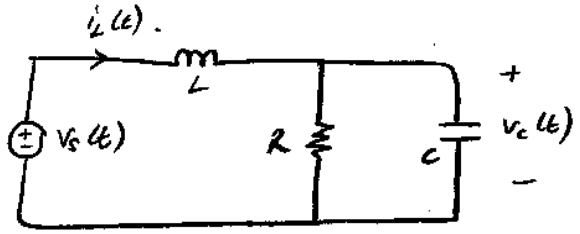
$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) i_1^{zs}(t) = \frac{i_s(t)}{LC}$$

$$i_1^{zs}(0^-) = 0 \ ; \ \frac{d}{dt} i_1^{zs}(0) = 0.$$

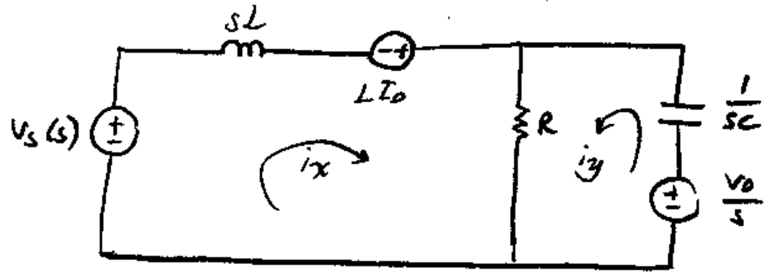
natural freq: roots of char. polynomial

poles of $I_1^{zs}(s)$ solution $\quad I_1^{zs}(s) = \dfrac{1/LC \ \ I_s(s)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$

Ex Mesh Analysis in S-domain



S domain $\Big\downarrow$ $\quad i_L(0^-) = I_o$, $\quad v_c(0^-) = V_o$



$$\begin{bmatrix} sL + R & R \\ sCR & 1 + sCR \end{bmatrix} \begin{bmatrix} i_x(s) \\ i_y(s) \end{bmatrix} = \begin{bmatrix} V_s(s) + LI_o \\ CV_o \end{bmatrix}$$

$$i_x(s) = \frac{\begin{vmatrix} V_s(s) + LI_o & R \\ CV_o & 1 + sRC \end{vmatrix}}{\begin{vmatrix} sL + R & R \\ sRC & 1 + sRC \end{vmatrix}} = \frac{(V_s(s) + LI_o)(1 + sRC) - RCV_o}{s^2 RLC + sL + R}$$

$$= \frac{\frac{V_s(s)}{RCL}(1 + sRC) + \left(\cancel{L}\frac{I_o}{RC} - \cancel{RC}\frac{V_o}{L}\right) + s\cancel{RCL}\, I_o}{s^2 \cancel{RLC} + \frac{s\cancel{L}}{RC} + \frac{\cancel{R}}{LC}}$$

$$I_X(s) = \frac{\left(\frac{1}{RCL} + \frac{s}{L}\right)}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} V_S(s) + \frac{\frac{I_0}{RC} - \frac{V_0}{L} + s I_0}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Let $R = 6\Omega$, $L = 6H$, $C = \frac{1}{48}F$

$$I_X(s) = \frac{1 + s/6}{s^2 + 6s + 8} V_S(s) + \frac{6I_0 - \frac{V_0}{6} + s I_0}{s^2 + 6s + 8}$$

$$\downarrow$$

$$\frac{-\frac{1}{2}\left(2I_0 - \frac{V_0}{6}\right)}{s+4} + \frac{\frac{1}{2}\left(4I_0 - \frac{V_0}{6}\right)}{s+2}$$

$$I_X^{zi}(s) = \frac{\frac{V_0}{12} - I_0}{s+4} + \frac{2I_0 - \frac{V_0}{12}}{s+2}$$

$$I_X^{zs}(s) = \frac{1 + s/6}{s^2 + 6s + 8} V_S(s)$$

→ Assume we have zero input case, find initial conditions to excite the mode with $\lambda = -2$.

$$i_X^{zi}(t) = \left[\left(\frac{V_0}{12} - I_0\right)e^{-4t} + \left(2I_0 - \frac{V_0}{12}\right)e^{-2t}\right]u(t)$$

then $V_0 = 12 I_0$ $\qquad \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \alpha \begin{bmatrix} 12 \\ 1 \end{bmatrix} \rightarrow$ excites $\lambda = -2$ mode only.

→ Assume $V_S(t) = 16 \cos 2t \, u(t)$ find $i_X^{zs}(t)$.

$$I_X^{zs}(s) = \frac{1 + s/6}{s^2 + 6s + 8} \cdot \frac{16s}{s^2 + 4}$$

$$= \frac{\frac{2/3\,(-32)}{2 \cdot 8}}{s+2} + \frac{\frac{1/3 \cdot 16\,(-4)}{-2 \cdot 20}}{s+4} + \frac{As + B}{s^2 + 4}$$

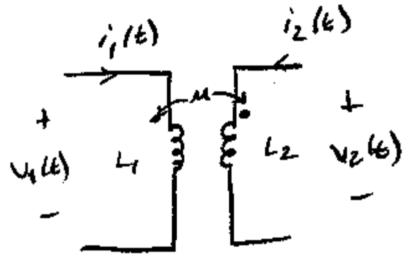
$$s = 0 \longrightarrow \quad 0 = -\frac{2}{3} + \frac{2}{15} + \frac{B}{4} \longrightarrow B = \frac{32}{15}$$

$$\frac{8}{3} = \frac{-16}{3} + \frac{16}{5} + 3 + 6A \longrightarrow A = \frac{16}{3}$$

$$I_X^{zs}(s) = \frac{-4/3}{s+2} + \frac{8/15}{s+4} + \frac{16/3\, s + 32/15}{s^2 + 4}$$

$$i_X^{zs}(t) = -\frac{4}{3}e^{-2t} + \frac{8}{15}e^{-4t} + \frac{16}{3}\cos(2t) + \frac{16}{15}\sin(2t)$$

Coupled Inductor



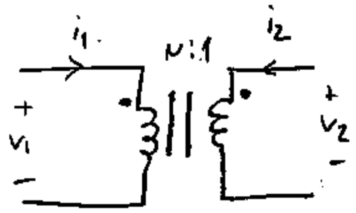
$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \dot{i}_1(t) \\ \dot{i}_2(t) \end{bmatrix}$$

$$i_1(0^-) = I_1$$

$$i_2(0^-) = I_2$$

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} sI_1(s) - I_1 \\ sI_2(s) - I_2 \end{bmatrix}$$

Transformer:



$$\frac{i_1(t)}{i_2(t)} = -\frac{1}{N} \qquad \frac{v_1(t)}{v_2(t)} = N$$

$$\frac{I_1(s)}{I_2(s)} = -\frac{1}{N} \qquad \frac{V_1(s)}{V_2(s)} = N$$

Network Functions

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$\underbrace{H(s)}_{\text{network function}} = \frac{\overbrace{\mathcal{L}\{y^{zs}(t)\}}^{\text{output at zero-state}}}{\underbrace{\mathcal{L}\{x(t)\}}_{\text{input}}}$$

Let $\dfrac{I_L^{zs}(s)}{V_s(s)} = \dfrac{s}{s+1} \longrightarrow (s+1)\, I_L^{zs}(s) = V_s(s) \cdot s$

$\mathcal{L}^{-1}\{ \}$

$$(D+1)\, \underbrace{i_L^{zs}(t)}_{\text{output (unknown)}} = \frac{d}{dt} \underbrace{v_s(t)}_{\text{input}}$$

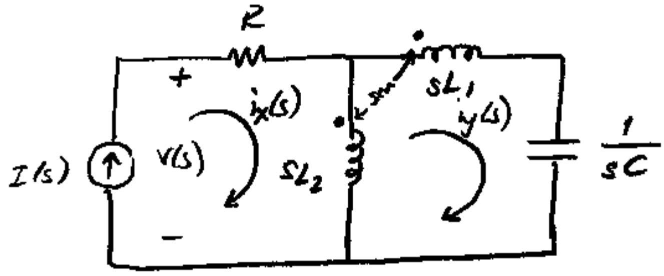
$$x_3 = \alpha x_1 + \beta x_2 \longrightarrow \boxed{LTI} \longrightarrow y_3 = \alpha y_1 + \beta y_2$$

$$\frac{\mathcal{L}\{y_3^{zs}(t)\}}{\mathcal{L}\{x_3(t)\}} = H(s)$$

$$\begin{aligned} \mathcal{L}\{y_3^{zs}(t)\} &= H(s)\, \mathcal{L}\{x_3(t)\} \\ &= H(s) \cdot \mathcal{L}\{\alpha x_1(t)\} + H(s)\, \mathcal{L}\{\beta x_2(t)\} \\ &= \mathcal{L}\{y_1^{zs}(t) + y_2^{zs}(t)\} \end{aligned}$$

The ZS solutions obey the superposition principle so ZS solution can be found by superposition method.
If initial conditions are non-zero, they should be treated separately by ZI solution.

Ex



Note that no initial conditions since network functions are always expressed in ZS.

$$Z(s) = \frac{V(s)}{I(s)}$$

$$I_x(s) = I(s)$$

$$V(s) = R \cdot I_s(s) + sL_2\left(I_s(s) - I_y(s)\right) + sM \cdot I_y(s)$$

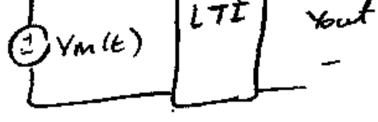
KVL: $-V_{L_2} + V_{L_1} + I_y(s) \cdot \frac{1}{sC} = 0$

$$-\left(sL_2\left(I(s) - I_y(s)\right) + sM\, I_y(s)\right)$$
$$+\left(sL_1 I_y(s) + sM\left(I(s) - I_y(s)\right)\right) + \frac{I_y(s)}{sC} = 0$$

Find $I_y(s)$ in terms of $I(s)$.
Replace $I_y(s)$ in the output relation and express output in terms of $I(s)$ (input)

$$Z(s) = \frac{V(s)}{I(s)} = \frac{R + s^2 RC(L_1 + L_2 - 2M) - s^3 \ldots}{s^2 C(L_1 + L_2 - 2M) + 1}$$

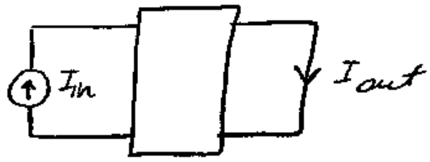
Network Functions



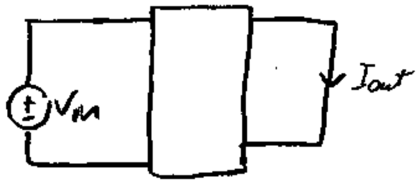
$$H_V(s) = \frac{V_{out}(s)}{V_{in}(s)}$$

$\downarrow$

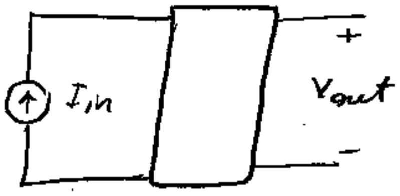
voltage transfer function.



$\longrightarrow$ current transfer function

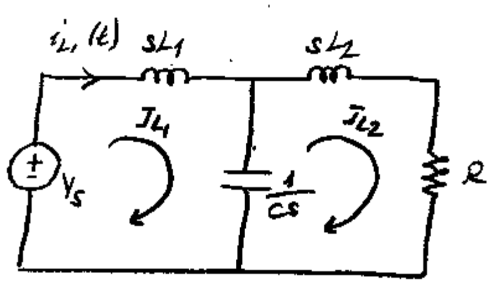


$\longrightarrow$ Transfer admittance : $\dfrac{I_{out}(s)}{V_{in}(s)}$



$\longrightarrow$ Transfer impedance : $\dfrac{V_{out}(s)}{I_{in}(s)}$

Ex



Find $\dfrac{I_{L_1}(s)}{V_s(s)}$

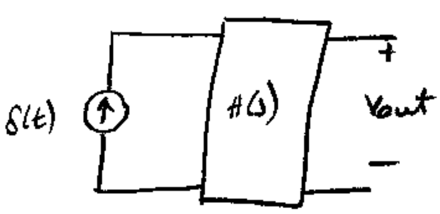
KVL: $-V_s(s) + I_{L_1}(s)\left(sL_1 + \frac{1}{sC}\right) - I_{L_2}\frac{1}{sC} = 0$

$$-I_{L_1}(s)\cdot\frac{1}{sC} + I_{L_2}(s)\left(sL_2 + R + \frac{1}{sC}\right) = 0$$

$$\begin{bmatrix} sL_1 + \frac{1}{sC} & -\frac{1}{sC} \\ -\frac{1}{sC} & \frac{1}{sC} + sL_2 + R \end{bmatrix}\begin{bmatrix} I_{L_1} \\ I_{L_2} \end{bmatrix} = \begin{bmatrix} V_s(s) \\ 0 \end{bmatrix}$$

$$I_{L_1}(s) = \frac{V_s(s)\cdot\left(\frac{1}{sC} + sL_2 + R\right)}{\frac{L_1}{C} + s^2L_1L_2 + sRL_1 + \frac{L_2}{C} + \frac{R}{sC}}$$

Network Functions and Impulse Response:



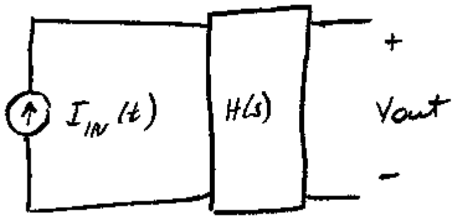
$V_{out}(s) = H(s)\cdot\mathcal{L}\{\delta(t)\}$

$V_{out}(s) = H(s)$

$H(s) = \mathcal{L}\{V_{out}(t) \text{ for } \delta(t) \text{ input}\}$

$\downarrow$

impulse response



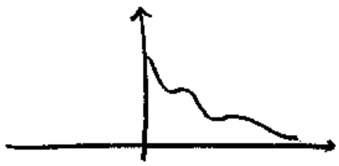
$V_{out}(s) = H(s) \cdot I_{in}(s)$

$\mathcal{L}^{-1}\{\}$ impulse response in s domain.

$$V_{out}(t) = \int_{-\infty}^{\infty} h(\tau) \cdot I_{in}(t-\tau)\, d\tau \leftarrow \text{convolution}$$

$$V_{out}(t) = \underbrace{h(t) * I_{in}(t)}_{\text{convolution}}$$

About the integral:



$h(t) = 0 \quad t<0$

$I_{IN}(t)$ is given for $t>0$.

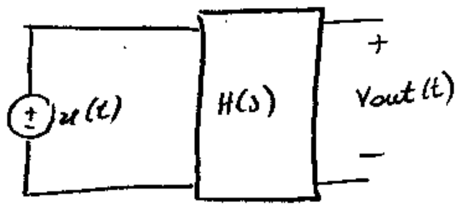
$I_{in}(t) = 0 \quad t<0$

$$V_{out}(t) = \int_{-\infty}^{\infty} h(\tau)\, I_{IN}(t-\tau)\, d\tau = \cancel{\int_{-\infty}^{0} \dots} + \int_{0}^{t} \dots + \cancel{\int_{t}^{\infty} \dots}$$

$h(\tau) = 0$ $\qquad$ $I_{in}(t-\tau) = 0$

$$\boxed{V_{out}(t) = \int_{0}^{t} h(\tau)\, I_{in}(t-\tau)\, d\tau}$$ (for causal systems)

Step Response



$V_{out}(s) = H(s) \cdot V_{in}(s)$

$\mathcal{L}\{u(t)\} = \frac{1}{s}$

$V_{out}(s) = H(s) \cdot \frac{1}{s}$

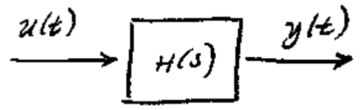
$$\xrightarrow{\mathcal{L}^{-1}\{\}} V_{out}^{step}(t) = \mathcal{L}^{-1}\left\{ \frac{H(s)}{s} \right\} = \int_{0}^{t} \underbrace{h(\tau)}_{\text{impulse response}} d\tau$$

or by convolution:

$$V_{out}^{step} = \int_{-\infty}^{\infty} h(\tau)\, v_{in}(t-\tau)\, d\tau = \int_{0}^{t} h(\tau)\, \underbrace{u(t-\tau)}_{1}\, d\tau$$

$$= \int_{0}^{t} h(\tau)\, d\tau$$

Ex



$$H(s) = \frac{1}{(s+2)(s+3)} = \frac{V_{out}^{zs}(s)}{V_{in}(s)}$$

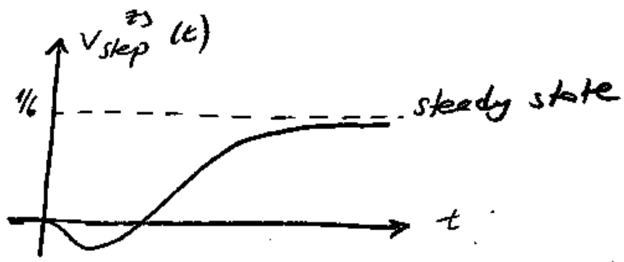
$$(s+2)(s+3)\,V_{out}^{zs}(s) = V_{in}(t)$$

$$(D+2)(D+3)\,V_{out}^{zs}(t) = V_{in}(t)$$

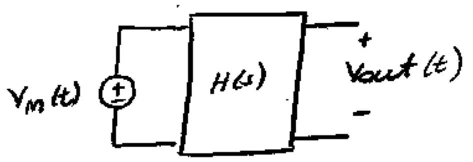
$$V_{out}^{step}(s) = H(s)\cdot\frac{1}{s}$$

$$= \frac{1}{(s+2)(s+3)\cdot s} = \underbrace{\frac{-1/2}{s+2} + \frac{+1/3}{s+3}}_{\text{response due to natural frequencies}} + \underbrace{\frac{1/6}{s}}_{\text{response due to external input}}$$

$$V_{out}(t) = -\frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} + \frac{1}{6}$$



Sinusoidal Steady State Response



$$V_{in}(t) = A\cdot\cos(\omega t+\phi)$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)}$$

$$\mathcal{L}\{V_{in}(t)\} = \mathcal{L}\{A\cdot\cos(\omega t+\phi)\}$$

$$= A\cdot\cos\phi\,\mathcal{L}\{\cos\omega t\} - A\sin\phi\,\mathcal{L}\{\sin\omega t\}$$

$$= A\cos\phi\,\frac{s}{s^2+\omega^2} - A\sin\phi\,\frac{\omega}{s^2+\omega^2} = \frac{A}{s^2+\omega^2}(s\cdot\cos\phi - \omega\sin\phi)$$

$$V_{out}(s) = H(s)\cdot V_{in}(s) = H(s)\cdot\left[A\cdot\frac{s\cos\phi-\omega\sin\phi}{s^2+\omega^2}\right]$$

$$= \underbrace{\frac{K}{s-j\omega} + \frac{K^*}{s+j\omega}}_{\text{due to sinusoidial input}} + \underbrace{\frac{A_1}{s-\lambda_1} + \frac{A_2}{s-\lambda_2} + \cdots \frac{A_N}{s-\lambda_N}}_{\text{resp. of nat. freq.}}$$

λ_k are the poles of $H(s)$; they are natural frequencies

Let's focus on the response due to sinusoidal input : $K = ?$

$$K = \left(H(s) A \cdot \frac{s\cos\phi - \omega\sin\phi}{s + j\omega} \right) \Big\downarrow_{s = j\omega}$$

$$K = H(j\omega) \cdot A \cdot \frac{j\omega\cos\phi - \omega\sin\phi}{2j\omega}$$

$$= H(j\omega) \frac{A}{2} \left[\cos\phi + j\sin\phi \right] = H(j\omega) \cdot \frac{A}{2} e^{j\phi}$$

$$\mathcal{L}^{-1}\left\{ \frac{K}{s - j\omega} + \frac{K^*}{s + j\omega} \right\} = K \cdot e^{j\omega t} + K^* e^{-j\omega t} = 2\,\mathrm{Re}\left\{ K e^{j\omega t} \right\}$$

$$= 2\,\mathrm{Re}\left\{ |K| \cdot e^{j(\omega t + \angle K)} \right\}$$

$$= 2|K| \cdot \cos(\omega t + \angle K)$$

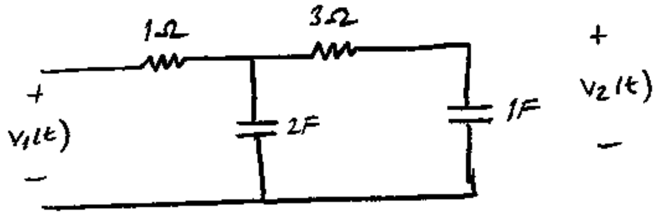
$$\mathcal{L}^{-1}\left\{ \frac{K}{s - j\omega} + \frac{K}{s + j\omega} \right\} = \underbrace{A \cdot |H(j\omega)| \cdot \cos(\omega t + \phi + \angle H(j\omega))}$$

the output at steady state
(free of natural response terms)

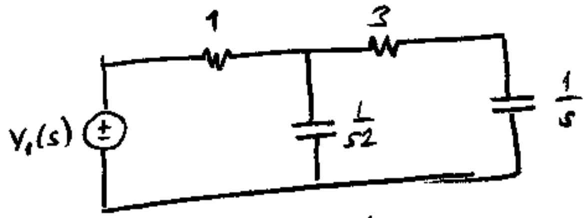
$A\cos(\omega t + \phi)$ 

$V_{out}^{SS} = A \cdot |H(j\omega)| \cos(\omega t + \phi + \angle H(j\omega))$

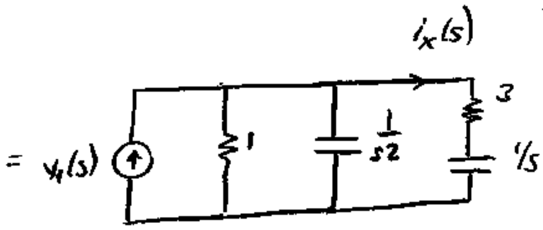
Ex/



a) $v_2^{SS}(t) = ?$ for $v_1(t) = 4\cos\left(\frac{1}{2}t + 30°\right)$



$=$



$$i_x(s) = \frac{1/\left(1/s + 3\right)}{1 + s2 + \frac{1}{1/s + 3}} \cdot V_1(s) = \frac{s}{(3s+1)(2s+1) + s} \cdot V_1(s)$$

$$V_2(s) = i_x(s) \cdot \frac{1}{s} = \frac{V_1(s)}{6s^2 + 6s + 1} = \underbrace{\frac{1/6}{s^2 + s + \frac{1}{6}}}_{H(s)} V_1(s)$$

a) If input is $4\cos\left(\frac{1}{2}t+30^\circ\right) \longrightarrow V_{out} = 4\left|H\left(j\tfrac{1}{2}\right)\right|\cos\left(\tfrac{1}{2}t+30^\circ + \angle H\left(j\tfrac{1}{2}\right)\right)$

$$H\left(j\tfrac{1}{2}\right) = \frac{1/6}{-\frac{1}{4}+j\frac{1}{2}+\frac{1}{6}} = \frac{1}{-\frac{1}{2}+j3} = \frac{1}{\sqrt{9+\frac{1}{4}}}\, e^{j(180^\circ + \tan^{-1} 3/0.5)}$$

If $\quad 4\cos\left(\tfrac{1}{2}t+30^\circ\right) \longrightarrow \boxed{H(s)} \longrightarrow \frac{8}{\sqrt{37}}\cos\left(\tfrac{1}{2}t - 150^\circ + \tan^{-1}6\right)$

FREQUENCY RESPONSE

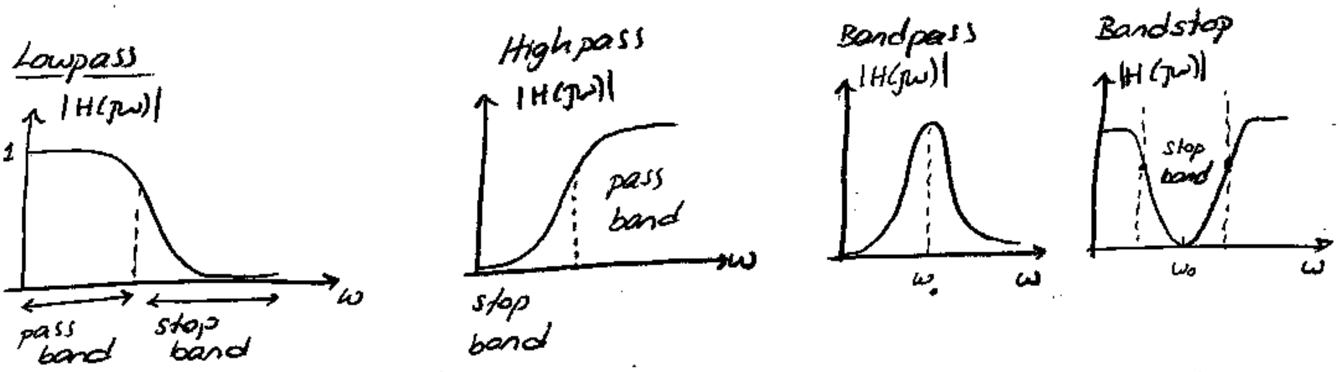
$$v_{in}(t) = A_{in}\cos(\omega t + \phi_{in})$$

$$V_{out}(t) = \underbrace{A_{in}\left|H(j\omega)\right|}\cos\left(\omega t + \phi_{in} + \angle H(j\omega)\right)$$

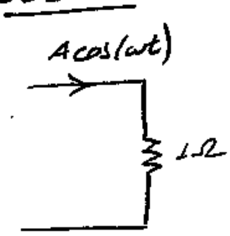
at the output, magnitude is multiplied by $|H(j\omega)|$

Gain function: $|H(j\omega)|$ (Magnitude Response)

Phase function: $\angle H(j\omega)$ (Phase Response)



Decibel



$$P_{1\Omega}^{AVG} = \frac{A^2}{2}R$$

$R = 1\Omega$

square of RMS of $A(\cos\omega t)$

The avg. power at the input is $A^2/2$.

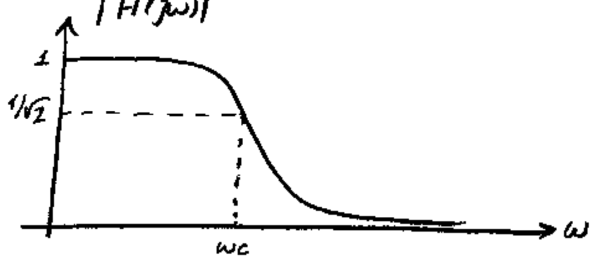
= AVG power dissipated over 1Ω resistor.

$v_1(t) = A\cos\omega t$

Decibel: $\quad 10\log_{10}\left(\dfrac{\text{Power at the output}}{\text{Power at the input}}\right)$

$$10 \log_{10} \left(\frac{\frac{|A(H(j\omega))|^2}{2} \cdot R}{\frac{A^2}{2} \cdot R} \right) = 10 \log_{10} \left(|H(j\omega)|^2 \right) \ dB$$

Low-pass



dB : dec Bell
↳ Graham Bell

↳ cut-off freq, half power freq, critical freq, -3dB freq

$\lvert H(j\omega)\rvert^2$	dB $\left(10 \log_{10} \lvert H(j\omega)\rvert^2\right)$
1	0
2	3
3	4.77
4	6
5	7
	8.45
7	9
8	9.54
9	10
10	

$$10 \log_{10} 3 = \frac{10}{4} \log_{10} 3^4$$

$$\approx \frac{10}{4} \log_{10} \underbrace{80}_{8 \times 10}$$

dB : $10 \log_{10} |H(j\omega)|^2 = 20 \log_{10} |H(j\omega)|$

↳ Remember that dB is related to power quantities.

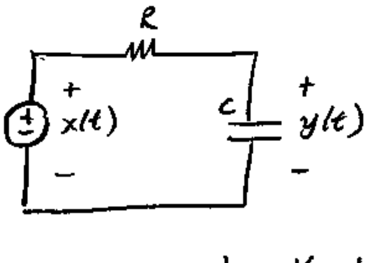
Frequency Response of 1st Order Circuits

a) Lowpass

$$H(s) = \frac{K}{s+\alpha} \qquad \frac{Y(s)}{X(s)} = \frac{K}{s+\alpha}$$

$$sY(s) + \alpha Y(s) = K X(s)$$

$$\frac{d}{dt} y(t) + \alpha y(t) = K x(t)$$



$$-x(t) + (C \cdot \dot{y}(t)) R + y(t) = 0$$

$$\dot{y}(t) + \underbrace{\frac{1}{RC}}_{\alpha} y(t) = \underbrace{\frac{1}{RC}}_{K} x(t)$$

$$|H(j\omega)| = \left| \frac{K}{\alpha + j\omega} \right| = \frac{|K|}{\sqrt{\alpha^2 + \omega^2}}$$

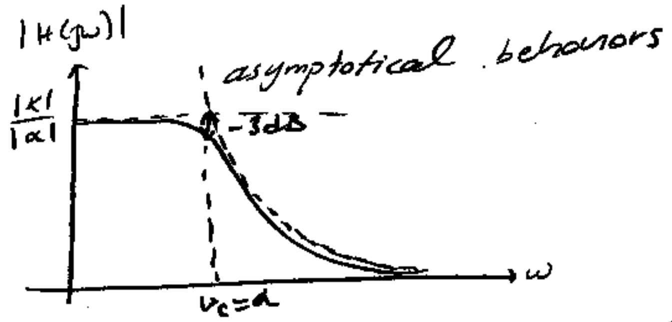
$$\angle H(j\omega) = \angle \left(\frac{K}{\alpha + j\omega} \right) = \angle K - \tan^{-1} \frac{\omega}{\alpha}$$

K is real number, but when $K<0 \longrightarrow \angle K = 180°$

$$|H(j\omega)| = \frac{|K|}{\omega\sqrt{1+\left(\frac{\alpha}{\omega}\right)^2}} = \frac{|K|}{|\alpha|\sqrt{1+\left(\frac{\omega}{\alpha}\right)^2}}$$

$$|H(j\omega)| \cong \begin{cases} \dfrac{|K|}{|\alpha|} & \omega \ll |\alpha| \\ \dfrac{|K|}{\omega} & \omega \gg |\alpha| \end{cases}$$

$(\omega = 10\alpha) \longrightarrow$ an order of magnitude more than α.



-3dB freq: A special freq at which $|H(j\omega)|$ is $1/\sqrt{2}$ times $|H(j\omega)|_{max}$.

$$\max|H(j\omega)| = \frac{|K|}{|\alpha|} \quad (\omega = 0)$$

$$\frac{\max|H(j\omega)|}{\sqrt{2}} = \frac{|K|}{|\alpha|\sqrt{1+\left(\frac{\omega}{\alpha}\right)^2}} = \frac{|K|}{|\alpha|\sqrt{2}} \longrightarrow \omega_c = \alpha$$

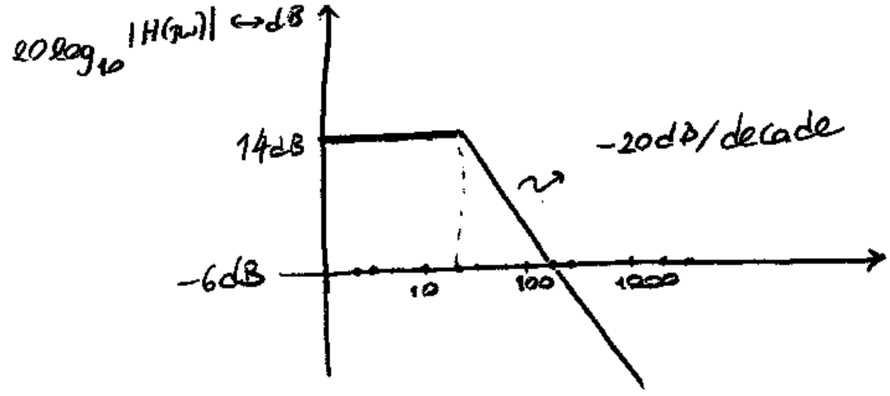
Interestingly the point where two graphs meet is the critical frequency

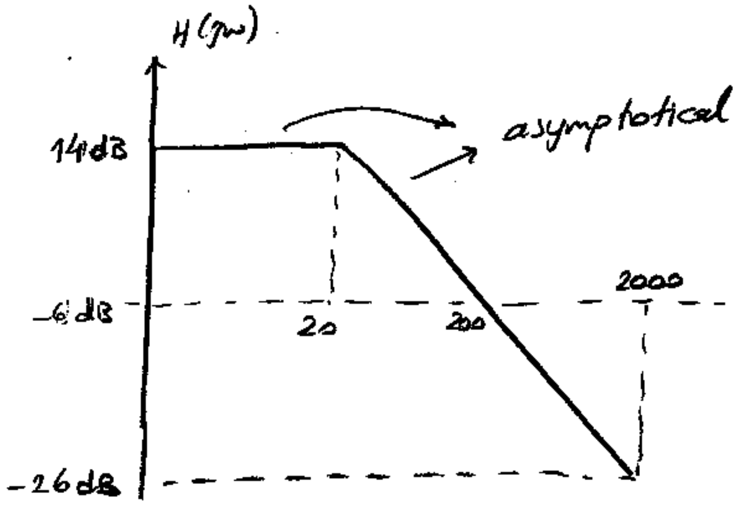
Low frequency asymptote : $\frac{|K|}{|\alpha|}$
High frequency asymptote : $\frac{|K|}{\omega}$
They are equal for $\omega = \alpha$

In magnitude response plots, we use dB scale and that changes the plot with linear scale that follows.

dB scale $\rightarrow$ Matlab notation loglog scale

$$H(s) = \frac{100}{s+20}$$



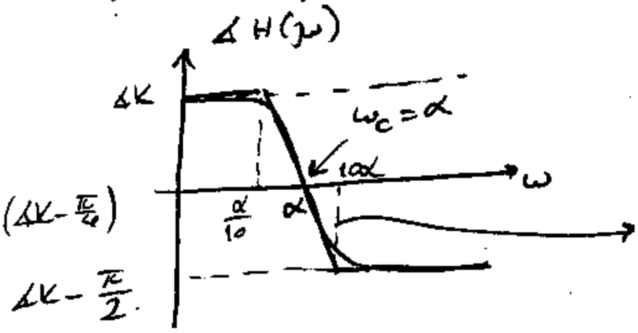


asymptotical approximations

$$\frac{100}{s+20}\Big|_{s=0}$$

$$14\,dB = 20\log_{10}(5)$$

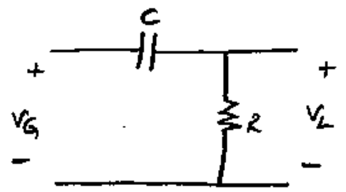
Phase Response:



$-45°/decade$

$$\measuredangle H(j\omega) = \measuredangle K - \tan^{-1}\left(\frac{\omega}{\alpha}\right)$$

10.5.2010 - Z

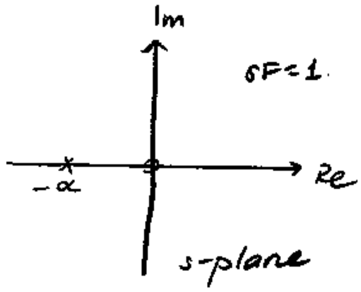


zero-state

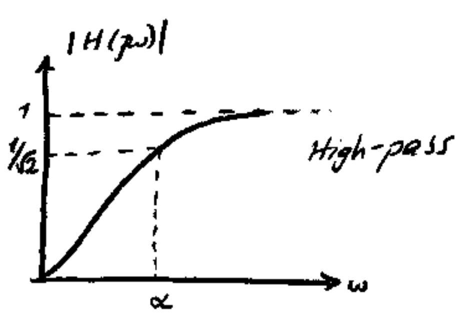
$$H(s) \triangleq \frac{V_L(s)}{V_G(s)} = \frac{R}{R+\frac{1}{sC}} = \frac{sRC}{sRC+1}$$

$$\tau \triangleq RC, \quad \alpha \triangleq \frac{1}{\tau}$$

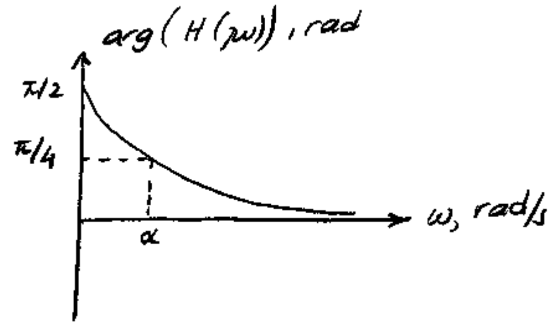
$$H(s) = \frac{\tau s}{\tau s+1} = \frac{s/\alpha}{1+s/\alpha} = \frac{1}{1+\frac{\alpha}{s}} = \frac{s}{s+\alpha}$$



$$H(j\omega) = \frac{j\omega\tau}{1+j\omega\tau} = \frac{j\omega}{\alpha+j\omega} = \frac{j\omega/\alpha}{1+j\omega/\alpha}$$

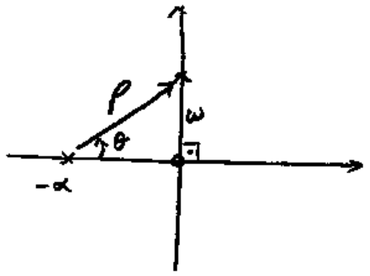


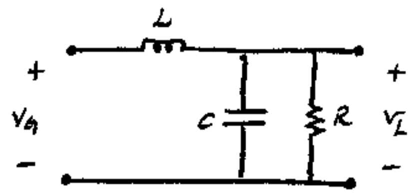
$$= \frac{\left|\frac{\omega}{\alpha}\right|}{\sqrt{1+\left(\frac{\omega}{\alpha}\right)^2}}\, e^{j\left(\frac{\pi}{2}\mathrm{sign}(\omega) - \tan^{-1}\frac{\omega}{\alpha}\right)}$$



$$|H(j\omega)| = \frac{\omega}{\rho}\, e^{j\left(\frac{\pi}{2}-\theta\right)}$$

$$\rho \to \frac{\omega}{\sqrt{a^2+\omega^2}}$$





$$R \parallel \frac{1}{sC} = \frac{\frac{R}{sC}}{R+\frac{1}{sC}} = \frac{R}{1+sRC}$$

$$H(s) = \frac{\frac{R}{1+sRC}}{\frac{R}{1+sRC}+sL} = \frac{R}{s^2RLC+sL+R}$$

$$= \frac{1}{LC}\,\frac{1}{s^2+\frac{s}{RC}+\frac{1}{LC}}$$

$$\omega_0 \triangleq \frac{1}{\sqrt{LC}}$$

$$2\alpha \triangleq \frac{1}{RC}$$

$$Q \triangleq \frac{\omega_0}{2\alpha} = \omega_0 RC$$

$$= \frac{\omega_0^2 RC}{\omega_0} = \frac{\frac{RC}{LC}}{\omega_0} = \frac{1}{\omega_0 \frac{L}{R}}$$

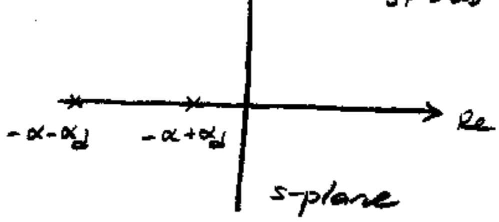
$$H(s) = \frac{\omega_0^2}{s^2+2\alpha s+\omega_0^2} = \frac{1}{\left(\frac{s}{\omega_0}\right)^2+\underbrace{\frac{2\alpha}{\omega_0}}_{\frac{1}{Q}}\frac{s}{\omega_0}+1}$$

$$s^2+2\alpha s+\omega_0^2 = 0 \longrightarrow s_{1,2} = -\alpha \mp \sqrt{\alpha^2-\omega_0^2}$$

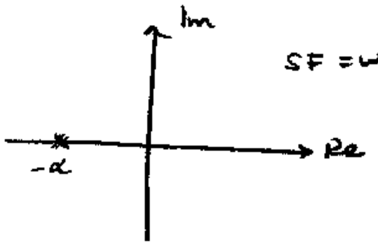
$Q < \frac{1}{2}$: $\alpha > \omega_0$; $\alpha_d \triangleq \sqrt{\alpha^2-\omega_0^2}$

$s_1 = -\alpha+\alpha_d \quad s_2 = -\alpha-\alpha_d$ OD → overdamped

$SF = \omega_0^2$



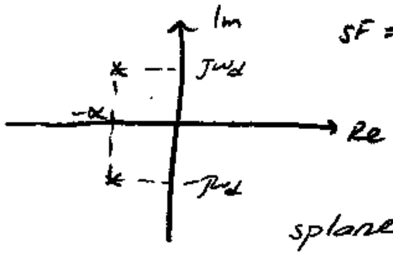
$Q = \frac{1}{2}$: $\alpha = \omega_0$ C.D. critically damped



$SF = \omega_0^2$

$Q > \frac{1}{2}$: $\omega_0 > \alpha$, $\omega_d \triangleq \sqrt{\omega_0^2 - \alpha^2}$

$s_{1,2} = -\alpha \mp j\omega_d$ U.D. underdamped



$SF = \omega_0^2$

splane

$\underline{Q < \frac{1}{2}}$: $H(s) = \dfrac{\omega_0^2}{(s-\alpha_1)(s-\alpha_2)} = \dfrac{1}{\left(1-\frac{s}{\alpha_1}\right)\left(1-\frac{s}{\alpha_2}\right)}$

$$H(j\omega) = \frac{1}{\left(1+j\frac{\omega}{\alpha_1}\right)\left(1+j\frac{\omega}{\alpha_2}\right)}$$

$$= \frac{1}{\sqrt{1+\left(\frac{\omega}{\alpha_1}\right)^2}\sqrt{1+\left(\frac{\omega}{\alpha_2}\right)^2}}\, e^{-j\left(\tan^{-1}\frac{\omega}{\alpha_1} + \tan^{-1}\frac{\omega}{\alpha_2}\right)}$$

$\underline{Q = \frac{1}{2}}$: $H(s) = \dfrac{1}{\left(1+\frac{s}{\alpha}\right)^2} \longrightarrow H(j\omega) = \dfrac{1}{\left(1+j\frac{\omega}{\alpha}\right)^2}$

$$= \frac{1}{1+\left(\frac{\omega}{\alpha}\right)^2}\, e^{-j\,2\tan^{-1}\frac{\omega}{\alpha}}$$

$\underline{Q > \frac{1}{2}}$: $H(j\omega) = \dfrac{1}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right] + j\frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)}$

$$= \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2\right\}^{\frac{1}{2}}}\, e^{-j\tan^{-1}\frac{\frac{1}{Q}\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}}$$

$x \triangleq \dfrac{\omega}{\omega_0}$, $P(x) \triangleq (1-x^2)^2 + \dfrac{1}{Q^2}x^2$

$y \triangleq x^2$, $R(y) = (1-y)^2 + \dfrac{1}{Q^2}y$

$R(0) = 1$, $|H(j0)| = 1$

$R(\infty) = \infty$, $|H(j\infty)| = 0$

$R(1) = \frac{1}{Q^2}$, $|H(j\omega_0)| = Q$

$$\frac{dR(y)}{dy} = 2y - 2 + \frac{1}{Q^2} \overset{?}{=} 0 \qquad y = 1 - \frac{1}{2Q^2}$$

if $1 - \frac{1}{2Q^2} \geq 0 \Rightarrow 1 \geq \frac{1}{2Q^2}$

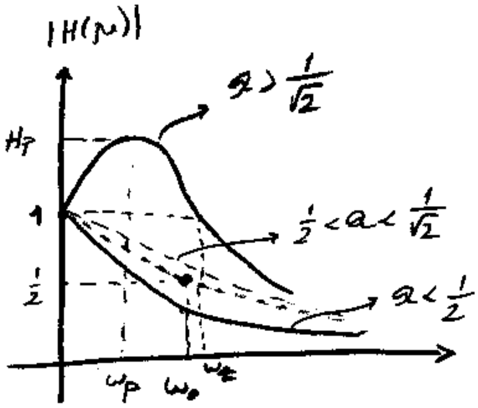
$\Rightarrow Q \geq \frac{1}{\sqrt{2}}$

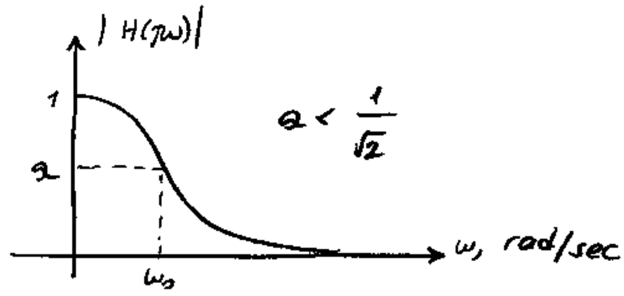
$$\omega_p = \omega_0 \sqrt{1 - \frac{1}{2Q^2}} \ , \quad Q \geq \frac{1}{\sqrt{2}}$$

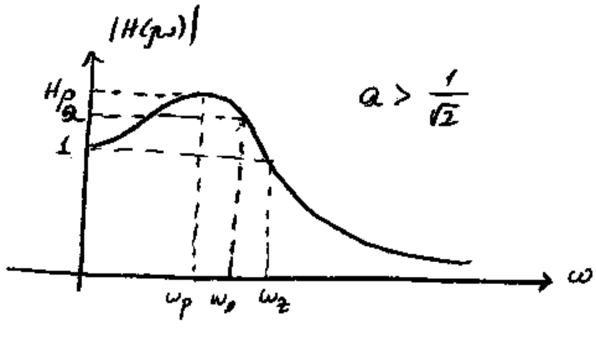
peak freq.

$$R(y_p) = \frac{1}{Q^2}\left(1 - \frac{1}{4Q^4}\right)$$

$$|H(j\omega_p)| = \frac{Q}{\sqrt{1 - \frac{1}{4Q^4}}}$$







$$T(x) = \tan^{-1} \frac{\frac{1}{Q}x}{1 - x^2}$$

$T(0) = 0$

$T(1) = \frac{\pi}{2}$

$T(\infty) = \pi$

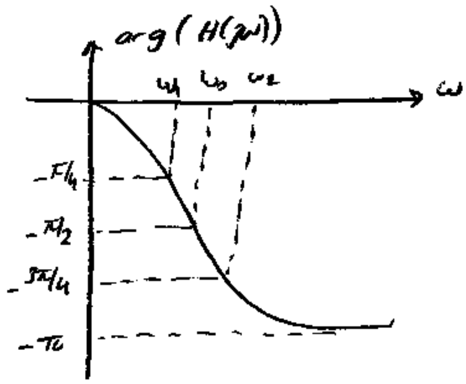
$$1 - x^2 = \mp \frac{1}{Q}x \quad \begin{matrix} \to 3\pi/4 \\ \to \pi/4 \end{matrix}$$

$$x^2 \pm \frac{1}{Q}x - 1 = 0$$

$$x_{1,2} = \mp \frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} + 1}$$

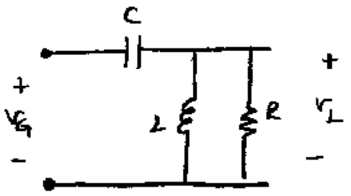
$$\omega_{1,2} = \omega_0 x_{1,2} = \mp \frac{\omega_0}{2Q} \pm \sqrt{\frac{\omega_0^2}{4Q^2} + \omega_0^2}$$

$$= \mp \alpha \pm \sqrt{\alpha^2 + \omega_0^2}$$



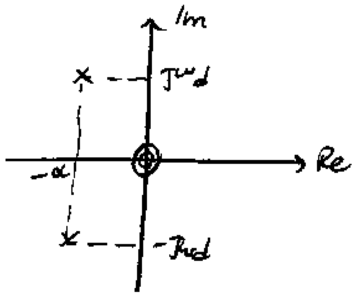
$\omega_1 = -\alpha + \sqrt{\alpha^2 + \omega_0^2} \Rightarrow \pi/4$ $\qquad \omega_2 - \omega_1 = 2\alpha$

$\omega_2 = \alpha + \sqrt{\alpha^2 + \omega_0^2} \Rightarrow 3\pi/4$ $\qquad \omega_1 \omega_2 = \omega_0^2$



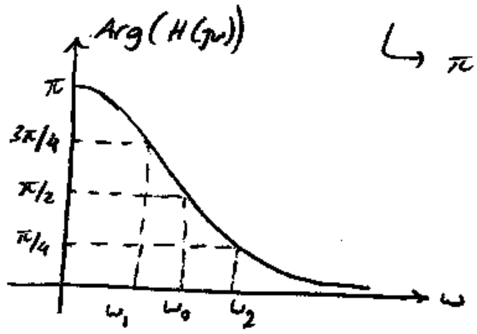
$$H(s) = \frac{s^2}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} = \frac{s^2}{s^2 + 2\alpha s + \omega_0^2}$$

$$= \frac{(s/\omega_0)^2}{(s/\omega_0)^2 + 1/Q\,(s/\omega_0) + 1}$$

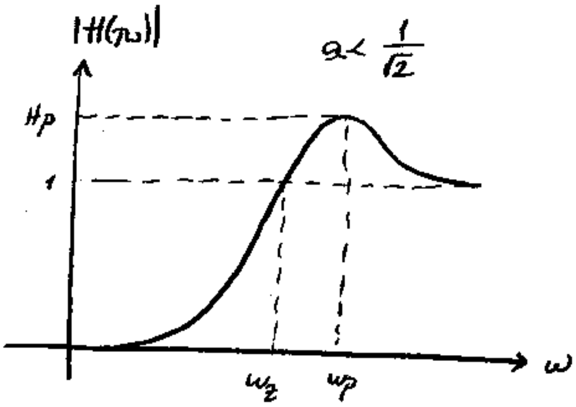


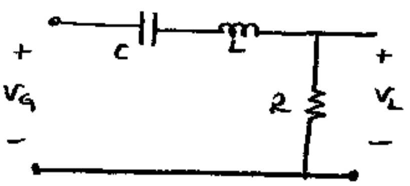
$$\arg(H(j\omega)) = \pi - \tan^{-1} \frac{\frac{1}{Q}\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}$$

$\hookrightarrow \pi$ + low pass



$Q < \frac{1}{\sqrt{2}}$

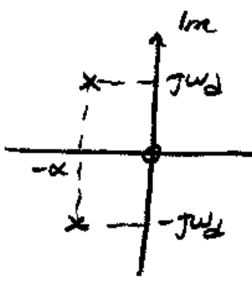




$$H(s) = \frac{R}{R + sL + \frac{1}{sC}} = \frac{R}{L}\,\frac{s}{s^2 + s\frac{R}{L} + \frac{1}{LC}} = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

$$H(s) = \frac{\frac{1}{Q}\left(\frac{s}{\omega_0}\right)}{\left(\frac{s}{\omega_0}\right)^2 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + 1} = \frac{1}{1 + Q\frac{(s/\omega_0)^2 + 1}{(s/\omega_0)}}$$

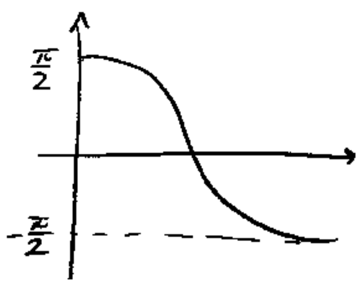


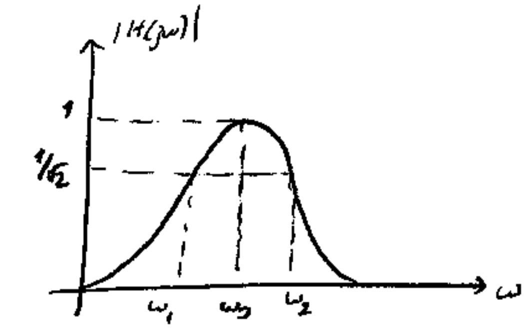
$SF = 2\alpha$

$Q > \frac{1}{2}$

$$H(j\omega) = \frac{1}{1 + jQ\,\frac{1-(\omega/\omega_0)^2}{\omega/\omega_0}}$$

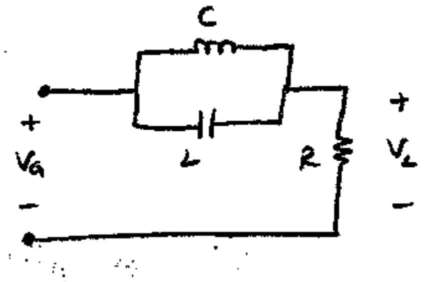
$$\mathrm{Arg}(H(j\omega)) = \underbrace{\frac{\pi}{2}\,\mathrm{sign}(\omega) - \tan^{-1}\frac{\frac{1}{Q}\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}}_{\frac{\pi}{2} + \text{Low pass}}$$





$\omega_2-\omega_1 = 2\alpha \rightarrow$ half power bandwith

$\omega_0=\sqrt{\omega_1\omega_2} \rightarrow$ center freq.



$$H(s)=\frac{R}{R+\frac{1}{sC+\frac{1}{sL}}}=\frac{s^2+\frac{1}{LC}}{s^2+s\frac{1}{RC}+\frac{1}{LC}}$$

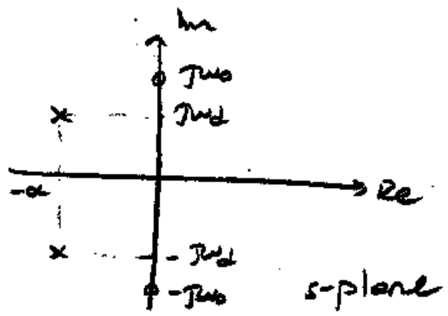
$$=\frac{s^2+\omega_0^2}{s^2+2\alpha s+\omega_0^2}$$

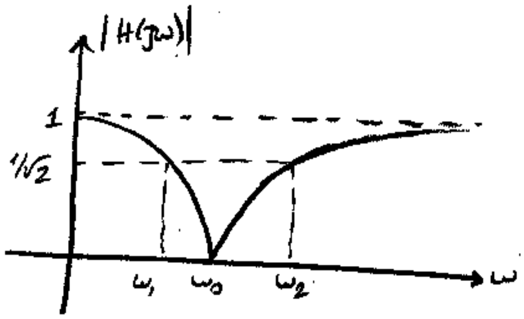
$$=\frac{(s/\omega_0)^2+1}{(s/\omega_0)^2+\frac{1}{Q}(s/\omega_0)+1}$$

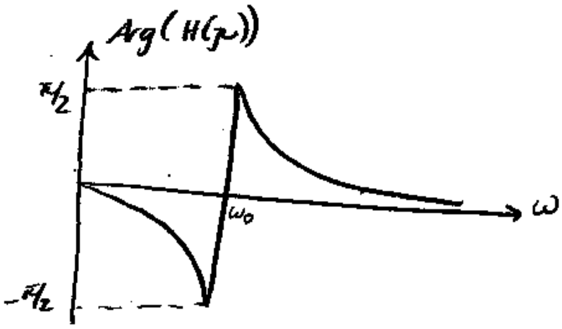
$$Y(j\omega_0)=j\omega_0 C+\frac{1}{j\omega_0 L}$$

$$=0$$

↳ no conductance

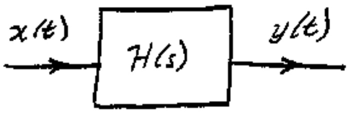






13.05.2010

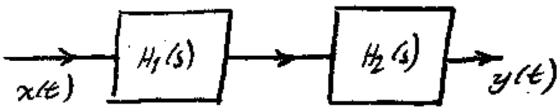
Band-pass - Band-stop Filters using 1st order Circuits



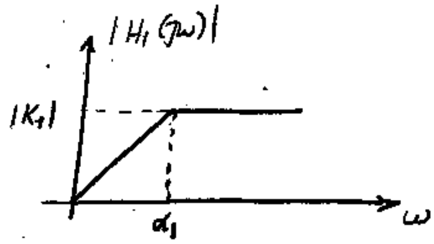
$H_1(s) = \frac{s}{s+\alpha_1} K_1 \longrightarrow$ high pass system

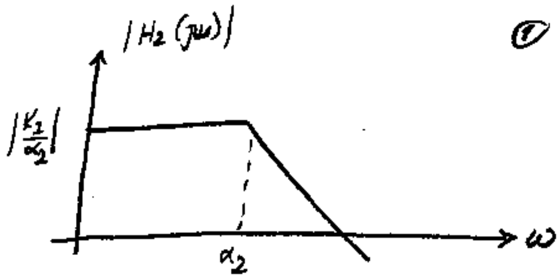
$H_2(s) = \frac{K_2}{s+\alpha_2} \longrightarrow$ lowpass system

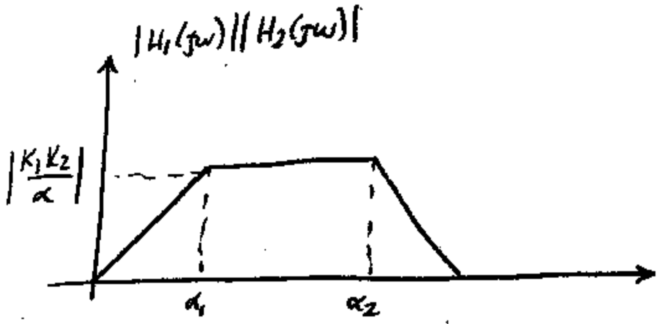
Band pass system



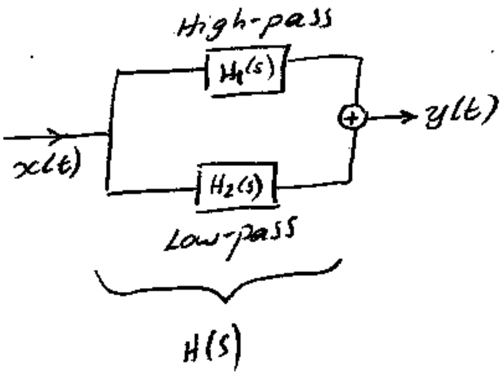
$H(s) = H_1(s)\,H_2(s) \rightarrow$ No loading effects between systems ① and ②

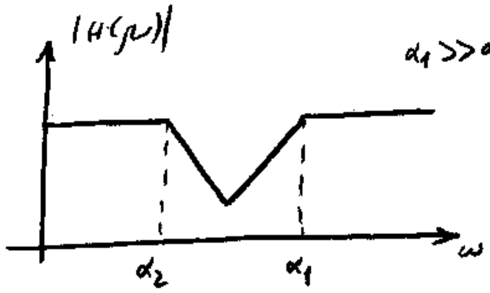






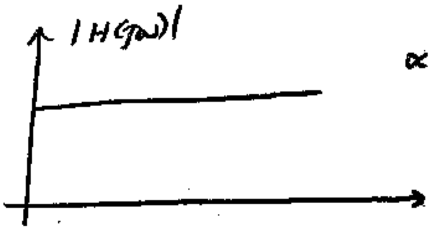
Parallel Combination





$\alpha_1 \gg \alpha_2$

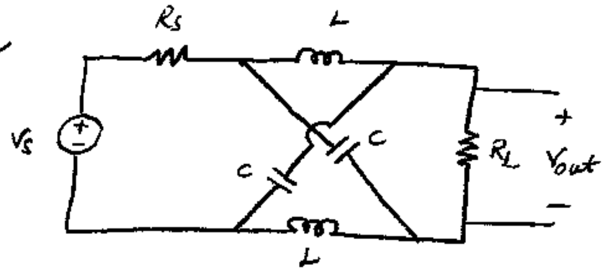
← Band-stop filter



$\alpha_2 \gg \alpha_1$

← All-pass filter

Ex



$$\frac{L}{C} = R_L^2$$

$$H(j\omega) = \frac{R_L}{R_L + R_S} \quad \frac{1 - j\omega C R_L}{1 + j\omega C R_L}$$

$$|H(j\omega)| = \frac{R_L}{R_L + R_S} \quad ; \quad \measuredangle H(j\omega) = 2\tan^{-1}(-\omega C R_L)$$

Second order circuits

(A) Band-Pass system

$$H(s) = K \frac{s}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$$

$s^2 + 2\gamma\omega_0 s + \omega_0^2$: char. polynomials

↳ distinct and real roots $\lambda_1 \neq \lambda_2$

identical roots $\lambda_1 = \lambda_2 \longrightarrow H(s) = \dfrac{Ks}{(s-\lambda_1)^2}$

complex conjugate roots $\lambda_1 = \lambda_2^*$

2nd order system with complex poles requires some special interest

$$H(s) = K \underbrace{\frac{s}{(s-\lambda_1)}}_{\text{high-pass}} \cdot \underbrace{\frac{1}{(s-\lambda_2)}}_{\text{low-pass}}$$

$$H(s)\Big|_{s=j\omega} = K \frac{j\omega}{-\omega^2 + 2\gamma\omega_0 j\omega + \omega_0^2} = K \frac{1}{\frac{\omega_0^2 - \omega^2}{j\omega} + 2\gamma\omega_0}$$

$$= K \frac{1}{\omega_0\left(2\gamma + j\left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right]\right)}$$

$$|H(j\omega)| = \frac{|K|}{\omega_0} \cdot \frac{1}{\sqrt{(2\gamma)^2 + \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}}$$

$$\measuredangle H(j\omega) = \angle K - \tan^{-1}\left[\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)/2\gamma\right]$$

$$s^2 + 2\gamma\omega_0 s + \omega_0^2 = 0$$

$$s_{1,2} = \left\{ -\gamma\omega_0 \mp \omega_0\sqrt{\gamma^2 - 1} \right\}$$

(1) $\gamma = 1 \longrightarrow s_{1,2} = \{-\omega_0\}$ double roots, critically damped

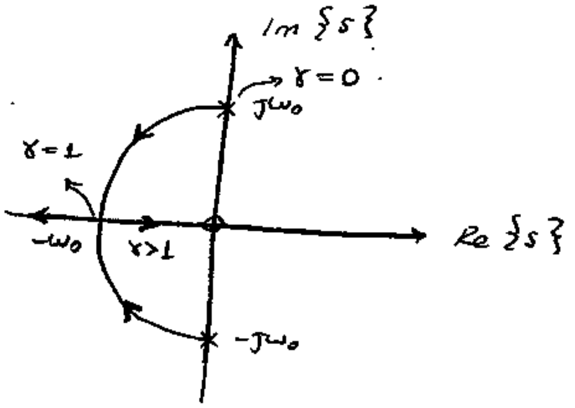
$x_{1,2} = -\omega_0$

② $\gamma > 1 \longrightarrow$ distinct roots (overdamped)

③ $\gamma = 0 \longrightarrow s_{1,2} = \{ \mp j\omega_0 \}$

④ $|\gamma| < 1$: complex roots

Pole-Zero Plot for $H(s)$

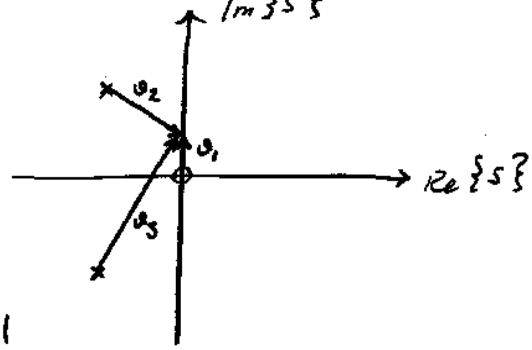


$$H(s) = \frac{Ks}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$$

γ : damping coefficient

ω_0 : resonant frequency (rad/sec)

$$H(s) = \frac{Ks}{(s-p_1)(s-p_2)}$$



$$|H(j\omega)| = \frac{|K| \cdot |j\omega|}{|j\omega - p_1| \cdot |j\omega - p_2|} = \frac{|K| \cdot |\theta_1|}{|\theta_2||\theta_3|}$$

<u>Returning to Algebraic Discussion</u>

$$H(j\omega) = \frac{K/\omega_0}{2\gamma + j\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

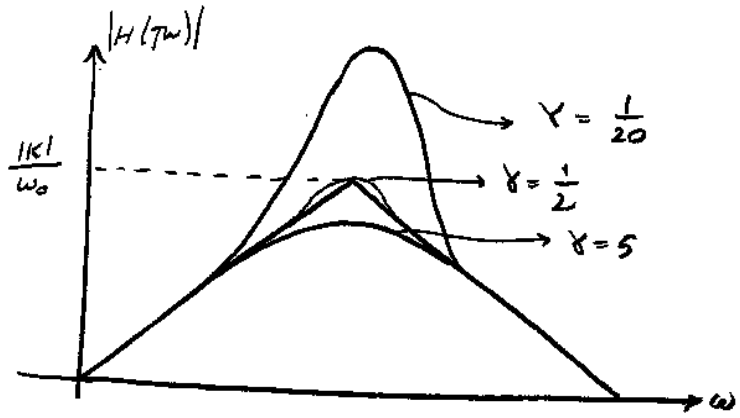
$$\arg\max_{\omega} (H(j\omega)) = \omega_0$$

$$|H(j\omega_0)| = \max_{\omega} |H(j\omega)| = \frac{|K|}{2\gamma\omega_0}$$

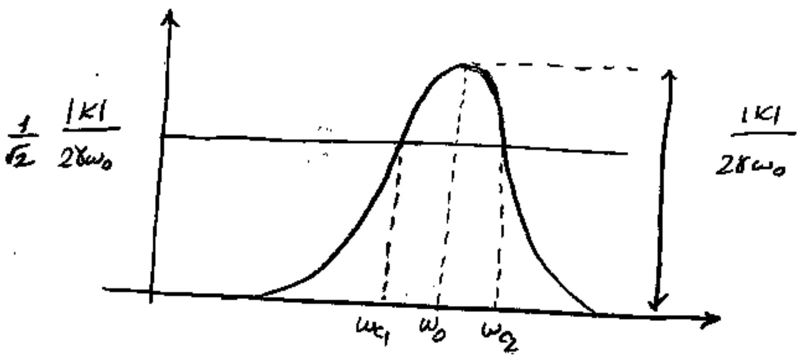
① $\omega \ll \omega_0 \longrightarrow H(j\omega) = \dfrac{K/\omega_0}{2\gamma + j\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \cong \dfrac{K/\omega_0}{2\gamma + j\left(-\frac{\omega_0}{\omega}\right)} \cong \dfrac{K/\omega_0}{-j\,\omega_0/\omega}$

$$\cong j\,\frac{K\omega}{\omega_0^2}$$

② $\omega \gg \omega_0 \longrightarrow H(j\omega) \cong \dfrac{K/\omega_0}{2\gamma + j\,\omega/\omega_0} \cong -j\,\dfrac{K}{\omega}$



Less damping (γ smaller) results in peaking freq. response.



ω_{c1}, ω_{c2} : cut off frequencies

$$|H(j\omega)| = \left| \frac{K/\omega_0}{2\gamma + j\underbrace{\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}_{\mp 2\gamma}} \right|_{\omega=\omega_{c1},\omega_{c2}} = \frac{|K|/\omega_0}{\sqrt{2}\cdot 2\gamma}$$

Then one of the cut-off frequencies satisfy the following property:

$$\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = 2\gamma$$

$\omega^2 - 2\gamma\omega_0\omega - \omega_0^2 = 0 \longrightarrow$ solution gives cut-off freq.

then $\omega_{c_{1,2}} = \omega_0\left(\gamma \mp \sqrt{1+\gamma^2}\right) \longrightarrow \omega_{c_2} = \omega_0\left(\gamma + \sqrt{1+\gamma^2}\right)$

For ω_{c_1} : we repeat the same calculation for $\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = -2\gamma$

and get $\omega_{c_1} = \omega_0\left(-\gamma + \sqrt{1+\gamma^2}\right)$

<u>Bandwidth</u> of the system $= \omega_{c_2} - \omega_{c_1} = 2\gamma\omega_0$

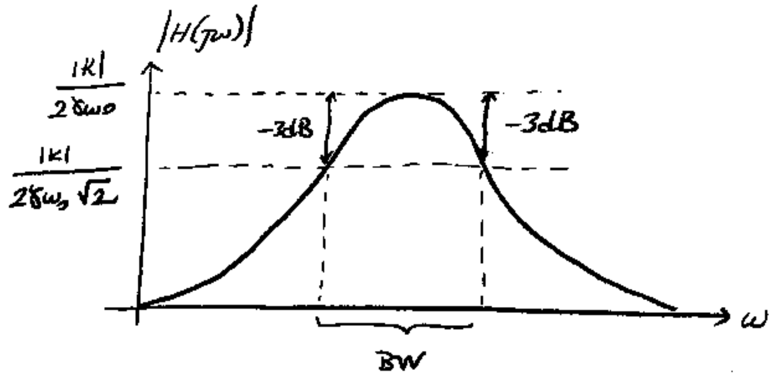
Q : quality Factor

$$Q = \frac{\omega_0}{BW} = \frac{1}{2\gamma}$$

17.09.2010

Second order Band-Pass Response:

$$H(s) = \frac{Ks}{s^2 + 2\gamma\omega_0 s + \omega_0^2} \longrightarrow \text{2nd order band-pass system}$$



$H(j\omega) \rightarrow$ ① $|H(j\omega)|$ has a peak at $\left(\omega_0, \frac{|K|}{2\gamma\omega_0}\right)$

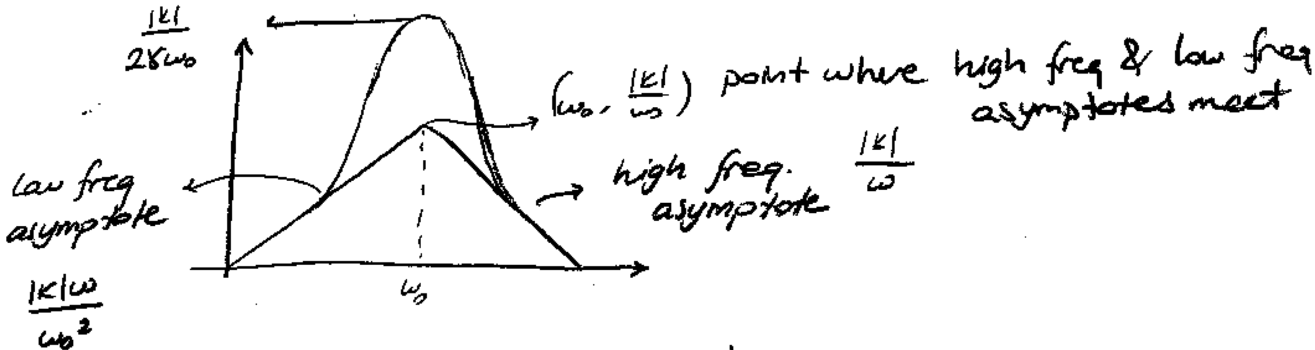
② $|H(j\omega_c)| = 70\%$ of the peak

$$\omega_{c_1} = \omega_0\left(-\gamma + \sqrt{1+\gamma^2}\right)$$

$$\omega_{c_2} = \omega_0\left(\gamma + \sqrt{1+\gamma^2}\right)$$

③ $BW = \omega_{c_2} - \omega_{c_1} = 2\gamma\omega_0$

④ $Q = \frac{\omega_0}{BW} = \frac{1}{2\gamma}$ (Q: quality factor)



1) γ small: peaky response $\rightarrow$ Q: high

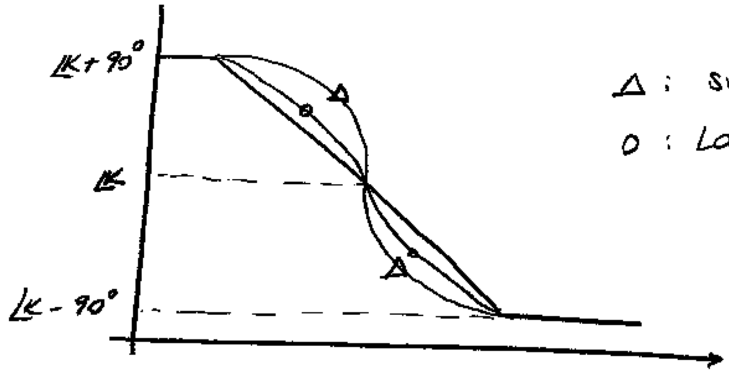
high Q filters $\rightarrow$ narrow band filters

2) γ large: broad response $\rightarrow$ Q: low

low Q filter $\rightarrow$ wide band filters

Phase Response (2nd order Bandpass Systems)

$$\angle H(j\omega) = \angle K - \tan^{-1}\left(\frac{\omega/\omega_0 - \omega_0/\omega}{2\gamma}\right)$$



Δ : smaller γ
o : Larger γ

Second Order Low Pass Systems

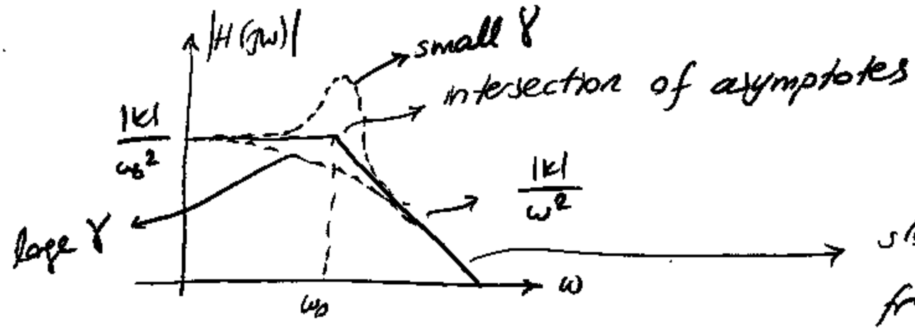
$$H(s) = \frac{K}{s^2 + 2\gamma\omega_0 s + \omega_0^2} \qquad H(j\omega) = \frac{K}{\omega_0^2 - \omega^2 + j\,2\gamma\omega_0\omega}$$

$$|H(j\omega)| = \frac{|K|}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\gamma\omega_0\omega)^2}} \qquad \angle H(j\omega) = \angle K - \tan^{-1}\left(\frac{2\gamma\omega_0\omega}{\omega_0^2-\omega^2}\right)$$

$$H(j\omega) = \frac{K}{\omega_0^2\left(1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\,2\gamma\frac{\omega}{\omega_0}\right)}$$

① $\omega \ll \omega_0 \longrightarrow |H(j\omega)| = \frac{|K|}{\omega_0^2}$

② $\omega \gg \omega_0 \longrightarrow |H(j\omega)| = \frac{|K|}{\omega^2}$

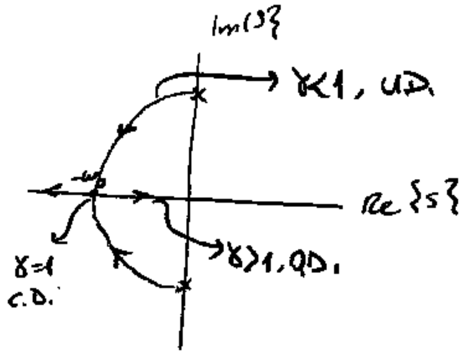


slope of the High freq. asymptote: -40 dB/decade

③ $|H(j\omega_0)| = \frac{|K|}{2\gamma\omega_0^2}$

Second order High Pass Systems

$$\mathcal{H}(s) = \frac{Ks^2}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$$



① $$H(j\omega) = \frac{-K\omega^2}{\omega_0^2 - \omega^2 + j\,2\gamma\omega_0\omega}$$

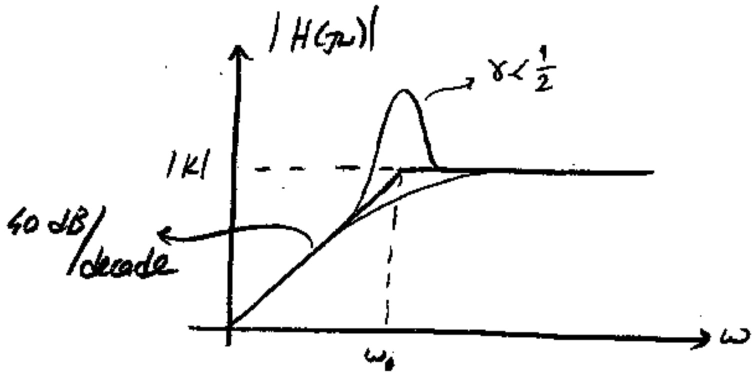
$$= \frac{-K\omega^2}{\omega_0^2\left(1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\,2\gamma\frac{\omega}{\omega_0}\right)}$$

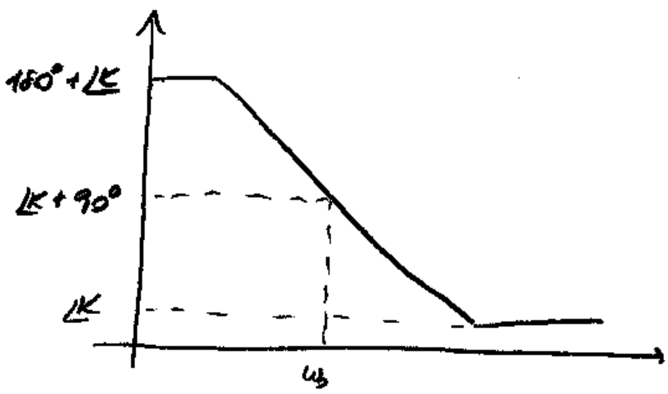
② $$\angle H(j\omega) = 180^\circ + \angle K - \tan^{-1}\frac{2\gamma\,\omega/\omega_0}{1 - (\omega/\omega_0)^2}$$

① $\omega \ll \omega_0 \rightarrow H(j\omega) = \frac{-K\omega^2}{\omega_0^2}$; $|H(j\omega)| = \frac{|K|\omega^2}{\omega_0^2}$

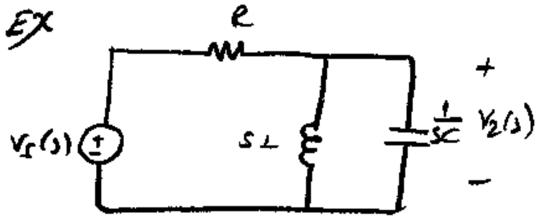
② $\omega \gg \omega_0 \rightarrow H(j\omega) = -K$; $|H(j\omega)| = |K|$

③ $|H(j\omega_0)| = \frac{|K|}{2\gamma}$





Ex



$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{s/RC}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

$$= K \frac{s}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$$

$L = 0.25\,H, \quad R = 1k\Omega, \quad C = 1\mu F$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2k \text{ rad/sec}$$

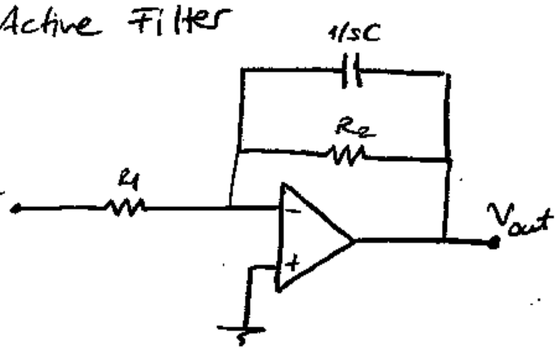
$$BW = 2\omega_0\gamma = \frac{1}{RC} = 1k\text{ rad/sec}$$

$$Q = \frac{\omega_0}{BW} = 2 \quad \text{(unitless)}$$

$Q = \frac{1}{2} \rightarrow$ critically damped (?)

$Q > \frac{1}{2} \rightarrow$ underdamped

Ex Active Filter



$$H(s) = \frac{-R_2}{R_1} \frac{1}{1 + sR_2C}$$

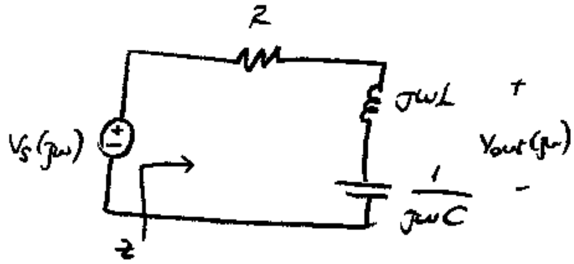
Scaling

24.05.2010

used to modify a given filter to

i) realizable filter (with components on the shelf)
ii) changing (scaling) gain-phase response

① magnitude scaling



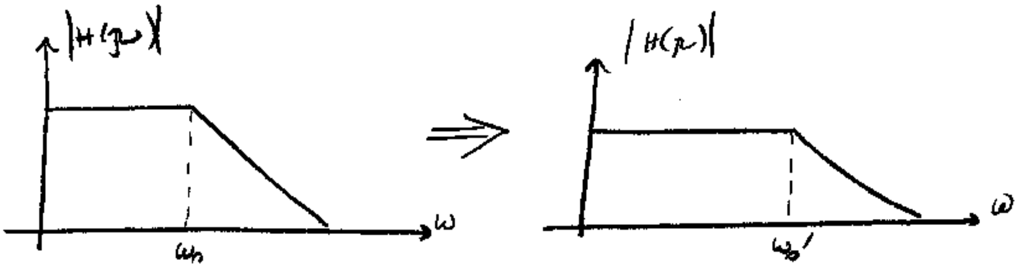
$$z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$I_s(j\omega) = \frac{V_s(j\omega)}{R + j\omega L + \frac{1}{j\omega C}}$$

$$\frac{V_{out}(j\omega)}{V_s(j\omega)} = \frac{j\omega L + \frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} \quad (*)$$

Replace
$$\left.\begin{aligned} R &\rightarrow k_m R \\ L &\rightarrow k_m L \\ C &\rightarrow C/k_m \end{aligned}\right\}$$
After replacement no change in the transfer function

② Frequency Scaling



$$H(j\omega) = \frac{j\omega L + \frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}$$

$|H(j\omega)|$ is the value at cut-off

$R \rightarrow R$

$L \rightarrow \frac{L}{k_f}$

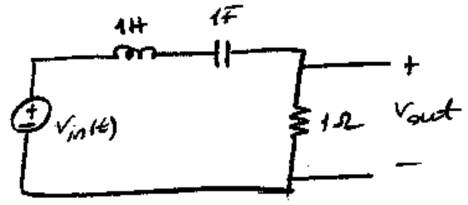
$C \rightarrow \frac{C}{k_f}$

after $H(j\omega)$

$$\frac{j\frac{\omega}{k_f}L + \frac{1}{j\frac{\omega}{k_f}C}}{R + j\frac{\omega}{k_f}L + \frac{1}{j\frac{\omega}{k_f}C}} = H\left(j\frac{\omega}{k_f}\right) \text{ before}$$

select $k_f = \frac{\omega_0'}{\omega_0}$ (?)

Ex



$$\omega_0 = \sqrt{1/LC} = 1 \text{ rad/s}$$

$$BW = 2\gamma\omega_0 = \frac{R}{L} = 1 \text{ rad/s}$$

$$Q = \frac{\omega_0}{BW} = 1$$

Scale the circuit such that the resonant frequency is at 500 Hz and use a 2μF capacitor.

$R \xrightarrow{k_f} R \xrightarrow{k_m} Rk_m \longrightarrow 160\Omega$

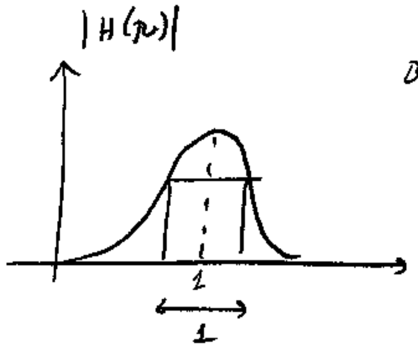
$L \xrightarrow{k_f} L/k_f \xrightarrow{k_m} L\frac{k_m}{k_f} \longrightarrow 50\,mH$

$C \xrightarrow{k_f} C/k_f \xrightarrow{k_m} \frac{C}{k_f k_m} \longrightarrow 2\,\mu F$

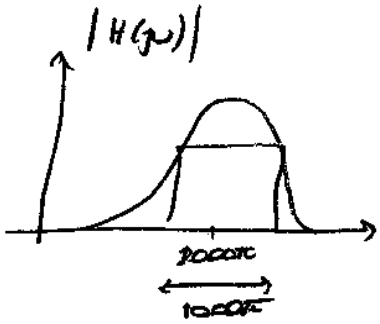
$\omega_0' = 2\pi\, 500 = 1000\pi$ rad/sec

$k_f = \frac{\omega_0'}{\omega_0} = 1000\pi$ $\qquad k_m = \frac{1F}{2\mu F \cdot 1000\pi} \simeq 160$

Q does not change!!



Before
$R = 1\Omega$
$L = 1H$
$C = 1F$



After
$R = 160\Omega$
$L = 50\,mH$
$C = 2\,\mu F$

Bode Plots

Ex $\quad H(s) = 12500 \dfrac{(s+10)}{(s+50)(s+500)}$

① Bring $H(s)$ into standard form

$$H^{std}(s) = K \frac{\left(1+\frac{s}{\alpha_1}\right)}{\left(1+\frac{s}{\alpha_2}\right)\left(1+\frac{s}{\alpha_3}\right)}$$

$$H(s) = 5 \frac{\left(1+\frac{s}{10}\right)}{\left(1+\frac{s}{50}\right)\left(1+\frac{s}{500}\right)}$$

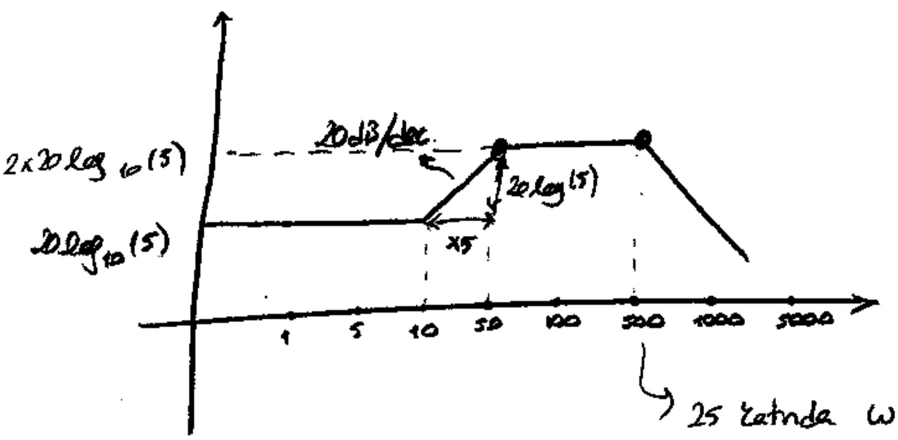
② Find critical frequencies:

$$\omega_{critical} = \{10, 50, 500\}$$

③ $$|H(j\omega)| = 5 \frac{|1+j\omega/10|}{|1+j\omega/50|\,|1+j\omega/500|}$$

dB $\downarrow$

$$20\log_{10}|H(j\omega)| = |H(j\omega)|_{dB} = 20\log_{10}(5) + 20\log|1+j\omega/10|$$
$$- 20\log|1+j\omega/50| - 20\log|1+j\omega/500|$$



25 katında ω ekseni keser.

Phase Plot

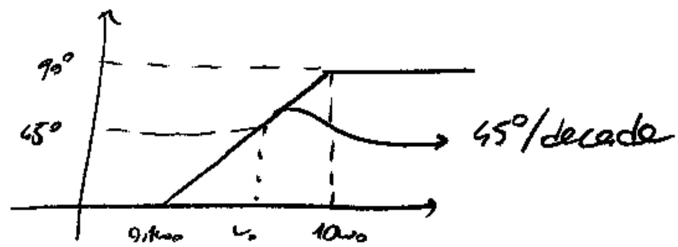
(1) $\measuredangle s$

(2) $\measuredangle (1 + j\omega/10)$

(III) $-\measuredangle (1 + j\omega/50)$

(IV) $-\measuredangle (1 + j\omega/500)$

$\measuredangle (1 + j\omega/\omega_0)$



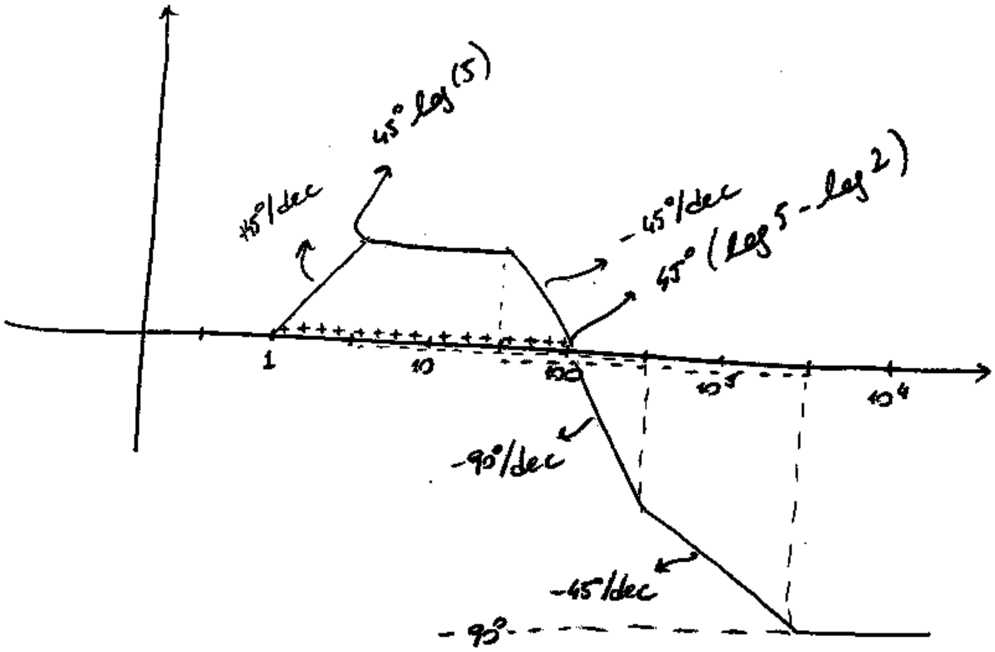
Edge points for graph drawing are at $\left\{ \frac{\omega_0}{10}, \omega_0, 10\omega_0 \right\}$

(1) 0

(2) +45°/dec btw [1, 100]

(3) −45°/dec [5, 500]

(4) −45°/dec btw [50, 5000]



27.05.2010

$$H(s) = \frac{ks}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$$

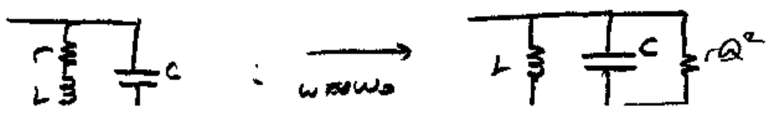
$BW = 2\gamma\omega_0$

$Q = \frac{\omega_0}{BW} = \frac{1}{2\gamma}$

$Q = \frac{1}{2}$; $(\gamma = 1) \longrightarrow (s + \omega_0)^2 \rightarrow$ critically damped $\quad \lambda_1 = \lambda_2 = -\omega_0$

$Q > \frac{1}{2} \longrightarrow$ overdamped $\quad \lambda_1, \lambda_2$: real distinct

$Q < \frac{1}{2} \rightarrow$ underdamped



Bode Plots with 2nd order Systems

Let's examine the following term:

$$A(s) = s^2 + 2\gamma\omega_0 s + \omega_0^2$$

Note: $A(s)$ can be factorized as $(s+\alpha_1)(s+\alpha_2)$ where α_1, α_2 real when $\gamma > 1$ and $\gamma > 1$ the reduces $A(s)$ to the multiplication of 1st order systems.

Then assume $\gamma < 1$; roots are imaginary

To start to analyze, write $A(s)$ in the standart form

$$A(s) = \omega_0^2 \left(1 + \frac{2\gamma}{\omega_0} s + \frac{s^2}{\omega_0^2} \right)$$

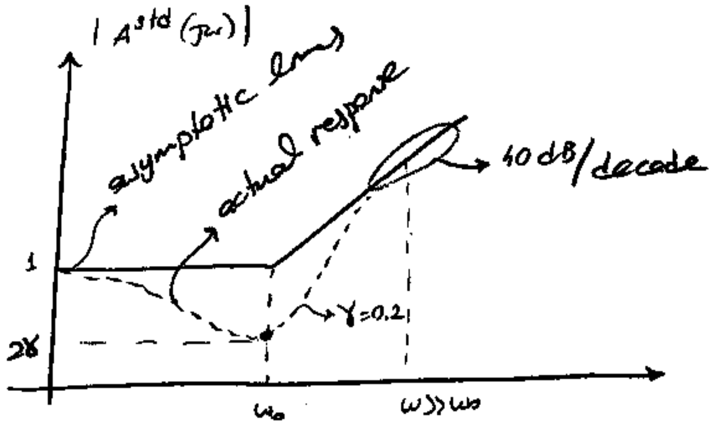
$$A^{std}(s) = 1 + \frac{2\gamma}{\omega_0} s + \frac{s^2}{\omega_0^2}$$

$s = j\omega$

$$\xrightarrow{|\cdot|} \sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{2\gamma\omega}{\omega_0}\right)^2}$$

$$\xrightarrow{\angle} \tan^{-1}\left(\frac{2\gamma\, \omega/\omega_0}{1 - \left(\frac{\omega}{\omega_0}\right)^2} \right)$$

$$20 \log_{10} \sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{2\gamma\omega}{\omega_0}\right)^2}$$



Case 1: $\omega \ll \omega_0$

Case 2: $\omega \gg \omega_0$

$$|A^{std}(j\omega)|_{dB} = 20\log_{10}\sqrt{\left(\frac{\omega}{\omega_0}\right)^4 + \cdots\left(\frac{\omega}{\omega_0}\right)^2 + \cdots\left(\frac{\omega^2}{\omega_0}\right)}$$

$$= 20\log_{10}\left(\frac{\omega}{\omega_0}\right)^2$$

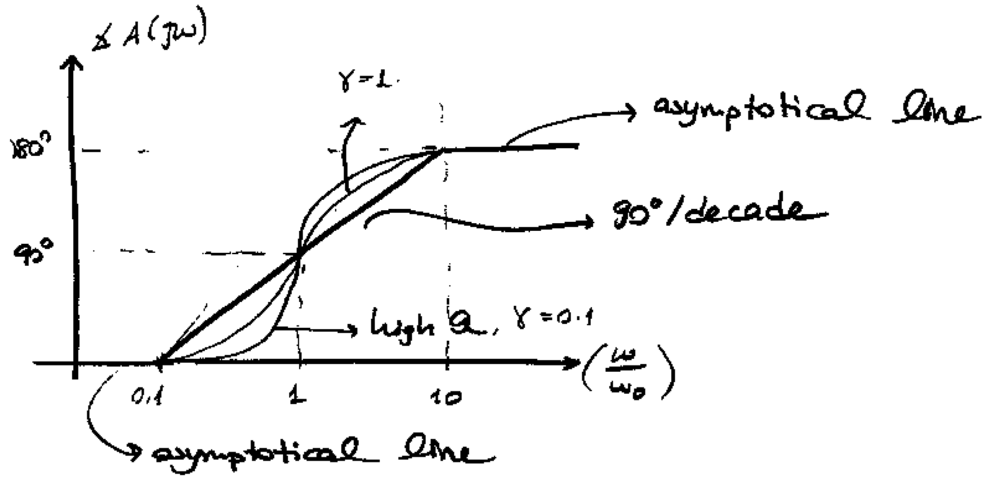
Case 3: $\omega = \omega_0$

$$|A^{std}(j\omega_0)| = 2\gamma$$

$\gamma < 1 \rightarrow$ for imaginary roots or underdamped systems

$Q = \frac{1}{2\gamma}$; High Q : small γ

Note: $\gamma = \frac{1}{2}$; $\xrightarrow[2\gamma = 1]{}$ actual curve passes through the intersection of asymptotes



Ex

$$H(s) = \frac{5000\,(s+100)}{s^2+400s+(500)^2} = \frac{5000\cdot 100\,(1+s/100)}{500^2\left(1+\frac{400s}{(500)^2}+\frac{s^2}{(500)^2}\right)}$$

$$= \frac{2\,(1+s/100)}{1+\frac{16}{10000}s+\left(\frac{s}{500}\right)^2}$$

$400 \rightarrow 2\gamma\omega_0 \rightarrow \gamma = 0.4 \rightarrow$ underdamped 2nd order system model

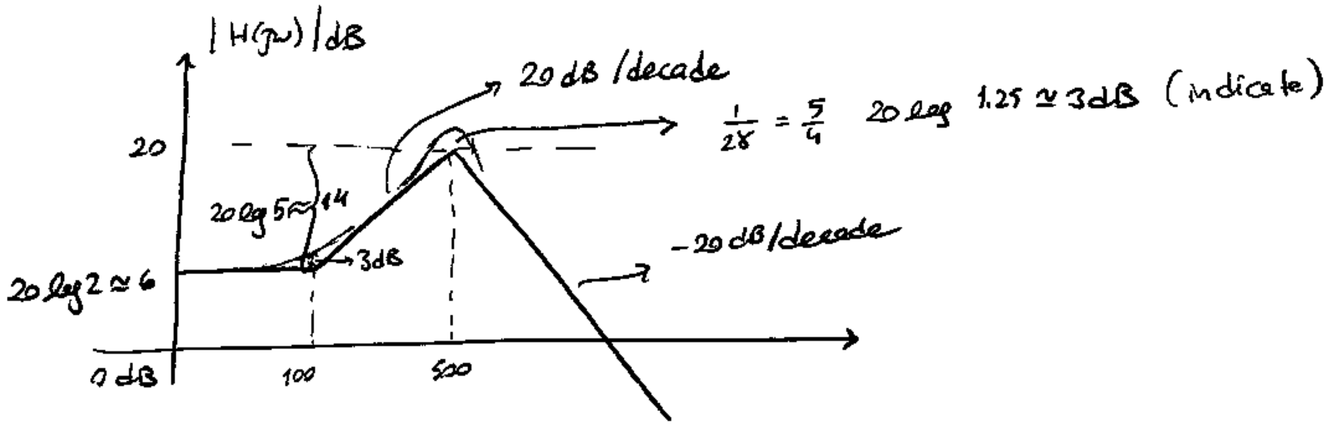
$(500)^2 \rightarrow \omega_0^2$

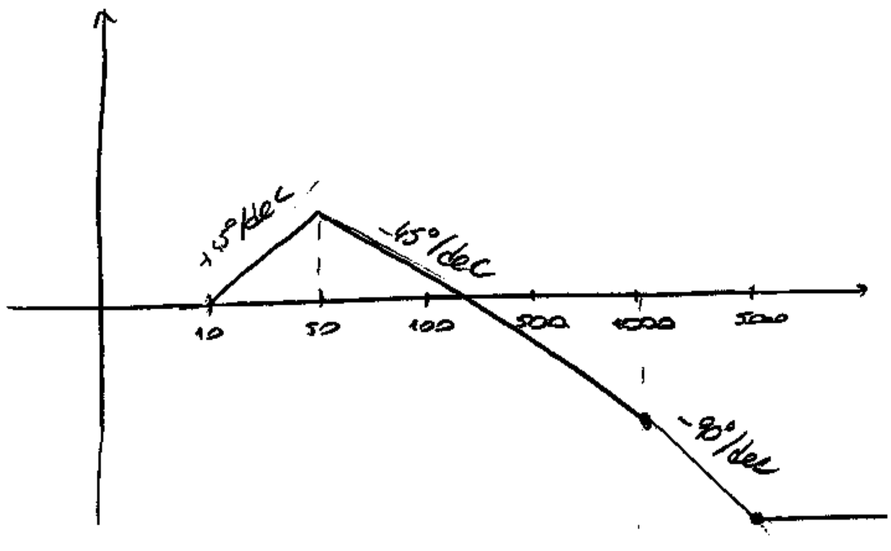
$\left(\frac{s}{500}\right)^2 \rightarrow \omega_0 = 500$

$\omega_{critical} = \{100, 500\}$

100 → zero location

500 → ω_0 (2nd order system)



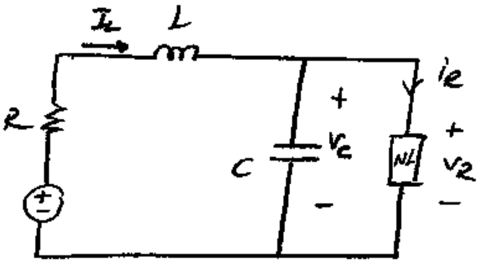


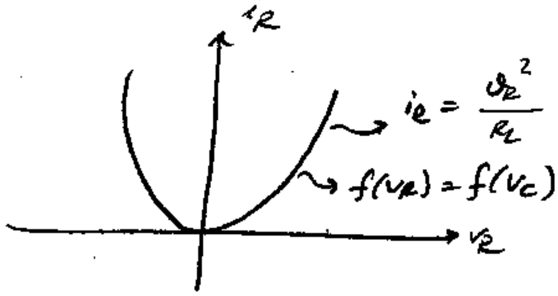
State Equations with non-linear elements

If an element is non-linear we would like to express the circuit in the state equation form, we can do the following.

1. Check whether the component is voltage / current controlled.
2. Try to include the current of the nonlinear component if it is current controlled (or its voltage otherwise)
3. Include variables of dynamic components as state variables
4. Try to combine 2 and 3

Ex





KCL: $C \cdot \dot{V_c}(t) = -i_R + i_L$

$\hookrightarrow \; -\dfrac{v_c^2(t)}{R_L}$

$$\dot{v_c}(t) = -\frac{1}{C \cdot R_L} v_c^2(t) + \frac{1}{C} i_L(t)$$

KVL $\quad -v_s(t) + R \cdot I_L(t) + L \dot{I_L}(t) + V_c(t) = 0$

$$\dot{I_L}(t) = -\frac{1}{L} V_c(t) - \frac{R}{L} I_L(t) + \frac{1}{L} v_s(t)$$

zero input:

$$\dot{v_c}(t) = -\frac{v_c^2(t)}{R_L \cdot C} + \frac{i_L(t)}{C} \qquad V_c(0^-) = V_0$$

$$\dot{I_L}(t) = -\frac{v_c(t)}{L} - \frac{R}{L} I_L(t) \qquad I_L(0^-) = I_0$$

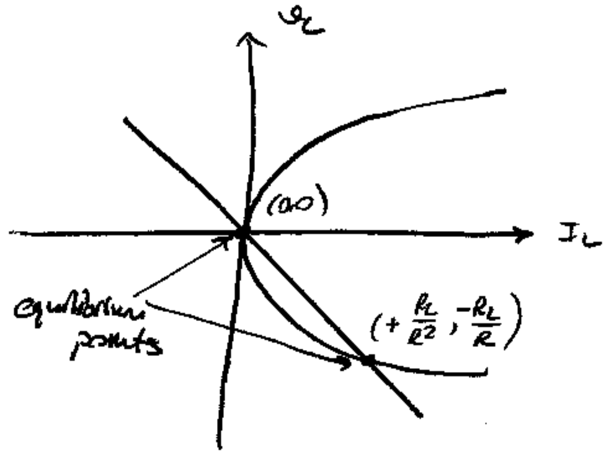
$V_c(t) \triangleq V_c \quad I_L(t) \triangleq I_L$

$\dot{v_c}(t) = 0 \longrightarrow \; -\dfrac{v_c^2}{R_L C} + \dfrac{I_L}{C} = 0$

$\dot{I_L}(t) = 0 \longrightarrow \; -\dfrac{v_c}{L} - \dfrac{R}{L} I_L = 0$

$v_c = -2\,I_L$

$v_c^2 = R_L\,I_L$



① In the following circuit $|V_L| = 240$ V (RMS) at all times. The following information about the loads are given:

L_1: absorbs 180 Watts and 240 VARs.

L_2: absorbs 600 VA at 0.6 pf lagging

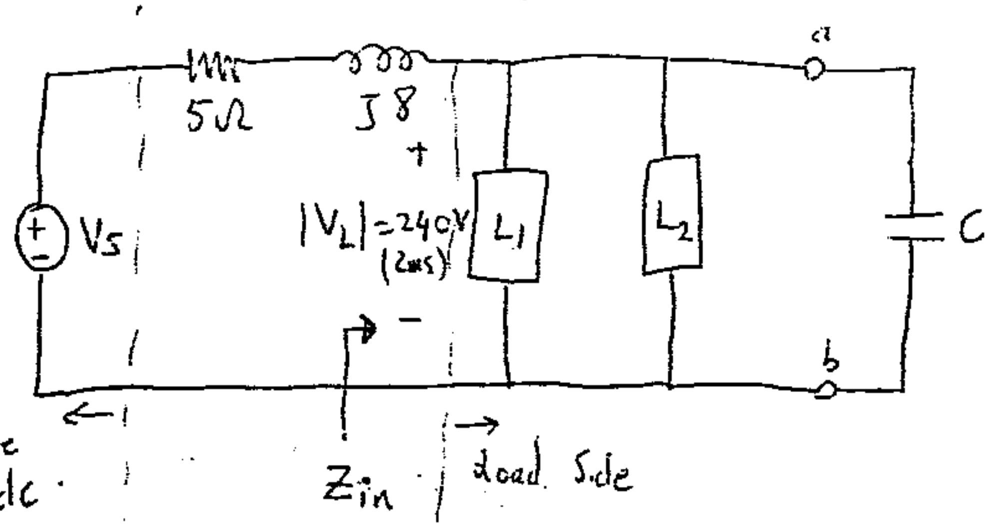
a) Assume there is no capacitor connected between a-b terminals. Find input impedance at the load side, source voltage $|V_S|$ (in RMS) and p.f. on source side.

b) Now, the capacitor is connected between a-b terminals. Find value of C such that average power absorbed by 5 ohm resistor is minimum. Find $|V_S|$ with the compensation capacitor.

c) Find value of C such that load side p.f. is 0.9 lagging.

Find $|V_S|$ with compensation capacitor.

$f = 50$ Hz.



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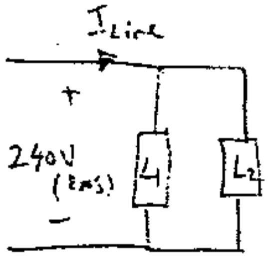


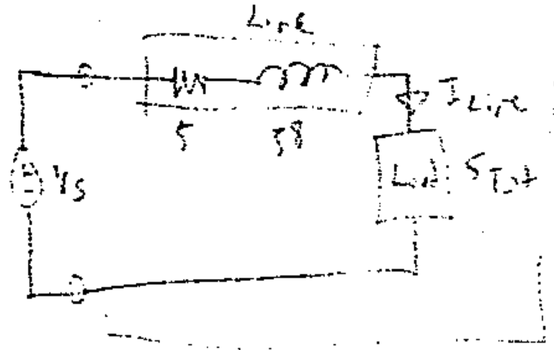
$$S_{L_1} = 180 + j\,240$$

$$+\ S_{L_2} = 600 \,\angle \cos^{-1}(0.6) = 360 + j\,480$$

$$S_{Total} = 540 + j\,720$$

a) $$|I_{Line}| = \frac{|S_{Total}|}{240} = \frac{900}{240} = 3.75 \text{ A} \quad (\text{RMS}).$$

$$S_{Total} = |I_{Line}|^2 \cdot Z \longrightarrow \quad Z = \frac{540 + j\,720}{(3.75)^2} = 38.4 + j\,51.2 = 64 \angle 53.1^\circ$$



$$S_{Line} = |I_{Line}|^2 \,(5 + j8) = 70.3125 + j\,112.5$$

$$S_{source} = S_{Line} + S_{Total} = 610.31 + j\,832.5$$

$$|S_{source}| = |I_{Line}|\,|V_s| \longrightarrow \quad |V_s| = \frac{|610.31 + j\,832.5|}{3.75} = 275.26 \text{ V RMS}$$

p.f on source side. $\cos\left(\tan^{-1}\dfrac{832.5}{610.31}\right) = 0.59$ lagging.

b) $S^{before} = 540 + j\,720$

$I_{Line}^{before} = 3.75 \text{ A}$

$$S^{after} = 540 + j\,720 - j\,720 = 540$$

compensator

$$I_{Line}^{after} = \frac{540}{240} = 2.25 \text{ A}$$

$$S_{Line} = (2.25)^2 (5 + j8) = 25.31 + j\,40.5$$

$$S_{source} = 540 + 25.31 + j\,40.5 \Longrightarrow$$

$S_{source} \overset{After}{=} 565.31 + j\,40.5$ (3)

$|S_{source}| = 566.75 = |V_S| \cdot |I_{Line}| \rightarrow$

$\rightarrow |V_S| \doteq 251.89$ V (RMS)

$S_{compensator} = -j\,720$

$S_{compensator} = \dfrac{|V_{cap}|^2}{X_c^*}$ ← 240

$X_c = \dfrac{|V_{cap}|^2}{S^*_{comp}} = \dfrac{(240)^2}{j\,720} = -j\,80.$

$X_c = \dfrac{-j}{\omega C}$

$X_c = -j\,80 \rightarrow C = \dfrac{1}{(2\pi\,50)\,80} = 39.8\,\mu F.$

(In this problem, to minimize power loss on line resistor, the current through the resistor should be minimized. Therefore load side pf should be 1.0 after compensation. to minimize loss)

c) After

$\hookrightarrow 540$

$\cos^{-1}(0.9) = 25.84$

$S^{After} = 540 + j\tan(25.84°)\,540$
$= 540 + j\,261$

$S^{After} = S^{before} + S^{compensation}$

$S_{compensation} = -j\,458.5.$

$X_c = -j\dfrac{(240)^2}{458.5} = -j\,125.63;\ C = \dfrac{1}{(100\pi)\,125.6}$
$= 25.3\,\mu F$

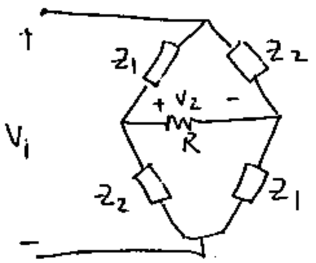
$$|I_{Line}|^{after} = \frac{|S^{after}|}{240} = \frac{540/0.9}{240} = 2.5\,A.$$ (4)

$$S^{Line} = (2.5)^2(5 + j8) = 31.25 + j50$$

$$S^{source} = 571.25 + j301.1 \quad \text{(supplied)}$$

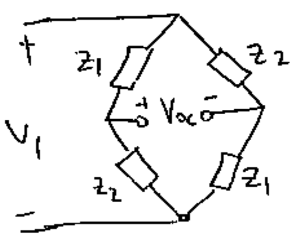
$$\underbrace{|S^{source}|}_{645} = |V_S| \cdot \underbrace{|I_{Line}|}_{2.5} \longrightarrow V_S = 258.27\,V. \quad \text{(RMS)}.$$

2nd Order All-Pass Circuit:



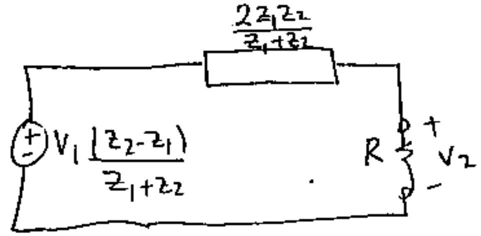
i) Z_{Th} seen by $R \Rightarrow (Z_1 \| Z_2)2$

ii)



$V_{oc} = V_1\left(\frac{Z_2 - Z_1}{Z_1 + Z_2}\right)$

Then.



$\frac{V_2(s)}{V_1(s)} = ?$

$$\frac{V_2(s)}{V_1(s)} = \frac{(Z_2^{(s)} - Z_1(s))}{Z_1^{(s)} + Z_2(s)} \cdot \frac{R}{R + \frac{2Z_1Z_2}{Z_1+Z_2}} = (Z_2 - Z_1) \cdot \frac{R}{R(Z_1+Z_2) + 2Z_1Z_2}$$

$$= \frac{(Z_2 - Z_1)}{(Z_1 + Z_2) + 2\frac{Z_1Z_2}{R}} \rightarrow$$

$\boxed{\text{let } Z_1 Z_2 = R^2}$ $\xrightarrow{\text{then}}$ $= \frac{(Z_2 - Z_1)}{(Z_1 + Z_2) + 2R}$

$$= \frac{Z_2(Z_2 - Z_1)}{Z_2\{(Z_1 + Z_2) + 2R\}}$$

$\boxed{H(s) = Z_2 - R}$ $= \frac{(Z_2 - R)(Z_2 + R)}{\ldots}$ $= \frac{Z_2^2 - R^2}{\ldots 2RZ_2 + R^2}$

here

$$H(j\omega) = \frac{Z_2(j\omega) - R}{Z_2(j\omega) + R}$$

$V_a(s) = \frac{sL - R}{sL + R} V_R(s)$

then ① $Z_2(j\omega)$: purely real $\forall \omega \xrightarrow[Z_2 = R_2]{} H(j\omega) = \frac{R_2 - R}{R_2 + R}$

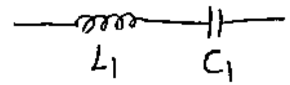
$\rightarrow |H(j\omega)|$ is not function of $\omega \rightarrow$ but system does not contain any dynamic elements. Therefore it is just a voltage divider, not a filter.

② $Z_2(j\omega)$: purely imaginary $\forall \omega \xrightarrow[Z_2 = sX_2]{} H(j\omega) = \frac{sX_2 - R}{sX_2 + R}$

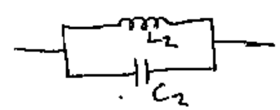
$\rightarrow |H(j\omega)| = 1 \rightarrow$ we have dynamic system with an all-pass structure.

Example:

Z_1:



Z_2:



$$\begin{cases} Z_1(s) = sL_1 + 1/sC_1 \\ Z_2(s) = \left(\frac{1}{sL_2} + sC_2\right)^{-1} \end{cases}$$

$Z_1 \cdot Z_2 = R^2 \rightarrow Z_1 = \frac{R^2}{Z_2} \rightarrow sL_1 + \frac{1}{sC_1} = R^2\left(\frac{1}{sL_2} + sC_2\right)$

let $\boxed{\begin{array}{l} L_1 = R^2 C_2 \\ C_1 = L_2/R^2 \end{array}}$

Then $H(s) = \frac{Z_2 - R}{Z_2 + R} = \frac{1 - RY_2}{1 + RY_2}$ $\left(Z_2^{-1} = Y_2 = \frac{1}{sL_2} + sC_2 \right)$

$$= \frac{1 - \frac{R}{sL_2} - sRC_2}{1 + \frac{R}{sL_2} + sRC_2}$$

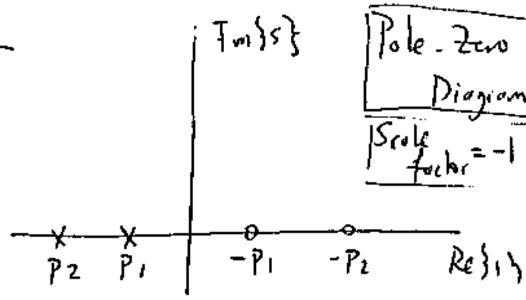
$$= - \frac{s^2 - \frac{1}{RC_2}s + \frac{1}{L_2C_2}}{s^2 + \underbrace{\left(\frac{1}{RC_2}\right)}_{2\alpha w_0}s + \underbrace{\left(\frac{1}{L_2C_2}\right)}_{w_0^2}}$$

$$H(s) = - \frac{s^2 - 2\alpha w_0 s + w_0^2}{s^2 + 2\alpha w_0 s + w_0^2}$$

Note: If s_1, s_2 are roots of $s^2 + 2\alpha w_0 s + w_0^2 = 0$.
then. $-s_1, -s_2$ are the roots of $s^2 - 2\alpha w_0 s + w_0^2 = 0$.
(Why ?: (Hint: $-2\alpha w_0$ = sum of roots ; w_0^2: product of roots))

then.

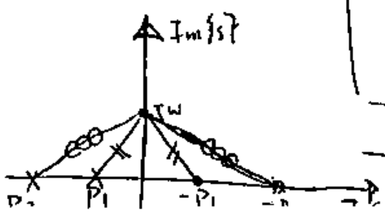
$$H(s) = - \frac{(s-p_1)(s-p_2)}{(s+p_1)(s+p_2)}$$



Pole-Zero Diagram

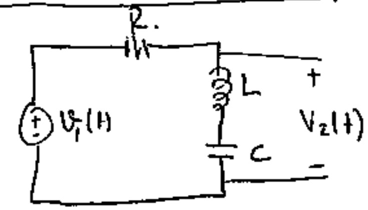
Scale factor = -1

Note:



← this picture shows that system is all-pass !.

Series RLC Band-Stop Filter

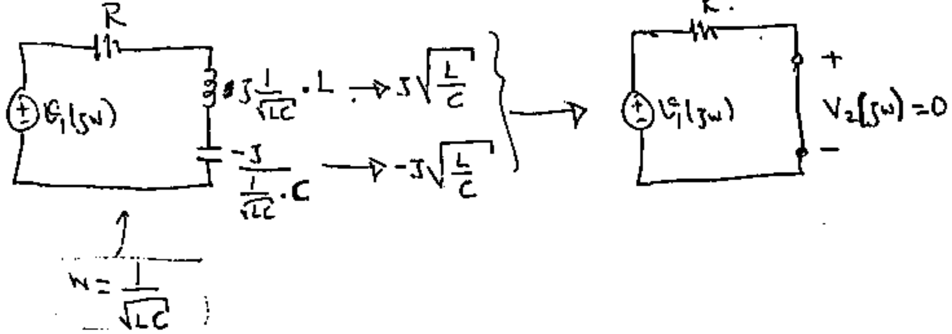


$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{sL + 1/sC}{sL + 1/sC + R}$$

$$= \frac{s^2 + 1/LC}{s^2 + \frac{R}{L}s + 1/LC}$$

$$H(jw) = \frac{\left(1/LC - w^2\right)}{\left(1/LC - w^2\right) + jw\frac{R}{L}}$$ → Note: $|H(jw)| = 0$ for $w = \pm\sqrt{1/LC}$

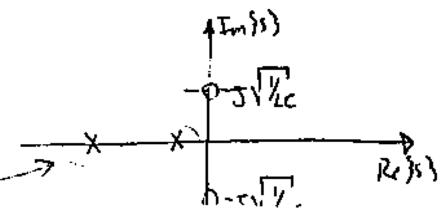
Let's redraw the circuit at $w = 1/\sqrt{LC}$



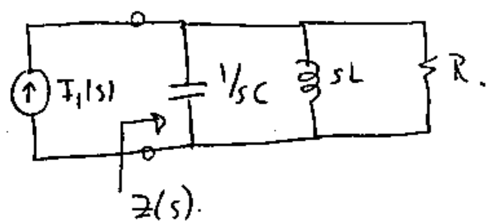
Note that $V_1(jw)$ source sees purely resistive circuit at $w = \frac{1}{\sqrt{LC}}$

This phenomena is called resonance. (more on this later).

Pole-Zero Diagram



Parallel RLC Resonance Circuit.



$$Z(s) = \frac{1}{1/R + 1/sL + sC} = \frac{s/C}{s^2 + \frac{1}{RC}s + 1/LC}$$

<u>Definition:</u> The frequency for which $Z(j\omega)$ is purely real is called the <u>resonance frequency</u>.

$$Z(j\omega) = \frac{j\omega/C}{-\omega^2 + 1/LC + j\omega\, 1/RC} = \frac{\omega/C}{j(\omega^2 - 1/LC) + \omega\, 1/RC}$$

is zero when $\omega = 1/\sqrt{LC}$ ← Resonance freq.

(We have studied this system under the title of 2nd Order Band-Pass circuits)

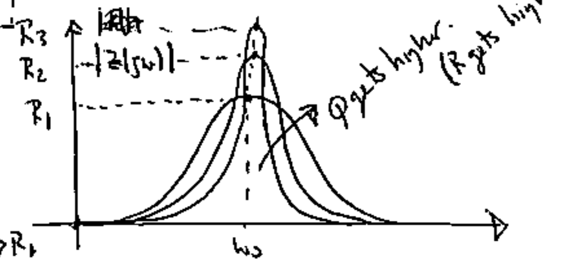
$$s^2 + \underbrace{1/RC}_{2\gamma\omega_0}\, s + \underbrace{1/LC}_{\omega_0^2} \longrightarrow \quad BW = \omega_2 - \omega_1 = 2\gamma\omega_0 = 1/RC$$

−3dB points

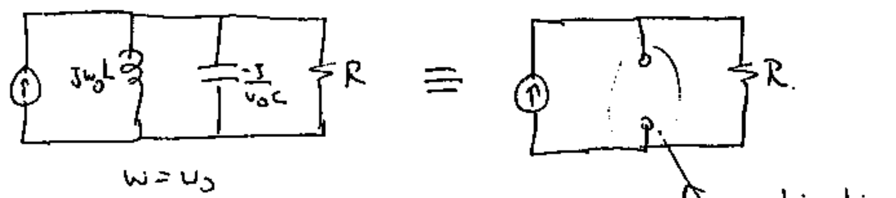
$$Q = \frac{\omega_0}{BW} = \frac{1}{2\gamma} = R\sqrt{\frac{C}{L}}$$

$$\omega_0 = \sqrt{1/LC}\,, \quad 2\gamma = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Picture shows the impedance seen by the source for 3 parallel RLC circuits with $R_3 > R_2 > R_1$ and the same L and C's.



The maximum impedance is seen at resonance freq. (for parallel RLC) and it is equal to R. Note that



Another Interpretation for Q: ← Quality factor.

For parallel RLC circuit, the average energy stored in capacitor is $E_C = \frac{1}{2} C V_{eff}^2$; similarly the average magnetic energy stored in inductor is $E_L = \frac{1}{2} L I_{eff}^2 \longrightarrow$

Let's calculate the total energy stored in L and C.

$$E_T = E_C + E_L = \frac{1}{2} C V_{eff}^2 + \frac{1}{2} L \left(\frac{v_{eff}}{\omega L}\right)^2$$

$$= \frac{1}{2}\left(C + \frac{1}{\omega^2 L}\right) v_{eff}^2$$

$$= \frac{1}{2} C \left(1 + \frac{1}{\omega^2 \underbrace{LC}_{1/\omega_0^2}}\right) v_{eff}^2$$

$$= \frac{1}{2} C \left(1 + \frac{\omega_0^2}{\omega^2}\right) v_{eff}^2$$

$$= \frac{1}{2} C \frac{\omega_0}{\omega}\left(\frac{\omega}{\omega_0} + \frac{\omega_0}{\omega}\right) v_{eff}^2$$

We will now show that at $w = w_0$

$$Q = 2\pi \cdot \frac{E_T}{T P_R} \quad \left(\begin{matrix} Q: \\ \text{unitless} \end{matrix}\right) \text{ where } \quad T = \frac{2\pi}{w_0}$$

where P_R is the average ~~power~~ power consumed by R and $T \cdot P_R$ is the energy dissipated over R in a period.

high Q, high E_T, small P_R.

Now

$$2\pi \frac{E_T}{T.P_R} = \frac{w E_T}{P_R} = \frac{w \frac{1}{2} C \frac{w_0}{w}\left(\frac{w}{w_0} + \frac{w_0}{w}\right) V_{eff}^2}{\frac{V_{eff}^2}{R}}$$

$$= \frac{RCw_0}{2}\left(\frac{w}{w_0} + \frac{w_0}{w}\right) \qquad 1/\sqrt{LC}$$

$$= \frac{R\sqrt{\frac{C}{L}}}{2}\left(\frac{w}{w_0} + \frac{w_0}{w}\right)$$

$$= \frac{Q}{2}\left(\frac{w}{w_0} + \frac{w_0}{w}\right)$$

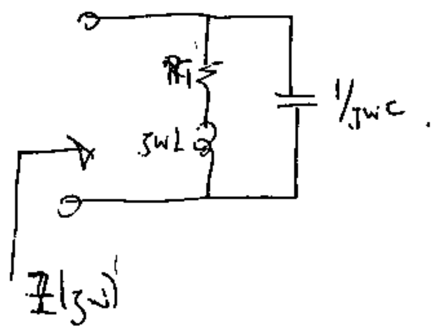
for $w = w_0 \longrightarrow 2\pi \frac{E_T}{T.P_R} = Q$, $\quad T = \frac{2\pi}{w_0}$.

Note: Resonant freq. is the freq. where maximum of $|Z(jw)|$ for parallel RLC circuit; BUT this is not

If $w_{..}$ is the freq of maxima of $|Z(jw)|$,

Finite Q Capacitors and Inductors.

In practice, it is not possible to have inductors which does not have any internal resistance. (Remember inductor ≡ coil) Then any resonance circuit, that is containing both capacitors and inductors are effected by the internal resistance value. Since resonance circuits are operated around the resonant freq, our analysis is focused on the behaviour around resonant freq.



$$Z(jw) = \frac{1}{\frac{1}{R_1 + jwL} + jwC}$$

$$Z(s) = \frac{1}{\frac{1}{R_1 + sL} + sC}$$

$$Z(s) = \frac{s/C + R_1/LC}{s^2 + s\left(R_1/L\right) + \left(1/LC\right)}$$

$2\gamma w_0$ $\quad w_0^2$

$$s^2 + s\frac{R_1}{L} + \frac{1}{LC} = s^2 + (2\gamma w_0)s + w_0^2$$

$BW = \frac{w_0}{Q}$

w_0 is not the resonance freq. or the frequency for which $Z(jw)$ has a maxima.

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$$Z(s)\Big|_{s=j\omega} = \frac{j\omega/C + r_1/LC}{\underbrace{(1/LC}_{\omega_a^2} - \omega^2) + j\omega \underbrace{(R_1/L)}_{\omega_a/Q}} = \frac{\frac{1}{C}\left(j\omega + \overset{\omega_a/Q}{r_1/L}\right)}{(\omega_a^2-\omega^2) + j\omega\,\omega_a/Q}$$

$$= \frac{\frac{1}{C}\left(j\omega + \omega_a/Q\right)}{(\omega_a^2-\omega^2) + j\omega\,\omega_a/Q}$$

for $Z(j\omega)$ to be purely real, angle of numerator and denominator of $Z(j\omega)$ should be same $\longrightarrow$ so

$$\frac{\cancel{\omega}}{\omega_a/Q} = \frac{\cancel{\omega}\,\omega_a/Q}{\omega_a^2-\omega^2} \longrightarrow \left(\frac{\omega_a}{Q}\right)^2 = \omega_a^2 - \omega^2$$

$$\frac{\omega_a L}{r}$$

$$\Downarrow$$

$$\omega^2 = \omega_a^2\left(1 - 1/Q^2\right)$$

$$\boxed{\omega = \omega_a\sqrt{1 - 1/Q^2}}$$

then $\boxed{\omega_0 = \omega_a\sqrt{1 - 1/Q^2}}$ is the resonant freq.

$$Z(j\omega_0) = \frac{1}{C}\,\frac{\cancel{(j\omega_0 + \omega_a/Q)}}{\omega_a/Q\,\cancel{(j\omega_0 + \frac{\omega_a^2-\omega^2}{\omega_a/Q})}} = \frac{Q\cdot 1}{C\omega_a} = \frac{Q\cdot\omega_a^2 L \cancel{C}}{\cancel{C}\omega_a}$$

$$= Q\cdot\frac{\omega_a r_1}{(r_1/L)\,Q}$$

$$= Q^2\cdot r_1$$

例: a) $r_1 = 50\,\Omega$, $L = 10\,mH$, $C = 1\,\mu F$

then

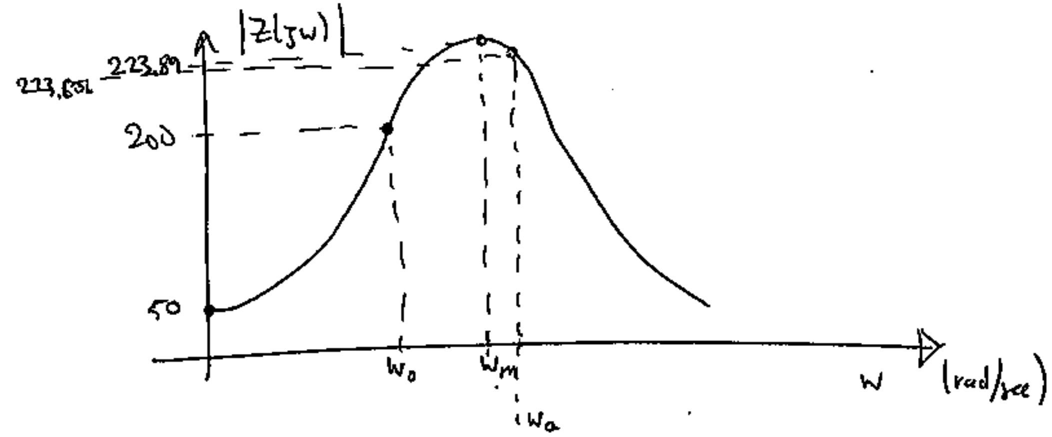
$\omega_a = 1/\sqrt{LC} = 10^4 = 10\,k$ rad/sec.

$Q = \frac{\omega_a}{r_1/L} = 2$ (unitless).

$\omega_0 = \omega_a\sqrt{1 - 1/Q^2} = 10^4\sqrt{1 - 1/4} = 8.66\,k$ rad/sec.

$Z(j0) = 50\,\Omega$, $Z(j\omega_0) = Q^2\cdot r_1 = 200\,\Omega$; $Z(j\omega_a) = 223.606\angle{-26^\circ}$

$Z(j\omega_m) = 223.89\angle{-23}$



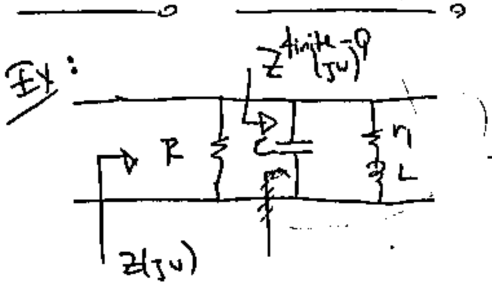
$\omega_a = \sqrt{1/LC}$

ω_m: $\longrightarrow$ should be found by taking derivative of $|Z(j\omega)|$ wrt. ω.

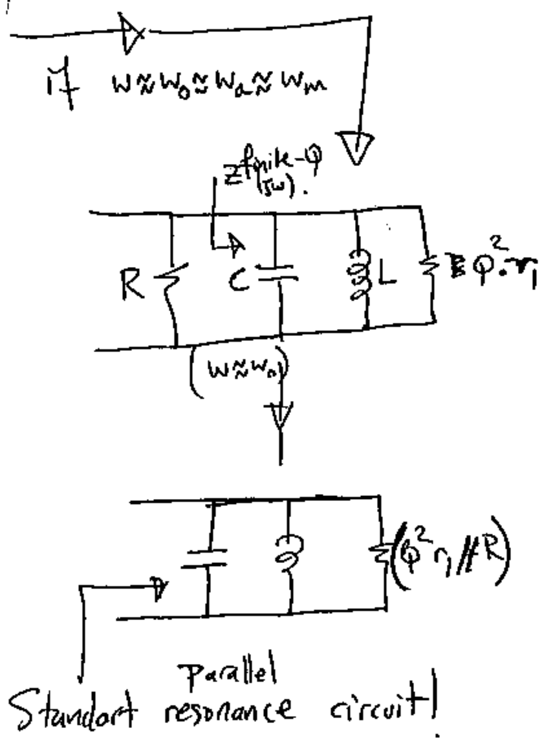
$\omega_0 = \omega_a\sqrt{1 - 1/Q^2} \longleftarrow$ resonant freq. of finite-Q circuit.

close to w_m. As $Q \to \infty$, w_a and w_m approaches w_0. But note that $Q = 2$ is close enough for many purposes in this example.

Ex:



Find $Z(jw)$, and assume that $w \approx$ resonant freq of this system.



$$w_p = w_0 \sqrt{1 - \frac{1}{2Q^2}}$$

$$Q > \frac{1}{\sqrt{2}}$$