EE 201

Tellegen's theorem

9. Consider the sinusoidal steady-state measurements performed on a linear time-invariant *RLC* network, shown in Fig. P9.9. In both instances the same voltage source is used (same frequency and same phasor). Show that $\hat{J}_1 = J_2$. (Hint: Show that $V_2\hat{J}_2 + V_1\hat{J}_1 = \hat{V}_2 J_2 + \hat{V}_1 J_1$.)

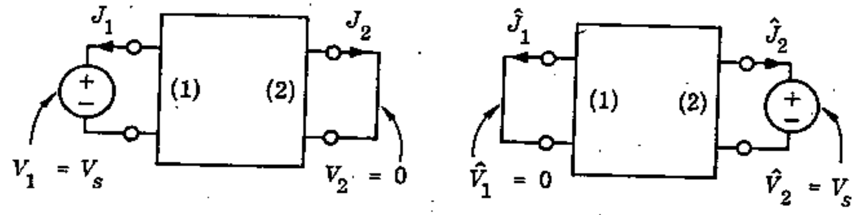


Fig. P9.9

Measurements and properties of RLC networks

10. Your technician measures the driving-point impedance (or admittance) at a fixed frequency ω_0 of a number of linear time-invariant networks made of passive elements. In each case, state whether or not you have any reasons to believe his results (in ohms or in mhos).

a. *RC* network: $Z = 5 + j2$

b. *RL* network: $Z = 5 - j7$

c. *RLC* network: $Y = 2 - j3$

d. *LC* network: $Z = 2 + j3$

e. *RLC* network: $Z = -5 - j19$

f. *RLC* network: $Z = -j7$

Whenever you accept a measurement as plausible, assume $\omega_0 = 1$ rad/sec, and give a linear time-invariant passive network which has the specified network function.

Basic Circuit Theory, (Desoer and Kuh)

Node and Mesh Analyses

This chapter and the succeeding two are devoted to *general* methods of network analysis. The problem of network analysis may be stated as follows: given the network graph, the branch characteristics, the input (i.e., the waveforms of the independent sources), and the initial conditions, calculate all branch voltages and branch currents. In these three chapters we shall consider only the formulation of network equations. The methods of solution and the properties of the solutions will be studied in Chaps. 13 to 16.

In the present chapter we shall systematically develop node and mesh analysis. This systematic treatment is particularly important at present when computers automatically perform the analysis of networks. Also from the results of these systematic analyses we shall obtain the tools that will allow us to develop the properties of these networks.

In Sec. 1, we present source transformations that we shall use in all the following methods of analysis. In Sec. 2, the consequences of Kirchhoff's laws are obtained in the context of node analysis. Section 3 develops the systematic analysis of linear time-invariant networks. Duality is developed in Sec. 4. Finally, mesh analysis is presented in Secs. 5 and 6. Again, only linear time-invariant networks are considered.

1 Source Transformations

In the general discussion of the problem of network analysis we assume that the number and the location of independent sources are arbitrary as long as Kirchhoff's laws are not violated (i.e., as long as no independent voltage sources form a loop and no independent current sources form a cut set—for in either case the waveforms of these sources would have to satisfy a linear constraint imposed by KVL and KCL, respectively).

To obviate separating the branches consisting only of sources from the other branches, it is useful to introduce first two network transformations that allow us to relocate sources in the network without affecting the problem. These transformations can be used for both independent and dependent sources. They are illustrated in Figs. 1.1 and 1.2.

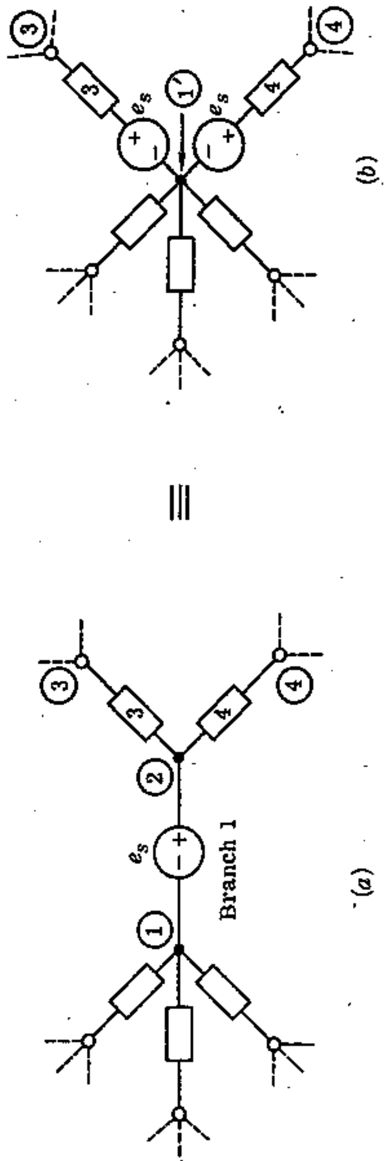


Fig. 1.1 Source transformation; a branch consisting of a voltage source alone is eliminated.

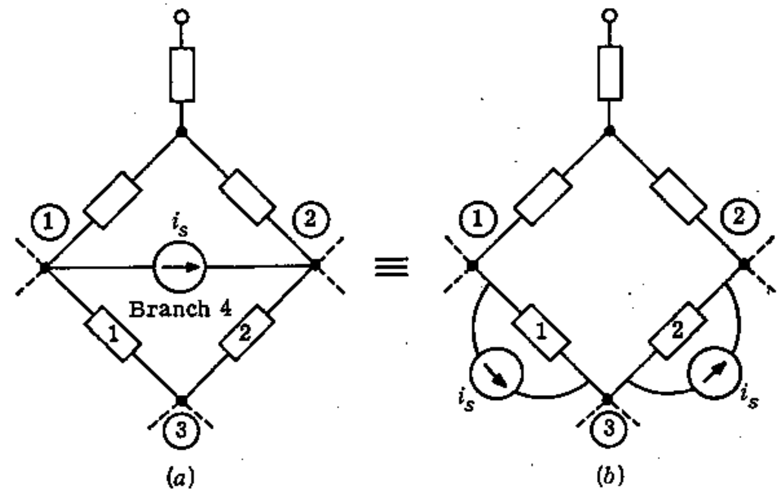


Fig. 1.2 Source transformation; a branch consisting of a current source alone is eliminated.

In Fig. 1.1*a*, branch 1 is a voltage source e_s which is connected between nodes ① and ②. Node ② is connected to node ③ through branch 3 and to node ④ through branch 4. If the current in branch 1 is of no interest to us, we can replace the circuit in Fig. 1.1*a* with its equivalent in Fig. 1.1*b*. In the new circuit, branch 1 has been eliminated, and a new node (1') is introduced. This new node (1') results from the merger of nodes ① and ② of the original circuit. To be equivalent, two sources e_s must be inserted in branch 3 and branch 4 of the new circuit.

Showing that this transformation does not change the solution of the problem is straightforward. We need only write the KVL equations for all the loops containing branch 3 and all the loops containing branch 4 in both networks. It is easily checked that the corresponding equations for the two networks are the same. Also KCL applied to node (1') is identical to the sum of the equations obtained by applying KCL to node ① and to node ② of the given network. Consequently, the KCL equations of both networks impose equivalent constraints.

In Fig. 1.2*a*, branch 4 is a current source i_s connected between nodes ① and ②. Nodes ① and ② are also connected to node ③ through branches 1 and 2, respectively. In Fig. 1.2*b* we show the equivalent circuit, where the current source in branch 4 has been removed; instead, two new current sources i_s are connected in parallel with branches 1 and 2. That this transformation does not change the solution of the problem can be seen by writing the KCL equations for nodes ①, ②, and ③ in both networks. Clearly, the corresponding equations are the same.

Exercise 1 Show that except for the element affected the following transformations do not affect the branch voltages and the branch currents of a network: (1) if a branch consists of a current source in series with an element, the element may be replaced by a short circuit; (2) if a branch consists of a current source in series with a voltage source, the voltage source may be replaced by a short circuit; (3) if a branch consists of a voltage source in parallel with an element, this element may be replaced by an open circuit; (4) if a branch consists of a voltage source in parallel with a current source, the current source may be replaced by an open circuit. Observe that cases (3) and (4) are the duals of (1) and (2), respectively.

In conclusion, by using these transformations, we can modify any given network in such a way that *each voltage source is connected in series with an element which is not a source and each current source is connected in parallel with an element which is not a source.*

Thus, we find that, without loss of generality, we can assume that for any network a typical branch, say, branch k, is of the form shown in Fig. 1.3, where v_{sk} is a voltage source, j_{sk} is a current source, and the rectangular box represents an element which is not a source. As before, the branch voltage is denoted by v_k and the branch current by j_k. The characterization of branch k thus includes possible source contributions. In particular, if there is no voltage source in branch k, we set $v_{sk} = 0$; similarly, if there is no current source, we set $j_{sk} = 0$.

Exercise 2 Suppose that, in Fig. 1.3, the nonsource element is a linear time-invariant resistor of resistance R_k. Show that the branch equation is

$$v_k = v_{sk} - R_k j_{sk} + R_k j_k \tag{1.1}$$

Show that this branch can be further simplified to look like Fig. 1.5.

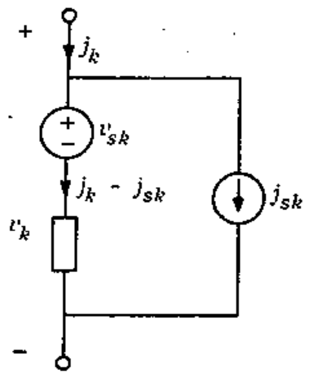


Fig. 1.3 Branch k, including voltage and current sources.

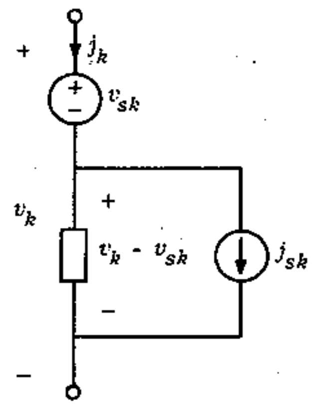


Fig. 1.4 Branch k, including voltage and current sources.

Exercise 3 Suppose that, in Fig. 1.4, the nonsource element is a linear time-invariant resistor of conductance G_k. Show that the branch equation is

$$j_k = j_{sk} - G_k v_{sk} + G_k v_k \tag{1.2}$$

Show that this branch can be further simplified to look like Fig. 1.6.

These two exercises illustrate useful equivalences. It is convenient, though not necessary, in node analysis for all independent sources to be current sources. Similarly, it is convenient, though not necessary, in loop or mesh analysis for all independent sources to be voltage sources.

From now on we shall assume that in dealing with resistive networks the branches are always of the form of Fig. 1.5 or Fig. 1.6, that is, a resistor in series with a voltage source or a resistor in parallel with a current source.

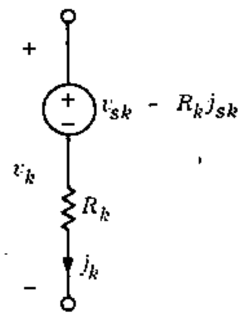


Fig. 1.5 A resistive branch with an equivalent voltage source.

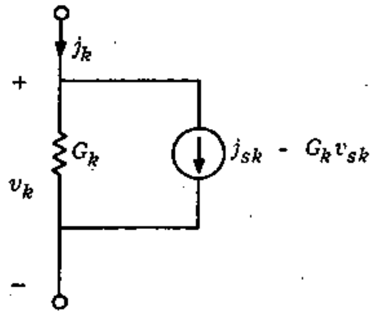


Fig. 1.6 A resistive branch with an equivalent current source.

In general networks, the resistor may be replaced by an inductor or a capacitor.

Exercise 4 Perform the transformation for the circuits in Fig. 1.7.

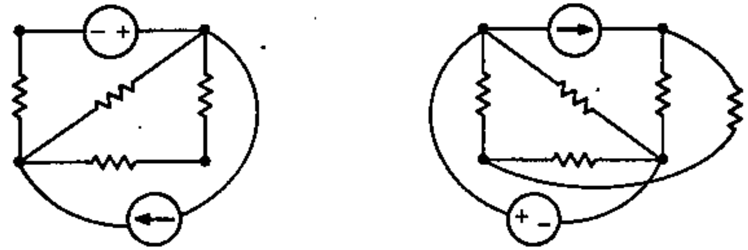


Fig. 1.7 Exercises on source transformation.

2 Two Basic Facts of Node Analysis

Let us consider any network $\mathfrak{N}$ and let it have n_t nodes and b branches. Altogether there are b branch voltages and b branch currents to be determined. Without loss of generality we may assume that the graph is connected, i.e., that it has one separate part only. (If the network were made of two separate parts, we could connect these two separate parts by tying them together at a common node.)

First, we pick arbitrarily a reference node. This reference node is usually called the **datum node.** We assign to the datum node the label (n_t) and to the remaining nodes the labels (1), (2), ..., (n), where $n \triangleq n_t - 1$.

2.1 Implications of KCL

Let us apply KCL to nodes (1), (2), ..., (n) (omitting the datum node) and let us examine the form of the equations obtained. Typically, as in

the case of node (k) shown in Fig. 2.1, we obtain a homogeneous linear algebraic equation in the branch currents; thus,

$$j_4 + j_6 - j_7 = 0$$

Thus, we have a system of n linear algebraic equations in b unknowns $j_1, j_2, \ldots, j_b$. The first basic fact of node analysis is the following statement:

The n linear homogeneous algebraic equations in $j_1, j_2, \ldots, j_b$, obtained by applying KCL to each node except the datum node, constitute a set of linearly independent equations.

Let us start by an observation. Consider the n_t equations obtained by writing KCL for each of the n_t nodes of $\mathfrak{N}$. Let $\mathcal{E}_k = 0$ represent the equation pertaining to node (k), $k = 1, 2, \ldots, n_t$. For the node (k) shown in Fig. 2.1, the equation $\mathcal{E}_k = 0$ gives $j_4 + j_6 - j_7 = 0$. We assert that $\sum_{k=1}^{n_t} \mathcal{E}_k$ reduces identically to zero. In other words, if we add all the n_t KCL equations (written in terms of the branch currents $j_1, j_2, \ldots, j_b$), all terms cancel out. This is obvious. Suppose branch 1 leaves node (2) and enters node (3). The term j_1 appears with a plus sign in $\mathcal{E}_2$ and a minus sign in $\mathcal{E}_3$, and, since j_1 appears in no other equation, j_1 cancels out in the sum. Since every branch of $\mathfrak{N}$ must leave one node and terminate on another node, all branch currents will cancel out in the sum. We conclude that *the n_t equations obtained by writing KCL for each of the nodes of the network $\mathfrak{N}$ are linearly dependent.*

Let us now prove that the n linear algebraic equations $\mathcal{E}_1 = 0$, $\mathcal{E}_2 = 0, \ldots, \mathcal{E}_n = 0$ are linearly independent. Suppose they were not; i.e., suppose that these n equations are linearly dependent. This would mean that, after some possibly necessary reordering of the equations, there is a linear combination of the first k equation $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k$ $(k \leq n)$ with respective *nonzero* weighting factors $\alpha_1, \alpha_2, \ldots, \alpha_k$, which sums *identically* to zero. Thus,

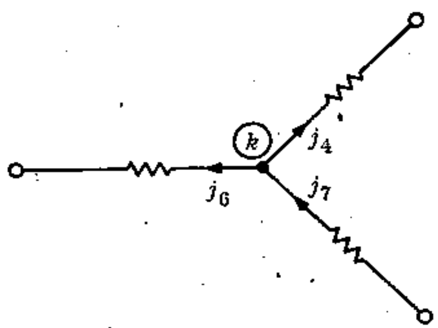


Fig. 2.1 A typical node to illustrate KCL.

(2.1) $\alpha_1 \mathcal{E}_1 + \cdots + \alpha_k \mathcal{E}_k \equiv 0$

Consider the set of all nodes ①, ②, . . . , ⓚ corresponding to $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k$ in Eq. (2.1). Consider the set of remaining nodes as shown in Fig. 2.2. The remainder of $\mathfrak{N}$ contains $n_t - k$ nodes; since $k \leq n$, the remainder includes at least one node. Since $\mathfrak{N}$ is connected, there is at least one branch, say branch l, which joins a node of the first set to a node of the second set. Then j_l can appear in one and only one of the equations $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k$ since the branch l is connected to only one of the nodes of the first set. Therefore, it cannot cancel out in the sum

$$\sum_{i=1}^{k} \alpha_i \mathcal{E}_i$$

Hence we arrive at a contradiction with Eq. (2.1). This argument holds for any possible linear combination. Therefore, the assumption that $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ are linearly dependent is false; hence the statement that these equations are linearly independent has been proved.

Consider again the system of n linear algebraic equations that express KCL for all the nodes of $\mathfrak{N}$ except the datum node. We assert that this system has the following matrix form:

(2.2) $\mathbf{Aj} = \mathbf{0}$ (KCL)

where $\mathbf{j}$ represents the branch current vector and is of dimension b; that is,

$$\mathbf{j} = \begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_b \end{bmatrix}$$

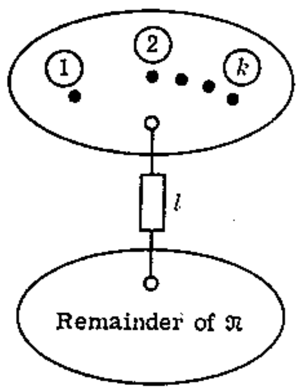


Fig. 2.2 Graph used to prove that n KCL equations are linearly independent.

and where $\mathbf{A} = (a_{ik})$ is an $n \times b$ matrix defined by

$$a_{ik} = \begin{cases} 1 & \text{if branch } k \textit{ leaves} \text{ node } (i) \\ -1 & \text{if branch } k \textit{ enters} \text{ node } (i) \\ 0 & \text{if branch } k \text{ is } \textit{not incident with} \text{ node } (i) \end{cases} \qquad (2.3)$$

$\mathbf{Aj}$ is therefore a vector of dimension n. This assertion is obvious since when we write that the ith component of the vector $\mathbf{Aj}$ is equal to zero, we merely assert that the sum of all branch currents leaving node ⓘ is zero.

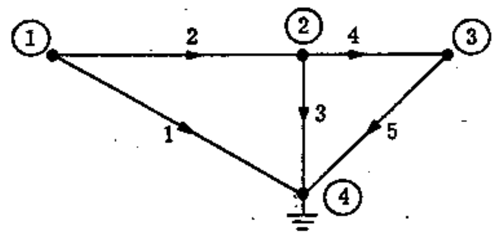
It is immediately observed that the rule expressed by Eq. (2.3) is identical with the rule specifying the elements of the node-to-branch incidence matrix $\mathbf{A}_a$ defined in the previous chapter. The only difference is that $\mathbf{A}_a$ has $n_t = n + 1$ rows. Obviously, $\mathbf{A}$ *is obtainable from* $\mathbf{A}_a$ *by deleting the row corresponding to the datum node.* $\mathbf{A}$ is therefore called the **reduced incidence matrix.**

Remark The fact that $\mathbf{Aj} = \mathbf{0}$ is a set of n linearly independent equations in the variables $j_1, j_2, \ldots, j_b$ implies that the $n \times b$ matrix $\mathbf{A}$ has rank n. Since we always have $b > n$, this conclusion can be restated as follows: the reduced incidence matrix $\mathbf{A}$ is of full rank.

Example 1 Consider the graph of Fig. 2.3, which contains four nodes and five branches ($n_t = 4, b = 5$). Let us number the nodes and branches, as shown on the figure, and indicate that node ④ is the datum node by the "ground" symbol used in the figure. The branch-current vector is

$$\mathbf{j} = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix}$$

The matrix $\mathbf{A}$ is obtained according to Eq. (2.3); thus,



$$
\begin{array}{r c l}
 & & \text{Node} \\
\mathbf{A} = & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} & \begin{array}{l} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \end{array} \\
 & \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{array} & \\
\text{Branch} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} &
\end{array}
$$

Thus, Eq. (2.2) states that

$$
\mathbf{Aj} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} = \mathbf{0}
$$

or

$$
\begin{aligned}
j_1 + j_2 &= 0 \\
-j_2 + j_3 + j_4 &= 0 \\
-j_4 + j_5 &= 0
\end{aligned}
$$

which are clearly the three node equations obtained by applying KCL to nodes 1, 2, and 3. In the present case it is easy to see that the three equations are linearly independent, since each one contains a variable *not* contained in any of the other equations.

Exercise 1 Verify that the 3×5 matrix A above is of full rank. (Hint: You need only exhibit a 3×3 submatrix whose determinant is nonzero.)

Exercise 2 Determine the incidence matrix $\mathbf{A}_a$ of the graph in Fig. 2.3. Observe that A is obtained from $\mathbf{A}_a$ by deleting the fourth row.

2.2 Implications of KVL

Let us call $e_1, e_2, \ldots, e_n$ the node voltages of nodes 1, 2, ..., n measured with respect to the datum node. The voltages $e_1, e_2, \ldots, e_n$ are called the **node-to-datum voltages.** We are going to use these node-to-datum voltages as variables in node analysis. KVL guarantees that the node-to-datum voltages are defined unambiguously; if we calculate the voltage of any node with respect to that of the datum node by forming an algebraic sum of branch voltages along a path from the datum to the node in question. KVL guarantees that the sum will be independent of the path chosen. Indeed, suppose that a first path from the datum to node k would give e_k as the node-to-datum voltage and that a second path would give $e_k' \neq e_k$. This situation contradicts KVL. Consider the loop formed by the first path followed by the second path; KVL requires that the sum of the branch voltages be zero, hence $e_k' = e_k$.

A somewhat roundabout but effective way of expressing the KVL constraints on the branch voltages consists in expressing the b branch voltages in terms of the n node voltages. Since for any networks $b > n$, the b branch voltages cannot be chosen arbitrarily, and they have *only* n degrees of freedom. Indeed, observe that the node-to-datum voltages $e_1, e_2, \ldots, e_n$ are linearly independent as far as the KVL is concerned; this is immediate since the node-to-datum voltages form no loops. Let $\mathbf{e}$ be the vector whose components are $e_1, e_2, \ldots, e_n$. We are going to show that the branch voltages are obtained from the node voltages by the equation

$$\mathbf{v} = \mathbf{A}^T\mathbf{e} \tag{2.4}$$

where $\mathbf{A}^T$ is the $b \times n$ matrix which is the transpose of the reduced incidence matrix A defined in Eq. (2.3).

To show this, it is necessary to consider two kinds of branches, namely, those branches which are incident with the datum node and those which are not. For branches which are incident with the datum node, the branch voltage is equal either to a node-to-datum voltage or its negative. For branches which are not incident with the datum node, the branch voltage must form a loop with two node-to-datum voltages, and hence it can be expressed as a linear combination of the two node-to-datum voltages by KVL. Therefore, in both cases the branch voltages depend linearly on the node-to-datum voltages. To show that the relation is that of Eq. (2.4), let us examine in detail the sign convention. Recall that v_k is the kth branch voltage, $k = 1, 2, \ldots, b$, and e_i is the node-to-datum voltage of node i, $i = 1, 2, \ldots, n$. Thus if branch k connects the ith node to the datum node, we have

$$
v_k = \begin{cases} e_i & \text{if branch } k \text{ } \textit{leaves} \text{ node } i \\ -e_i & \text{if branch } k \text{ } \textit{enters} \text{ node } i \end{cases}
$$

On the other hand, if branch k leaves node i and enters node j, then we have as is easily seen from Fig. 2.4

$$v_k = e_i - e_j$$

Since in all cases v_k can be expressed as a linear combination of the voltages $e_1, e_2, \ldots, e_n$, we may write

$$
\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_b \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \cdots c_{1n} \\ c_{21} & c_{22} \cdots c_{2n} \\ \cdots & \cdots \\ c_{b1} & c_{b2} \cdots c_{bn} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}
$$

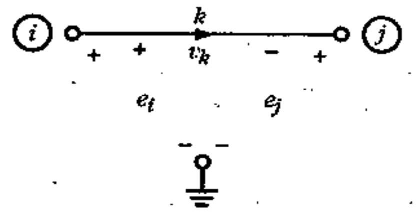


Fig. 2.4 Calculation of the branch voltage v_k in terms of the node voltages e_i and e_j; $v_k = e_i - e_j$.

where the c_{ki}'s are 0, 1, or −1 according to the rules above. A little thought will show that

$$c_{ki} = \begin{cases} 1 & \text{if branch } k \textit{ leaves} \text{ node } (i) \\ -1 & \text{if branch } k \textit{ enters} \text{ node } (i) \\ 0 & \text{if branch } k \text{ is } \textit{not incident with} \text{ node } (i) \end{cases} \tag{2.5}$$

A comparison of Eq. (2.5) with Eq. (2.3) shows immediately that $c_{ki} = a_{ik}$ for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, b$. Therefore, the matrix **C** (whose elements are the c_{ki}'s) is in fact the transpose of the reduced incidence matrix **A**. Hence Eq. (2.4) is established.

Example 2 For the circuit in Fig. 2.3,

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} \qquad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

According to Eq. (2.5), we have

$$\mathbf{A}^T = \begin{array}{c} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \\ \text{Node} \ (1) \quad (2) \quad (3) \end{array} \begin{array}{l} \text{Branch} \\ \longleftarrow 1 \\ \longleftarrow 2 \\ \longleftarrow 3 \\ \longleftarrow 4 \\ \longleftarrow 5 \\ \\ \end{array}$$

Thus, Eq. (2.4) states that

$$\mathbf{v} = \mathbf{A}^T\mathbf{e} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

or

$$v_1 = e_1$$
$$v_2 = e_1 - e_2$$
$$v_3 = e_2$$
$$v_4 = e_2 - e_3$$
$$v_5 = e_3$$

These five scalar equations are easily recognized as expressions of the KVL.

Summary Equations (2.2) and (2.4) give

$$\mathbf{Aj} = \mathbf{0} \qquad \text{(KCL)} \qquad \mathbf{v} = \mathbf{A}^T\mathbf{e} \qquad \text{(KVL)}$$

and are the two basic equations of node analysis. They are obtained from the network graph and the two Kirchhoff laws, which make them independent of the nature of the elements of the network. Eq. (2.2) expresses KCL and consists of n independent linear homogeneous algebraic equations in the b branch currents $j_1, j_2, \ldots, j_b$. Equation (2.4) expresses KVL and expresses the b branch voltages $v_1, v_2, \ldots, v_b$ in terms of the n node-to-datum voltages $e_1, e_2, \ldots, e_n$.

Obviously, to solve for the n network variables $e_1, e_2, \ldots, e_n$, we need to know the branch characterization of the network, i.e., the b branch equations which relate the branch voltages **v** to branch currents **j**. Only in these branch equations does the nature of the network elements come into the analysis. Thus, the remaining problem is to combine Eqs. (2.2) and (2.4) with the branch equations and obtain n equations in n unknowns, $e_1, e_2, \ldots e_n$. This requires some elimination. For nonlinear and time-varying networks the elimination problem is usually difficult, and we shall postpone its discussion until later. However, for linear time-invariant networks the branch equations can be combined with Eqs. (2.2) and (2.4), and the elimination can easily be performed. We shall therefore treat exclusively the linear time-invariant networks in Sec. 3.

2.3 Tellegen's Theorem Revisited

As an application of the fundamental equations (2.2) and (2.4), let us use them to give a short proof of Tellegen's theorem. Let $v_1, v_2, \ldots, v_b$ be a set of b *arbitrarily chosen* branch voltages that satisfy all the constraints imposed by KVL. From these v_k's, we can uniquely define node-to-datum voltages $e_1, e_2, \ldots, e_n$, and we have [from (2.4)]

$$\mathbf{v} = \mathbf{A}^T\mathbf{e}$$

Let $j_1, j_2, \ldots, j_b$ be a set of b *arbitrarily chosen* branch currents that satisfy all the constraints imposed by KCL. Since we use *associated* reference directions for these currents to those of the v_k's, KCL requires that [from (2.2)]

$$\mathbf{Aj} = \mathbf{0}$$

Now we obtain successively

$$\begin{aligned}\sum_{k=1}^{b} v_k j_k &= \mathbf{v}^T\mathbf{j}\\ &= (\mathbf{A}^T\mathbf{e})^T\mathbf{j}\\ &= e^T(\mathbf{A}^T)^T j\\ &= \mathbf{e}^T\mathbf{Aj}\end{aligned}$$

Hence, by (2.2)

$$\mathbf{v}^T\mathbf{j} = 0 \tag{2.6}$$

Thus, we have shown that $\sum_{k=1}^{b} v_k j_k = 0$. This is the conclusion of the Tellegen theorem for our arbitrary network.

Let us draw some further conclusions from (2.2), (2.4), and (2.6). Consider $\mathbf{j}$ and $\mathbf{v}$ as vectors in the *same* b-dimensional linear space R^b. From (2.2) it follows that *the set of all branch-current vectors that satisfy KCL form a linear space:* call it $\mathcal{V}_I$. (See Appendix A for the definition of linear space.) To prove this, observe that if $\mathbf{j}_1$ is such that $\mathbf{Aj}_1 = \mathbf{0}$, then $\mathbf{A}(\alpha\mathbf{j}_1) = \alpha\mathbf{Aj}_1 = \mathbf{0}$ for all real numbers α; $\mathbf{Aj}_1 = \mathbf{0}$ and $\mathbf{Aj}_2 = \mathbf{0}$ imply that $\mathbf{Aj}_1 + \mathbf{Aj}_2 = \mathbf{A}(\mathbf{j}_1 + \mathbf{j}_2) = \mathbf{0}$.

It can similarly be shown that *the set of all branch voltage vectors* $\mathbf{v}$ *that satisfy KVL form a linear space; let us call the space* $\mathcal{V}_V$.

Tellegen's theorem may be interpreted to mean that any vector in $\mathcal{V}_I$ is orthogonal to every vector of $\mathcal{V}_V$. In other words, *the subspaces* $\mathcal{V}_I$ *and* $\mathcal{V}_V$ *are orthogonal subspaces of* R^b.

We now show that *the direct sum of the orthogonal subspaces* $\mathcal{V}_I$ *and* $\mathcal{V}_V$ *is* R^b *itself.* In other words, *any* vector in R^b, say $\mathbf{x}$, can be written *uniquely* as the sum of a vector in $\mathcal{V}_V$, say, $\mathbf{v}$, and a vector in $\mathcal{V}_I$, say, $-\mathbf{j}$.

To prove this, consider the graph specified by the reduced incidence

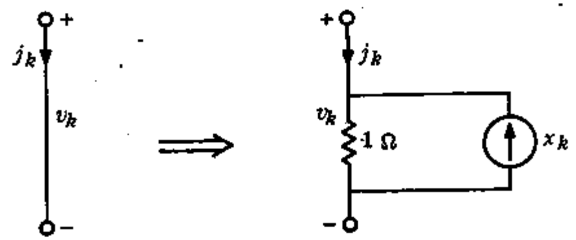


Fig. 2.5 Branch k is replaced by a 1-ohm resistor and a constant current source x_k.

matrix **A**. For $k = 1, 2, \ldots, b$, replace branch k by a 1-ohm resistor in parallel with a current source of x_k amp (here $\mathbf{x} = x_1, x_2, \ldots, x_b$ represents an arbitrary vector in R^b); this replacement is illustrated in Fig. 2.5. Call $\mathfrak{N}$ the network resulting from the replacement. As we shall prove later (remark 2, Sec. 3.2), this resistive network $\mathfrak{N}$ has a unique solution whatever the values of the current sources $x_1, x_2, \ldots x_b$. Note that the branch equations read

$$\mathbf{v} = \mathbf{j} + \mathbf{x}$$

In other words, we have shown that *any* vector $\mathbf{x}$ in R^b can be written in a *unique* way as the sum of a vector in $\mathcal{V}_V$ and a vector in $\mathcal{V}_I$. Hence, the direct sum of $\mathcal{V}_V$ and $\mathcal{V}_I$ is R^b.

Remark The subspaces $\mathcal{V}_I$ and $\mathcal{V}_V$ depend only on the graph. They are completely determined by the incidence matrix, and consequently they are independent of the nature of the branches and waveforms of the sources.

3 Node Analysis of Linear Time-invariant Networks

In linear time-invariant networks all elements except the independent sources are linear and time-invariant. We have studied in detail, but separately, the branch equations of linear time-invariant resistors, capacitors, inductors, coupled inductors, ideal transformers, and controlled sources. The problem of general node analysis is to combine these branch equations with the two basic equations

$$\mathbf{Aj} = \mathbf{0} \qquad \text{(KCL)} \tag{3.1}$$

and

$$\mathbf{v} = \mathbf{A}^T\mathbf{e} \qquad \text{(KVL)} \tag{3.2}$$

The resulting equations will, in general, take the form of *linear* simultaneous differential equations or integrodifferential equations with n network variables $e_1, e_2, \ldots, e_n$. The purpose of this section is to study the formulation of these equations and to develop some important properties of the resulting equations. For simplicity we consider first the case in which only resistors and independent sources are allowed in the network. In this case the resulting equations will be linear *algebraic* equations. We next consider the sinusoidal steady-state analysis of networks using phasors and impedances. Finally, we consider the formulation of general differential and integrodifferential equations.

3.1 Analysis of Resistive Networks

Consider a linear time-invariant resistive network with b branches, n_t nodes, and one separate part. A typical branch is shown in Fig. 3.1. Note that it includes independent sources. The branch equations are of the form

(3.3a) $v_k = R_k j_k + v_{sk} - R_k j_{sk} \qquad k = 1, 2, \ldots, b$

or, equivalently,

(3.3b) $j_k = G_k v_k + j_{sk} - G_k v_{sk} \qquad k = 1, 2, \ldots, b$

In matrix notation, we have from Eq. (3.3*b*)

(3.4) $\mathbf{j} = \mathbf{G}\mathbf{v} + \mathbf{j}_s - \mathbf{G}\mathbf{v}_s$

where $\mathbf{G}$ is called the **branch conductance matrix.** It is a *diagonal* matrix of order b; that is,

$$\mathbf{G} = \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & G_b \end{bmatrix}$$

The vectors $\mathbf{j}_s$ and $\mathbf{v}_s$ are source vectors of dimension b; that is,

$$\mathbf{j}_s = \begin{bmatrix} j_{s1} \\ j_{s2} \\ \vdots \\ j_{sb} \end{bmatrix} \qquad \mathbf{v}_s = \begin{bmatrix} v_{s1} \\ v_{s2} \\ \vdots \\ v_{sb} \end{bmatrix}$$

It is only necessary to combine Eqs. (3.1) (3.2), and (3.4) to eliminate the branch variables and obtain a vector equation in terms of the vector network variable $\mathbf{e}$. Premultiplying Eq. (3.4) by the matrix $\mathbf{A}$, substituting $\mathbf{v}$ by $\mathbf{A}^T\mathbf{e}$, and using Eq. (3.1), we obtain

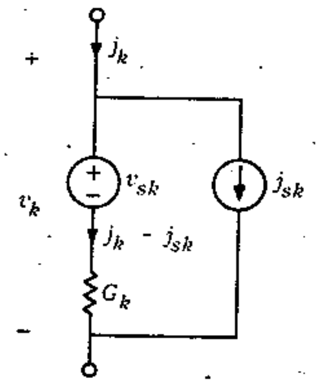


Fig. 3.1 The kth branch.

(3.5) $\mathbf{A}\mathbf{G}\mathbf{A}^T\mathbf{e} + \mathbf{A}\mathbf{j}_s - \mathbf{A}\mathbf{G}\mathbf{v}_s = \mathbf{0}$

or

(3.6) $\mathbf{A}\mathbf{G}\mathbf{A}^T\mathbf{e} = \mathbf{A}\mathbf{G}\mathbf{v}_s - \mathbf{A}\mathbf{j}_s$

In Eq. (3.6) $\mathbf{A}\mathbf{G}\mathbf{A}^T$ is an $n \times n$ square matrix, whereas $\mathbf{A}\mathbf{G}\mathbf{v}_s$ and $-\mathbf{A}\mathbf{j}_s$ are n-dimensional vectors. Let us introduce the following notations:

(3.7a) $\mathbf{Y}_n \triangleq \mathbf{A}\mathbf{G}\mathbf{A}^T$

(3.7b) $\mathbf{i}_s \triangleq \mathbf{A}\mathbf{G}\mathbf{v}_s - \mathbf{A}\mathbf{j}_s$

then Eq. (3.6) becomes

(3.8) $\mathbf{Y}_n\mathbf{e} = \mathbf{i}_s$

The set of equations (3.8) is usually called the **node equations; $\mathbf{Y}_n$** is called the **node admittance matrix,**† and $\mathbf{i}_s$ is the **node current source vector.**

The node equations (3.8) are very important: they deserve careful examination. First, observe that since the graph specifies the reduced incidence matrix $\mathbf{A}$ and since the branch conductances specify the branch admittance matrix $\mathbf{G}$, the node admittance matrix $\mathbf{Y}_n$ is a known matrix: indeed, $\mathbf{Y}_n \triangleq \mathbf{A}\mathbf{G}\mathbf{A}^T$.

Similarly, the vectors $\mathbf{v}_s$ and $\mathbf{j}_s$, which specify the sources in the branches, are given: therefore, the node current source vector $\mathbf{i}_s$ is also

† We call $\mathbf{Y}_n$ the node *admittance* matrix rather than the node *conductance* matrix even though we are dealing with a purely resistive network. It will be seen that in sinusoidal steady-state analysis we have exactly the same formulation; hence it is more convenient to introduce the more general term *admittance*.

known by (3.7*b*). Thus, Eq. (3.8) relates the unknown n-vector $\mathbf{e}$ to the known $n \times n$ matrix $\mathbf{Y}_n$ and the known n-vector $\mathbf{i}_s$. The vector equation (3.8) consists of a system of n *linear algebraic* equations in the n unknown node-to-datum voltages $e_1, e_2, \ldots, e_n$. Once $\mathbf{e}$ is found, it is a simple matter to find the b branch voltages $\mathbf{v}$ and the b branch currents $\mathbf{j}$. Indeed, (3.2) gives $\mathbf{v} = \mathbf{A}^T\mathbf{e}$, and having $\mathbf{v}$, we obtain $\mathbf{j}$ by the branch equation (3.4); that is, $\mathbf{j} = \mathbf{G}\mathbf{v} + \mathbf{j}_s - \mathbf{G}\mathbf{v}_s$.

Example 1 Let us consider the circuit in Fig. 3.2, where all element values are given. The graph of the circuit is the same one illustrated in Fig. 2.3. Let us give the detailed procedure for writing the equations and solving the branch variables.

Step 1 Pick a datum node, say ④, and label the remaining nodes ①, ②, and ③. Call e_1, e_2, and e_3 the voltages of nodes ①, ②, and ③, respectively, with respect to the datum node.

Step 2 Number the branches 1, 2, 3, 4, and 5, and assign each one a reference direction. The variable G_i is the conductance of branch numbered i.

Step 3 Write the three linearly independent equations expressing KCL for nodes ①, ②, and ③; thus,

$$\mathbf{A}\mathbf{j} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.9}$$

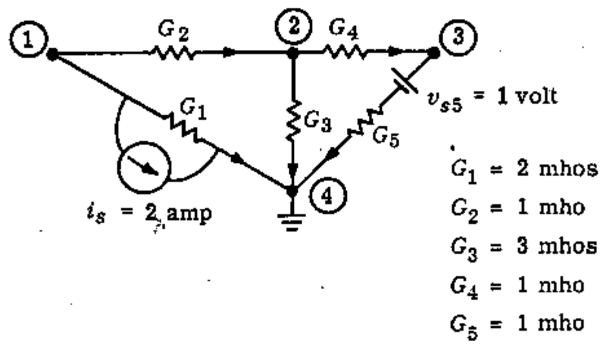


Fig. 3.2 Network for Example 1.

Note that the reduced incidence matrix $\mathbf{A}$ is the same as in Example 1 of Sec. 2.1.

Step 4 Use KVL to express the branch voltage v_k in terms of the node voltages e_i; thus,

$$\mathbf{v} = \mathbf{A}^T\mathbf{e} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{3.10}$$

Step 5 Write the branch equations in the form

$$\mathbf{j} = \mathbf{G}\mathbf{v} + \mathbf{j}_s - \mathbf{G}\mathbf{v}_s \tag{3.11a}$$

Thus,

$$\begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.11b}$$

For example, the fifth scalar equation of (3.11*b*) reads

$$j_5 = v_5 - 1$$

Step 6 Substitute into (3.11) the expression for $\mathbf{v}$ given by (3.10), and multiply the result on the left by the matrix $\mathbf{A}$; according to (3.9), the result is $\mathbf{0}$. After reordering terms, we can put the answer in the form

$$\mathbf{Y}_n\mathbf{e} = \mathbf{i}_s \tag{3.12a}$$

where

$$\mathbf{Y}_n \triangleq \mathbf{A}\mathbf{G}\mathbf{A}^T$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{3.12b}$$

and

$$\mathbf{i}_s \stackrel{\Delta}{=} \mathbf{AGv}_s - \mathbf{Aj}_s = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \tag{3.12c}$$

Thus, the node equation

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \tag{3.12d}$$

Step 7 Solve Eq. (3.12*d*) for **e**. The numerical solution of such equations is done by the Gauss elimination method whenever $n > 5$. Formally, we may express the answer in terms of the inverse matrix $\mathbf{Y}_n^{-1}$; thus,

$$\mathbf{e} = \mathbf{Y}_n^{-1}\mathbf{i}_s = \frac{1}{25}\begin{bmatrix} 9 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 14 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -17 \\ -1 \\ 12 \end{bmatrix} \tag{3.13}$$

Once the node voltages **e** are found, the branch voltages **v** are obtained from (3.10) as follows:

$$\mathbf{v} = \mathbf{A}^T\mathbf{e} = \frac{1}{25}\begin{bmatrix} -17 \\ -16 \\ -1 \\ -13 \\ 12 \end{bmatrix} \tag{3.14}$$

The next step is to use **v** to obtain the branch currents **j** by Eq. (3.11); then we have

$$\mathbf{j} = \mathbf{Gv} + \mathbf{j}_s - \mathbf{Gv}_s = \frac{1}{25}\begin{bmatrix} 16 \\ -16 \\ -3 \\ -13 \\ -13 \end{bmatrix} \tag{3.15}$$

This completes the analysis of the network shown in Fig. 3.2; that is, all branch voltages and currents have been determined.

3.2 Writing Node Equations by Inspection

The step-by-step procedure detailed above is very important for two reasons. First, it exhibits quite clearly the various facts that have to be used to analyze the network, and second, it is completely general in the sense that it works in all cases and is therefore suitable for automatic computation.

In the case of networks made only of resistors and *independent* sources (in particular, with no coupling elements such as dependent sources), the node equations can be written by inspection. Let us call y_{ik} the (i,k) element of the node admittance matrix $\mathbf{Y}_n$; then the vector equation

$$\mathbf{Y}_n\mathbf{e} = \mathbf{i}_s \tag{3.16a}$$

written in scalar form becomes

$$\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} i_{s1} \\ i_{s2} \\ \vdots \\ i_{sn} \end{bmatrix} \tag{3.16b}$$

The following statements are easily verified in simple examples, and can be proved for networks without coupling elements.

1. *y_{ii} is the sum of the conductances of all branches connected to node (i); y_{ii} is called the* ***self-admittance*** *of node (i).*
2. *y_{ik} is the negative of the sum of the conductances of all branches connecting node (i) and node (k); y_{ik} is called the* ***mutual admittance between node (i) and node (k)***.
3. *If we convert all voltage sources into current sources, then i_{sk} is the algebraic sum of all source currents entering node (k).*

Exercise Prove statements 1 and 2 above. Hint: $\mathbf{Y}_n = \mathbf{AGA}^T$; consequently,

$$y_{ii} = \sum_{j=1}^{b} a_{ij}G_ja_{ij} = \sum_{j=1}^{b} (a_{ij})^2G_j$$

and

$$y_{ik} = \sum_{j=1}^{b} a_{ij}G_ja_{kj}$$

Note that $(a_{ij})^2$ can only be zero or 1; similarly, $a_{ij}a_{kj}$ can only be zero or -1.

Example 1 (continued) Consider again the network of Fig. 3.2. Let us, by inspection, write every branch current in terms of the node voltages; thus.

$$j_1 = G_1 e_1 + 2$$
$$j_2 = G_2(e_1 - e_2)$$
$$j_3 = G_3 e_2$$
$$j_4 = G_4(e_2 - e_3)$$
$$j_5 = G_5 e_3 - 1$$

In the last equation, we used the equivalent current source for branch 5, as shown in Fig. 3.3. Substituting the above into the node equations, we obtain

$$(G_1 + G_2)e_1 - G_2 e_2 = -1$$
$$-G_2 e_1 + (G_2 + G_3 + G_4)e_2 - G_4 e_3 = 0$$
$$-G_4 e_3 + (G_4 + G_5)e_3 = 1$$

By inspection, it is easily seen that statements 1, 2, and 3 above hold for the present case. Also if the numerical values for the G_k's are substituted, the answer checks with (3.12*a*).

Remarks

1. For networks made of resistors and independent sources, the node admittance matrix $\mathbf{Y}_n = (y_{ik})$ in Eq. (3.8) is a *symmetric matrix;* i.e., $y_{ik} = y_{ki}$ for $i,\ k = 1, 2, \ldots, n$. Indeed, since $\mathbf{Y}_n \triangleq \mathbf{AGA}^T$, then $\mathbf{Y}_n{}^T = (\mathbf{AGA}^T)^T = \mathbf{AG}^T\mathbf{A}^T = \mathbf{AGA}^T = \mathbf{Y}_n$. In the last step we used the fact that $\mathbf{G}^T = \mathbf{G}$ because $\mathbf{G}$, the branch conductance matrix of a resistive network with *no coupling elements,* is a diagonal matrix.
2. If all the conductances of a linear resistive network are *positive,* it is easy to show that $\det(\mathbf{Y}_n) > 0$.† Cramer's rule then guarantees that whatever $\mathbf{i}_s$ may be, Eq. (3.16) has a *unique solution.* The fact that $\det(\mathbf{Y}_n) > 0$ also follows from $\mathbf{Y}_n = \mathbf{AGA}^T$, where $\mathbf{G}$ is an $n \times n$ diagonal matrix with *positive* elements and $\mathbf{A}$ is an $n \times b$ matrix with

† See Sec. 2.4, Appendix B.

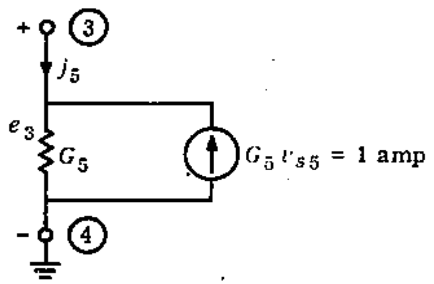


Fig. 3.3 Branch 5 of Fig. 3.2 in terms of current source.

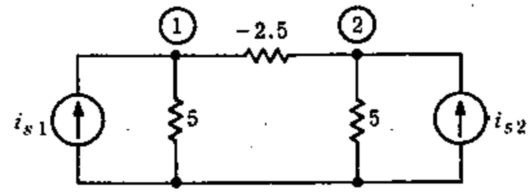


Fig. 3.4 A resistive network with a negative conductance. The conductances are given in mhos.

rank n. Note that the assumption that *all* conductances are positive is crucial. For example, consider the circuit in Fig. 3.4, where one of the conductances is negative. The node admittance matrix is singular (i.e., its determinant is zero). Indeed, the node equations are

$$\begin{bmatrix} 2.5 & 2.5 \\ 2.5 & 2.5 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} i_{s1} \\ i_{s2} \end{bmatrix}$$

and, for example, with $i_{s1} = i_{s2} = 5$, $e_1 = 1 - \alpha$ and $e_2 = 1 + \alpha$, solutions whatever α may be! Thus, this network has an infinite number of solutions!

Exercise Assume that all elements of the graph of Fig. 2.3 have conductances of 5 mhos and that a current source squirts 1 amp into node 1 and sucks it out of node 2. Write the node equations by inspection.

3.3 Sinusoidal Steady-state Analysis

Consider now a linear time-invariant network containing resistors, inductors, capacitors, and independent sources; such networks are usually referred to as *RLC* networks. Suppose that we are only interested in the sinusoidal steady-state analysis. Let all the independent sources be sinusoidal at the *same* angular frequency ω, and suppose that the branch currents and voltages have reached the steady state. Consequently, we shall use voltage phasors, current phasors, impedances, and admittances, to write the branch equations and Kirchhoff's laws. Again we assume that each branch includes a voltage source and a current source in addition to its nonsource element.

Thus, a typical branch contains an admittance, say Y_k (in the kth branch), which is one of the forms G_k, $j\omega C_k$, or $1/j\omega L_k$, depending on whether the kth branch is a resistor, capacitor, or inductor, respectively. The branch equation is

$$J_k = Y_k V_k + J_{sk} - Y_k V_{sk} \qquad k = 1, 2, \ldots, b \tag{3.17}$$

where J_k and V_k are the kth branch current phasor and voltage phasor.

and J_{sk} and V_{sk} are the kth branch phasors representing the current and voltage sources of branch k. In matrix form Eq. (3.17) can be written as

$$\mathbf{J} = \mathbf{Y}_b\mathbf{V} + \mathbf{J}_s - \mathbf{Y}_b\mathbf{V}_s \tag{3.18}$$

The matrix $\mathbf{Y}_b$ is called the **branch admittance matrix,** and the vectors **J** and **V** are, respectively, the branch-current phasor vector and the branch-voltage phasor vector. The analysis is exactly the same as that of the resistive network in the preceding section. The node equation is of the form

$$\mathbf{Y}_n\mathbf{E} = \mathbf{I}_s \tag{3.19}$$

where the phasor **E** represents the node-to-datum voltage vector, the phasor $\mathbf{I}_s$ represents the current-source vector, and $\mathbf{Y}_n$ is the node admittance matrix. In terms of **A** and $\mathbf{Y}_b$, $\mathbf{Y}_n$ is given by

$$\mathbf{Y}_n = \mathbf{A}\mathbf{Y}_b\mathbf{A}^T \tag{3.20a}$$

$$\mathbf{I}_s = \mathbf{A}\mathbf{Y}_b\mathbf{V}_s - \mathbf{A}\mathbf{J}_s \tag{3.20b}$$

Remark Consider on the one hand the steady-state analysis of an *RLC* network driven by sinusoidal sources having the same frequency, and, on the other hand, the analysis of a resistive network. In both cases node analysis leads to a set of linear algebraic equations in the node voltages. In the sinusoidal steady-state case the unknowns are node-to-datum voltage *phasors,* and the coefficients of the equations are complex numbers which depend on the frequency. Finally, recall that the node-to-datum voltages are obtained from the phasors by

$$e_k(t) = \operatorname{Re}(E_k\, e^{j\omega t}) \qquad k = 1, 2, \ldots, n$$

In the resistive network case, the equations had real numbers as coefficients and their solution gave the node-to-datum voltages directly.

If we had only networks without coupling elements, the inspection method of the preceding section would suffice. We are going to tackle an example that has both *dependent* sources and *mutual inductances.* It will become apparent that for this case the inspection method does not suffice, and the value of our systematic procedure will become apparent. After the example, we shall sketch out the general procedure.

Example Consider the linear time-invariant network shown in Fig. 3.5. The independent current source is sinusoidal and is represented by the phasor I: its waveform is $|I| \cos(\omega t + \measuredangle I)$. We assume that the network is in the sinusoidal steady state, consequently, all waveforms will be represented by phasors **V, J, E,** etc. Note the presence of two dependent sources. The three inductors L_3, L_4, and L_5 are magnetically coupled, and their inductance matrix is

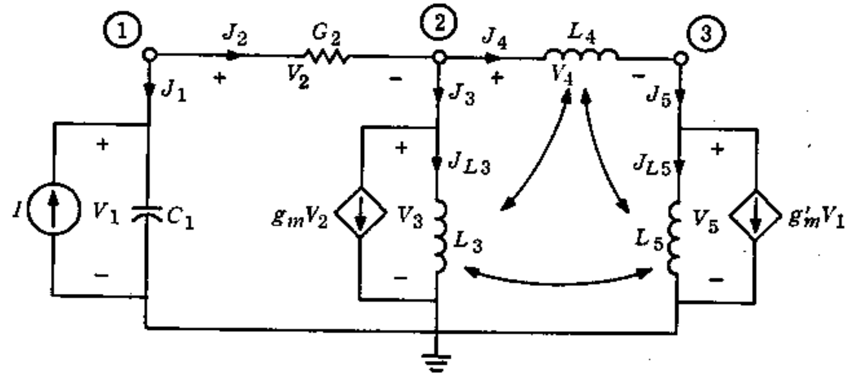


Fig. 3.5 Example of sinusoidal steady-state analysis when mutual magnetic coupling and dependent sources are present.

$$\mathbf{L} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & \frac{5}{3} & -\frac{4}{3} \\ 1 & -\frac{4}{3} & \frac{5}{3} \end{bmatrix}$$

First, to express Kirchhoff's laws, we need the node incidence matrix **A**; by inspection

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{3.21}$$

Second, we need the branch equations. We write them in terms of phasors; thus,

$$\begin{aligned} J_1 &= j\omega C_1 V_1 - I \\ J_2 &= GV_2 \end{aligned} \tag{3.22a}$$

To write J_3, J_4, and J_5 in terms of the V_k's is not that simple; indeed the inductance matrix tells us only the relation between V_3, V_4, and V_5 and the *inductor currents* J_{L3}, J_4, and J_{L5} (see Fig. 3.5 for the definition of J_{L3} and J_{L5}). Therefore,

$$\mathbf{V}' = j\omega \mathbf{L}\mathbf{J}_L$$

or

$$\begin{bmatrix} V_3 \\ V_4 \\ V_5 \end{bmatrix} = j\omega \begin{bmatrix} 1 & -1 & 1 \\ -1 & \frac{5}{3} & -\frac{4}{3} \\ 1 & -\frac{4}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} J_{L3} \\ J_4 \\ J_{L5} \end{bmatrix}$$

Clearly, for our purposes we need the currents as a function of the voltages. Hence, we invert the inductance matrix and get an expression of the form

$$\mathbf{J}_L = \frac{1}{j\omega}\mathbf{L}^{-1}\mathbf{V}'$$

or

$$\begin{bmatrix} J_{L3} \\ J_4 \\ J_{L5} \end{bmatrix} = \frac{1}{j\omega}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} V_3 \\ V_4 \\ V_5 \end{bmatrix}$$

Having obtained these relations and noting that $J_3 = J_{L3} + g_m V_2$ and $J_5 = J_{L5} + g'_m V_1$, we can write our last three branch equations as follows:

(3.22b)
$$J_3 = g_m V_2 + \frac{3}{j\omega} V_3 + \frac{1}{j\omega} V_4 - \frac{1}{j\omega} V_5$$
$$J_4 = \frac{1}{j\omega} V_3 + \frac{2}{j\omega} V_4 + \frac{1}{j\omega} V_5$$
$$J_5 = g'_m V_1 - \frac{1}{j\omega} V_3 + \frac{1}{j\omega} V_4 + \frac{2}{j\omega} V_5$$

The branch equations (3.22a) and (3.22b) constitute a system of five equations of the form, as in (3.18).

(3.23) $\mathbf{J} = \mathbf{Y}_b\mathbf{V} + \mathbf{J}_s$

Note that the matrix $\mathbf{Y}_b$ has complex numbers as elements, is no longer diagonal (because of both mutual coupling and dependent sources), and is no longer symmetric (because of dependent sources). Let us calculate the node admittance matrix $\mathbf{Y}_n \triangleq \mathbf{A}\mathbf{Y}_b\mathbf{A}^T$ as follows:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} j\omega C_1 & 0 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 & 0 \\ 0 & g_m & \frac{3}{j\omega} & \frac{1}{j\omega} & -\frac{1}{j\omega} \\ 0 & 0 & \frac{1}{j\omega} & \frac{2}{j\omega} & \frac{1}{j\omega} \\ g'_m & 0 & -\frac{1}{j\omega} & \frac{1}{j\omega} & \frac{2}{j\omega} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} j\omega C_1 & G_2 & 0 & 0 & 0 \\ 0 & -G_2 + g_m & \frac{4}{j\omega} & \frac{3}{j\omega} & 0 \\ g'_m & 0 & \frac{-2}{j\omega} & \frac{-1}{j\omega} & \frac{1}{j\omega} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} j\omega C_1 + G_2 & -G_2 & 0 \\ -G_2 + g_m & G_2 - g_m + \frac{7}{j\omega} & \frac{-3}{j\omega} \\ g'_m & \frac{-3}{j\omega} & \frac{2}{j\omega} \end{bmatrix} = \mathbf{Y}_n$$

The right-hand term of the node equation, from (3.19) and (3.20b), is $-\mathbf{A}\mathbf{J}_s$ since $\mathbf{V}_s$ is identically zero. It is easily found to be a vector whose first component is I (all the others are zero). Thus, the node equation is

(3.24)
$$\begin{bmatrix} j\omega C_1 + G_2 & -G_2 & 0 \\ -G_2 + g_m & G_2 - g_m + \frac{7}{j\omega} & \frac{-3}{j\omega} \\ g'_m & \frac{-3}{j\omega} & \frac{2}{j\omega} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}$$

After substituting numerical values, we can solve these three equations for the phasors E_1, E_2, and E_3. Successively, we get $\mathbf{V}$ by

$\mathbf{V} = \mathbf{A}^T\mathbf{E}$

and $\mathbf{J}$ by (3.23).

Systematic procedure

This example exhibits clearly the value of the systematic method; obviously, it is hazardous to try to write Eq. (3.24) by inspection.

The detailed example above suggests a method for writing the sinusoidal steady-state equations of *any* linear time-invariant network driven by sinusoidal sources of the *same* frequency. Note that the network may include R's, L's, C's, mutual inductances, dependent sources, and independent sources. The steps are as follows:

Step 1 Perform (if needed) source transformations as indicated in Sec. 1.

Step 2 Write the requirements of Kirchhoff's laws; thus,

(3.25a) $\mathbf{A}\mathbf{J} = \mathbf{0}$

(3.25*b*) $\mathbf{V} = \mathbf{A}^T\mathbf{E}$

Step 3 Write the branch equations [from (3.18)]

$$\mathbf{J} = \mathbf{Y}_b(j\omega)\,\mathbf{V} - \mathbf{Y}_b(j\omega)\,\mathbf{V}_s + \mathbf{J}_s$$

where $\mathbf{Y}_b(j\omega)$ is the branch admittance matrix. Note that $\mathbf{Y}_b$ is evaluated at $j\omega$, where ω represents the angular frequency of the sinusoidal sources.

Step 4 Perform the substitution to obtain the node equations labeled (3.19)

$$\mathbf{Y}_n(j\omega)\mathbf{E} = \mathbf{I}_s$$

where [from (3.20*a*) and (3.20*b*)]

$$\mathbf{Y}_n(j\omega) \triangleq \mathbf{A}\mathbf{Y}_b(j\omega)\mathbf{A}^T$$
$$\mathbf{I}_s \triangleq \mathbf{A}\mathbf{Y}_b(j\omega)\,\mathbf{V}_s - \mathbf{A}\mathbf{J}_s$$

Step 5 Solve the node equations (3.19) for the phasor **E**.

Step 6 Obtain **V** by (3.25*b*) and **J** by (3.18).

Properties of the node admittance matrix $\mathbf{Y}_n(j\omega)$

From the basic equation

$$\mathbf{Y}_n(j\omega) = \mathbf{A}\mathbf{Y}_b(j\omega)\mathbf{A}^T$$

we obtain the following useful properties:

1. Whenever there are no coupling elements (i.e., neither mutual inductances nor *dependent* sources), the $b \times b$ matrix $\mathbf{Y}_b(j\omega)$ is diagonal, and consequently the $n \times n$ matrix $\mathbf{Y}_n(j\omega)$ is *symmetric.*
2. Whenever there are no *dependent* sources (mutual inductances are allowed), both $\mathbf{Y}_b(j\omega)$ and $\mathbf{Y}_n(j\omega)$ are *symmetric.*

3.4 Integrodifferential Equations

In general, node analysis of linear networks leads to a set of simultaneous integrodifferential equations, i.e., equations involving unknown functions, say $e_1, e_2, \ldots,$ some of their derivatives $\dot{e}_1, \dot{e}_2, \ldots,$ and some of their integrals $\int_0^t e_1(t')\,dt'$, $\int_0^t e_2(t')\,dt', \ldots$. We shall present a systematic method for obtaining the node integrodifferential equations of any linear time-invariant network. These equations are necessary if we have to compute the complete response of a given network to a given input and a given initial state. The method is perfectly general, but in order not to get bogged down in notation we shall present it by way of an example.

Example We are given (1) the linear time-invariant network shown in Fig. 3.6; (2) the element values G_1, G_2, C_3, and g_m, and the reciprocal inductance matrix

$$\begin{bmatrix} \Gamma_{44} & \Gamma_{45} \\ \Gamma_{45} & \Gamma_{55} \end{bmatrix}$$

(note that this matrix corresponds to the reference directions for j_{L4} and j_5 in Fig. 3.6); (3) the input waveform $j_{s1}(t)$ for $t \geq 0$; and (4) the initial values of initial capacitor voltage $v_3(0)$ and initial inductor currents $j_{L4}(0)$ and $j_5(0)$. We shall proceed in the same order as that used for the sinusoidal steady-state.

Step 1 Perform (if needed) source transformations as indicated in Sec. 1.

Step 2 Write the requirement of Kirchhoff's laws as follows:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(KCL)} \tag{3.26}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \qquad \text{(KVL)} \tag{3.27}$$

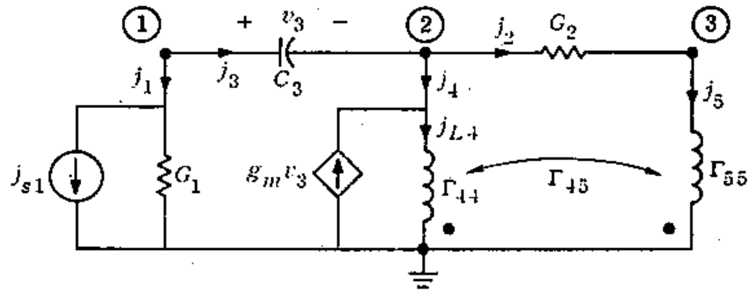


Fig. 3.6 Network for which the node integrodifferential equations are obtained.

Note that the j_k's, v_k's, and e_i's are time functions and *not* phasors. To lighten the notation, we have written $j_1, j_2, \ldots, e_1, e_2, \ldots$, etc., instead of $j_1(t), j_2(t), \ldots$, etc.

Step 3 Write the branch equations (expressing the branch currents in terms of the branch voltages); thus,

$$j_1 = G_1 v_1 + j_{s1}$$

$$j_2 = G_2 v_2$$

$$j_3 = C_3 \dot{v}_3$$

Noting that j_{L4} is the current in the fourth inductance and that $j_4 = j_{L4} - g_m v_3$, we obtain the branch equation of the inductors as follows:

$$j_4 = -g_m v_3 + \Gamma_{44} \int_0^t v_4(t')\,dt' + \Gamma_{45} \int_0^t v_5(t')\,dt' + j_{L4}(0)$$

$$j_5 = \Gamma_{45} \int_0^t v_4(t')\,dt' + \Gamma_{55} \int_0^t v_5(t')\,dt' + j_5(0)$$

It is convenient to use the notation D to denote the differentiation operator with respect to time: for example,

$$Dv_3 = \dot{v}_3 = \frac{dv_3}{dt}$$

and the notation D^{-1} to denote the *definite* integral $\int_0^t \cdot$; for example,

$$\frac{1}{D} v_4 = \int_0^t v_4(t')\,dt'$$

With these notations the branch equations can be written in matrix form as follows:

$$\begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} = \begin{bmatrix} G_1 & 0 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 & 0 \\ 0 & 0 & C_3 D & 0 & 0 \\ 0 & 0 & -g_m & \Gamma_{44}D^{-1} & \Gamma_{45}D^{-1} \\ 0 & 0 & 0 & \Gamma_{45}D^{-1} & \Gamma_{55}D^{-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} + \begin{bmatrix} j_{s1} \\ 0 \\ 0 \\ j_{L4}(0) \\ j_5(0) \end{bmatrix} \tag{3.28}$$

Note that the matrix is precisely that obtained in the sinusoidal steady state if we were to replace D by $j\omega$. Note also the contribution of the initial state, the terms $j_{L4}(0)$ and $j_5(0)$ in the right-hand side. In the following we shall not perform algebra with the D symbols among themselves: we shall only multiply and divide them by constants.†

† It is *not* legitimate to treat D and D^{-1} as algebraic symbols analogous to real or complex numbers. Whereas it is true that D and D^{-1} may be added and multiplied by real numbers (constants), it is also true that $D\,D^{-1} \neq D^{-1}D$. Indeed, apply the first operator to a function f; thus,

Step 4 Eliminate the v_k's and j_k's from the system of (3.26), (3.27), and (3.28). If we note that they have, respectively, the form

$$\mathbf{A}\mathbf{j} = \mathbf{0} \tag{3.29}$$

$$\mathbf{v} = \mathbf{A}^T\mathbf{e} \tag{3.30}$$

$$\mathbf{j} = \mathbf{Y}_b(D)\,\mathbf{v} + \mathbf{j}_s \tag{3.31}$$

the result of the elimination is of the familiar form

$$\mathbf{A}\mathbf{Y}_b(D)\,\mathbf{A}^T\mathbf{e} = -\mathbf{A}\mathbf{j}_s \tag{3.32}$$

or

$$\mathbf{Y}_n(D)\,\mathbf{e} = \mathbf{i}_s$$

Let us calculate for the example the *node admittance matrix operator;* thus,

$$\mathbf{Y}_n(D) \triangleq \mathbf{A}\mathbf{Y}_b(D)\,\mathbf{A}^T$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_1 & 0 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 & 0 \\ 0 & 0 & C_3 D & 0 & 0 \\ 0 & 0 & -g_m & \Gamma_{44}D^{-1} & \Gamma_{45}D^{-1} \\ 0 & 0 & 0 & \Gamma_{45}D^{-1} & \Gamma_{55}D^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} G_1 + C_3 D & -C_3 D & 0 \\ -C_3 D - g_m & G_2 + C_3 D + g_m + \Gamma_{44}D^{-1} & -G_2 + \Gamma_{45}D^{-1} \\ 0 & -G_2 + \Gamma_{45}D^{-1} & G_2 + \Gamma_{55}D^{-1} \end{bmatrix}$$

$$D\,D^{-1}f = \frac{d}{dt}\left[\int_0^t f(\tau)\,d\tau\right] = f(t)$$

where the last step used the fundamental theorem of the calculus. Now

$$D^{-1}Df = \int_0^t f'(\tau)\,d\tau = f(\tau)\Big|_0^t = f(t) - f(0).$$

On the other hand, D and the *positive* integral powers of D can be manipulated by the usual rules of algebra. In fact, for any *positive* integers m and n,

$$D^m D^n = D^{m+n}$$

and for any real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$

$$(\alpha_1 D^n + \beta_1)(\alpha_2 D^m + \beta_2) = \alpha_1\alpha_2 D^{m+n} + \alpha_1\beta_2 D^n + \alpha_2\beta_1 D^m + \beta_1\beta_2$$

Therefore, for the present example, node analysis gives the following integrodifferential equation:

$$
\begin{bmatrix} G_1 + C_3D & -C_3D & 0 \\ -C_3D - g_m & C_3D + G_2 + g_m + \Gamma_{44}D^{-1} & -G_2 + \Gamma_{45}D^{-1} \\ 0 & -G_2 + \Gamma_{45}D^{-1} & G_2 + \Gamma_{55}D^{-1} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -j_{s1} \\ -j_{L4}(0) \\ -j_5(0) \end{bmatrix} \tag{3.33}
$$

The required initial conditions are easily obtained; writing the cut-set law for branches 1, 4, and 5 we have

$$
e_1(0) = \frac{1}{G_1}[-j_{s1}(0) - j_{L4}(0) + g_m v_3(0) - j_5(0)] \tag{3.34}
$$

where all terms in brackets are known. Finally,

$$
e_2(0) = e_1(0) - v_3(0) \tag{3.35}
$$

and

$$
e_3(0) = e_2(0) - G_2 j_5(0) \tag{3.36}
$$

Remarks

1. Except for the initial conditions, the writing of node equations in the integrodifferential equation form is closely related to that used in the sinusoidal steady-state analysis. It is easily seen that if we replace D by $j\omega$ in $\mathbf{Y}_n(D)$, we obtain the node admittance matrix $\mathbf{Y}_n(j\omega)$ for the sinusoidal steady-state analysis. Therefore, in the absence of coupling elements, $\mathbf{Y}_n(D)$ can always be written by inspection.

2. Although the equations of (3.33) correspond to the rather formidable name of "integrodifferential equations," they are in fact no different than differential equations; it is just a matter of notation! To prove the point, introduce new variables, namely fluxes ϕ_2 and ϕ_3 *defined* by

$$
\phi_2(t) \triangleq \int_0^t e_2(t')\,dt' \qquad \phi_3(t) \triangleq \int_0^t e_3(t')\,dt' \tag{3.37}
$$

Clearly,

$$
De_2 = D^2\phi_2 \qquad e_2 = D\phi_2 \qquad D^{-1}e_2 = \phi_2
$$

The system of (3.33) becomes

$$
\begin{bmatrix} G_1 + C_3D & -C_3D^2 & 0 \\ -C_3D - g_m & C_3D^2 + (G_2 + g_m)D + \Gamma_{44} & -G_2D + \Gamma_{45} \\ 0 & -G_2D + \Gamma_{45} & G_2D + \Gamma_{55} \end{bmatrix} \begin{bmatrix} e_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} -j_{s1} \\ -j_{L4}(0) \\ -j_5(0) \end{bmatrix}
$$

and the initial conditions are

$e_1(0)$ given by (3.34)

$\phi_2(0) = 0$ see (3.37) $\qquad \dot\phi_2(0) = e_2(0)$ given by (3.35)

$\phi_3(0) = 0$ see (3.37) $\qquad \dot\phi_3(0) = e_3(0)$ given by (3.36)

3.5 Shortcut Method

When the network under study involves only a few dependent sources, the equations can be written by inspection if one uses the following idea: treat the *dependent* source as an independent source, and only in the last step express the source in terms of the appropriate variables.

Example 1 Let us write the sinusoidal steady-state equations for the network of Fig. 3.7.

Step 1 Replace the dependent sources by independent ones, and call them J_{s3} and J_{s5}.

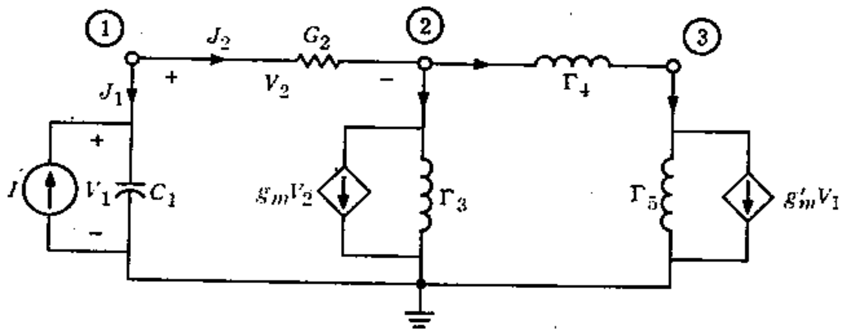


Fig. 3.7 Network including dependent sources.

Step 2 Write the equations by inspection as follows:

$$\begin{bmatrix} G_2 + j\omega C_1 & -G_2 & 0 \\ -G_2 & G_2 + \dfrac{\Gamma_3 + \Gamma_4}{j\omega} & -\dfrac{\Gamma_4}{j\omega} \\ 0 & -\dfrac{\Gamma_4}{j\omega} & \dfrac{\Gamma_4 + \Gamma_5}{j\omega} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} I \\ -J_{s3} \\ -J_{s5} \end{bmatrix}$$

Step 3 Express the dependent source waveforms in terms of the appropriate variables as follows:

$$J_{s3} = g_m V_2 = g_m(E_1 - E_2)$$

and

$$J_{s5} = g'_m V_1 = g'_m E_1$$

Step 4 Substitute and rearrange terms; thus we have

$$\begin{bmatrix} G_2 + j\omega C_1 & -G_2 & 0 \\ g_m - G_2 & G_2 - g_m + \dfrac{\Gamma_3 + \Gamma_4}{j\omega} & -\dfrac{\Gamma_4}{j\omega} \\ g'_m & -\dfrac{\Gamma_4}{j\omega} & \dfrac{\Gamma_4 + \Gamma_5}{j\omega} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}$$

Example 2 Let us write by inspection the integrodifferential equations of the network shown in Fig. 3.6 under the assumption that $\Gamma_{45} = 0$. The network is redrawn in Fig. 3.8.

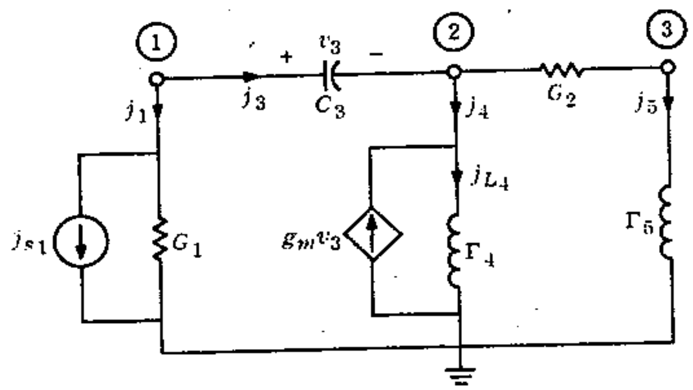


Fig. 3.8 Network analyzed in Example 2.

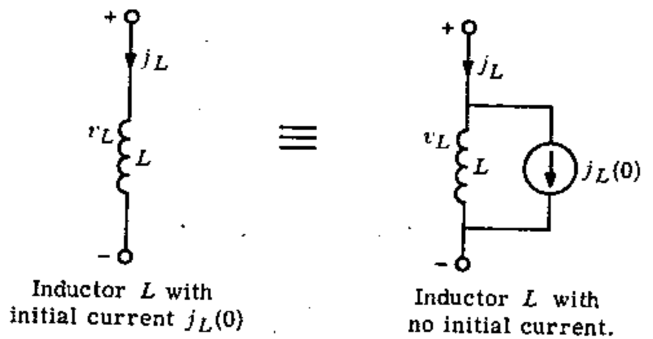


Fig. 3.9 Useful equivalence when writing equations by inspection.

Step 1 We must observe that the branch equations of inductors include initial currents (see Fig. 3.9); thus,

$$j_L(t) = \Gamma \int_0^t v_L(t')\,dt' + j_L(0)$$

Thus, every inductor can be replaced by an inductor without initial current in parallel with a *constant* current source $j_L(0)$. After replacing the dependent source $g_m v_3$ by an independent source j_{s4}, we have the network shown in Fig. 3.10.

Step 2 By inspection, the equations are

$$\begin{bmatrix} G_1 + C_3 D & -C_3 D & 0 \\ -C_3 D & G_2 + \Gamma_4 D^{-1} + C_3 D & -G_2 \\ 0 & -G_2 & G_2 + \Gamma_5 D^{-1} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -j_{s1} \\ j_{s4} - j_{L4}(0) \\ -j_5(0) \end{bmatrix}$$

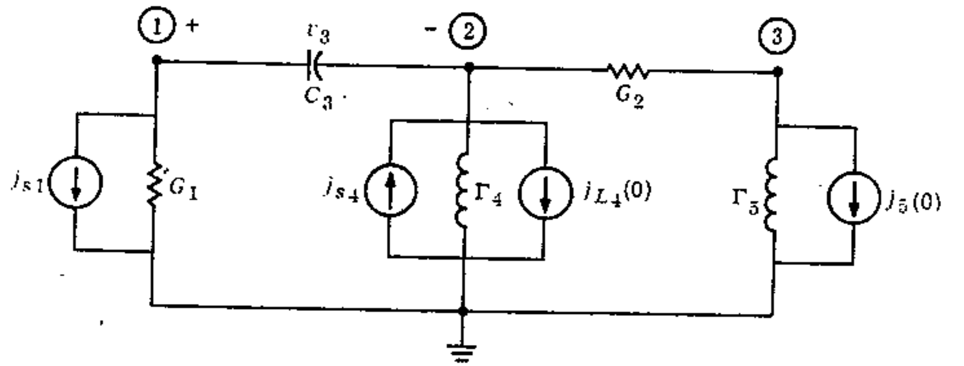


Fig. 3.10 Network of Fig. 3.8 in which the initial currents have been replaced by constant current sources.

Step 3 Express the dependent source waveform in terms of the appropriate variable as follows:

$$j_{s4} = g_m v_3 = g_m(e_1 - e_2)$$

Step 4 Substitute and rearrange terms; thus,

$$\begin{bmatrix} G_1 + C_3D & -C_3D & 0 \\ -g_m - C_3D & g_m + G_2 + \Gamma_4 D^{-1} + C_3D & -G_2 \\ 0 & -G_2 & G_2 + \Gamma_5 D^{-1} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -j_{s1} \\ -j_{L4}(0) \\ -j_5(0) \end{bmatrix}$$

In summary, it is easy in many instances to write the equations by inspection. It is important to know that the systematic method of Sec. 3.1 and 3.3 always works and hence can be used as a check in case of doubt.

4 Duality

In this section we propose to develop the concept of duality which we shall use repeatedly in the remainder of this chapter and in Chap. 11.

4.1 Planar Graphs, Meshes, Outer Meshes

By the very definition of a *graph* that we adopted, namely, a set of nodes and a set of branches each terminated at each end into a node, it is clear that a given graph may be drawn in several different ways. For example, the three figures shown in Fig. 4.1 are representations of the same graph. Indeed, they have the same incidence matrix. Similarly, a loop is a concept that does not depend on the way the graph is drawn: for example, the branches f, b, c, and e form a loop in the three figures shown in Fig. 4.1.

If we use the term "mesh" intuitively, we would call the loop $bcef$ a

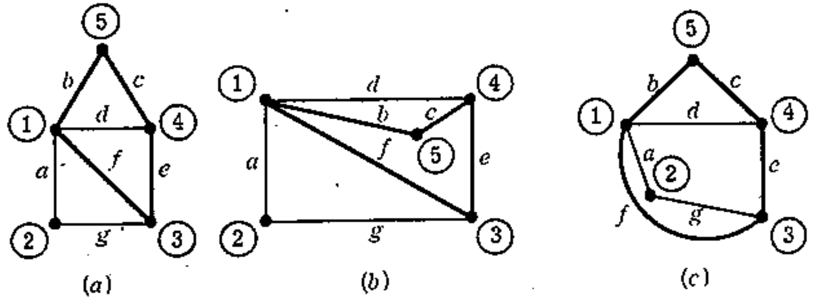


Fig. 4.1 The figures (*a*), (*b*), and (*c*) represent the same *graph* in the form of three different *topological graphs*.

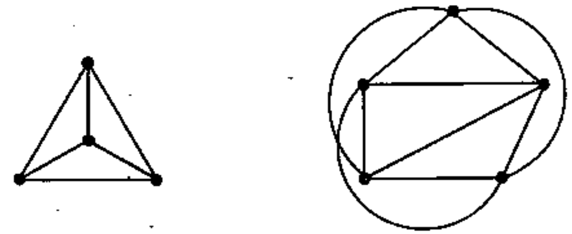


Fig. 4.2 (*a*) Planar graph; (*b*) nonplanar graph.

mesh in Fig. 4.1*b*, but not in Fig. 4.1*a* or *c*. For this reason, we need to consider graphs drawn in a specific way. When we do consider a graph $\mathcal{G}$ drawn in a way specified by us, we refer to it as the **topological graph** $\mathcal{G}$. For example, the three figures in Fig. 4.1 may be considered to be the same *graph* or three different *topological graphs*.

A graph $\mathcal{G}$ is said to be **planar** if it *can* be drawn on the plane in such a way that no two branches intersect at a point which is not a node. The graph in Fig. 4.2*a* is a planar graph, whereas the graph in Fig. 4.2*b* is not.

Consider a topological graph $\mathcal{G}$ which is planar. We call any loop of this graph for which there is no branch in its interior a **mesh**. For example, for the topological graph shown in Fig. 4.1*a* the loop $fbce$ is not a mesh; for the topological graph shown in Fig. 4.1*b* the loop $fbce$ is a mesh. In Fig. 4.1*c* the loop $fbce$ contains no branches in its exterior; it is called the **outer mesh** of the topological graph.

If you imagine the planar graph as your girl friend's hair net and if you imagine it slipped over a transparent sphere of lucite, then as you stand in the center of the sphere and look outside, you see that there is no significant difference between a mesh and the outer mesh.

We next exhibit a type of network whose graphs have certain properties that lead to simplification in analysis. Consider the three graphs shown in Fig. 4.3*a*–*c*. Each of these graphs has the property that it can be partitioned into *two* nondegenerate subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ which are connected together by *one* node.† Graphs which have this property are called **hinged graphs.** A graph that is not hinged is called unhinged (or sometimes, nonseparable): thus, an **unhinged graph** has the property that whenever it is partitioned into two connected nondegenerate subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$, the subgraphs have at least two nodes in common. Determining whether a graph is hinged or not is easily done by inspection (see Fig. 4.3 for examples).

From a network analysis point of view, if a network has a graph that is hinged and if there is no coupling (by mutual inductances or dependent

† By nondegenerate we mean that the subgraph is not an isolated node.

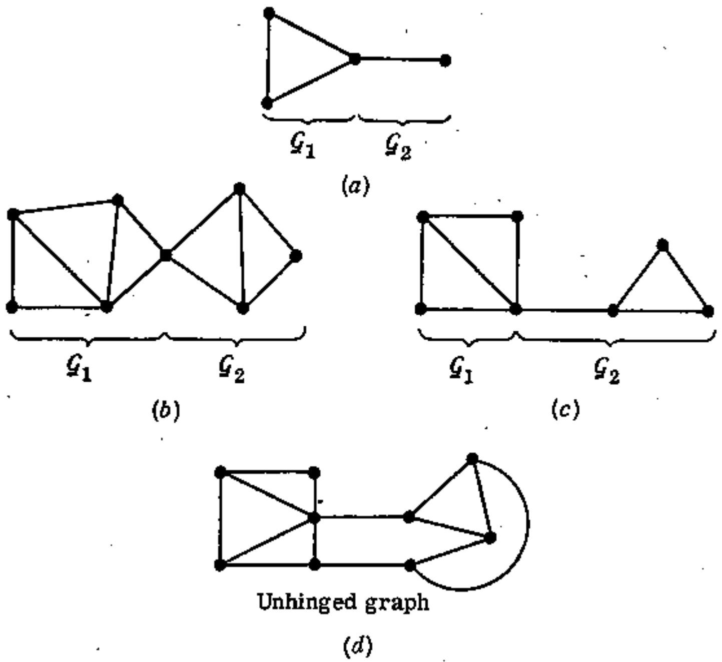


Fig. 4.3 Examples of hinged graphs, (a), (b), and (c), and an unhinged graph, (d).

sources) between the elements of $\mathcal{G}_1$ and $\mathcal{G}_2$, the analysis of the network reduces to the analysis of two independent subnetworks, namely the networks corresponding to the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected together by one node, KCL requires that the net current flow from $\mathcal{G}_1$ to $\mathcal{G}_2$ be zero at all times, so there is no exchange of current between the two subnetworks. Also the fact that the two subnetworks have a node in common does not impose any restriction on the branch voltages.

Counting meshes

It is easy to see that for *a connected unhinged planar graph the number of meshes is equal to* $b - n_t + 1$ where b is the number of branches and n_t the number of nodes. The proof can be given by mathematical induction. Let the number of meshes be l; thus we want to show that

$$\boxed{l = b - n_t + 1} \tag{4.1}$$

Consider the graph in Fig. 4.4a, where $l = 1$. Here it is obvious that Eq. (4.1) is true. Next consider a graph which has l meshes, and we assume that Eq. (4.1) is true. We want to show that (4.1) is still true if the graph is

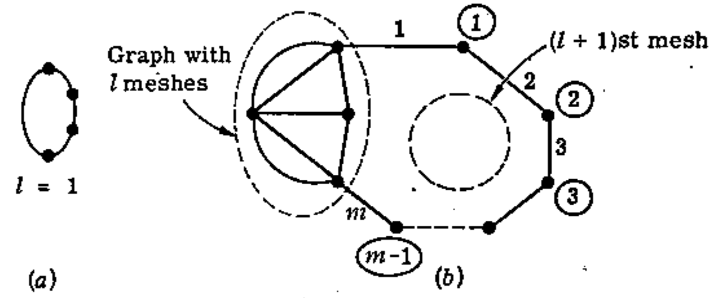


Fig. 4.4 Indication of proof of $l = b - n_t + 1$.

changed to have $l + 1$ meshes. We can increase the number of meshes by 1 by adding a branch between two existing nodes or by adding m branches in series which are connected to the existing graph through $m - 1$ new nodes, as shown in Fig. 4.4b. For the new graph with $l + 1$ meshes Eq. (4.1) is still satisfied, because $m - 1$ nodes and m branches have been added, resulting in one additional mesh. Therefore, by induction, Eq. (4.1) is true in general.

The matrix $\mathbf{M}_a$

A fundamental property of a connected unhinged planar graph is that each branch of the graph belongs to exactly two meshes if we include the outer mesh. Consider such a graph with specified branch orientation. We shall assign by convention the following reference directions for the meshes: the clockwise direction for each mesh and the counterclockwise direction for the outer mesh. This is illustrated in the graph of Fig. 4.5. Thus, an oriented planar graph $\mathcal{G}$ that is connected and unhinged can be described

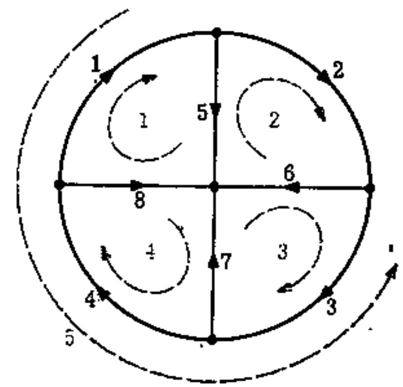


Fig. 4.5 An oriented planar graph with eight branches and five meshes (including the outer mesh).

analytically by a matrix $\mathbf{M}_a$. Let $\mathcal{G}$ have b branches and $l + 1$ meshes (including the outer mesh); then $\mathbf{M}_a$ is defined as the rectangular matrix of $l + 1$ rows and b columns whose (i,k)th element m_{ik} is defined by

$$m_{ik} = \begin{cases} 1 & \text{if branch } k \text{ is in mesh } i \text{ and if their reference directions } \textit{coincide} \\ -1 & \text{if branch } k \text{ is in mesh } i \text{ and if their reference directions } \textit{do not coincide} \\ 0 & \text{if branch } k \textit{ does not belong} \text{ to mesh } i \end{cases}$$

For the graph shown in Fig. 4.5, $b = 8$, $l + 1 = 5$, and the matrix $\mathbf{M}_a$ is

$$\begin{array}{cc} & \overbrace{\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}}^{\text{Branches}} \\ \text{Meshes}\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Observe that the matrix $\mathbf{M}_a$ has a property in common with the incidence matrix $\mathbf{A}_a$: that is, in each column all elements are zero except for one $+1$ and one -1. In the succeeding subsection the concept of dual graphs will be introduced so that we may further explore the relation between these matrices.

4.2 Dual Graphs

Before giving a precise formulation of the concept of dual graphs and dual networks, let us consider an example. By pointing to some features of the following example, we shall provide some motivation for the later formulations.

Example 1 Consider the linear time-invariant networks shown in Fig. 4.6. For simplicity we assume that the sources are sinusoidal and have the same frequency, and that the networks are in the sinusoidal steady state. In the first network, say $\mathfrak{N}$, we represent the two sinusoidal node-to-datum voltages by the *phasors* E_1 and E_2. By inspection, we obtain the following node equations:

$$\left(j\omega C_1 + \frac{1}{j\omega L}\right)E_1 - \frac{1}{j\omega L}E_2 = I_s \tag{4.2a}$$

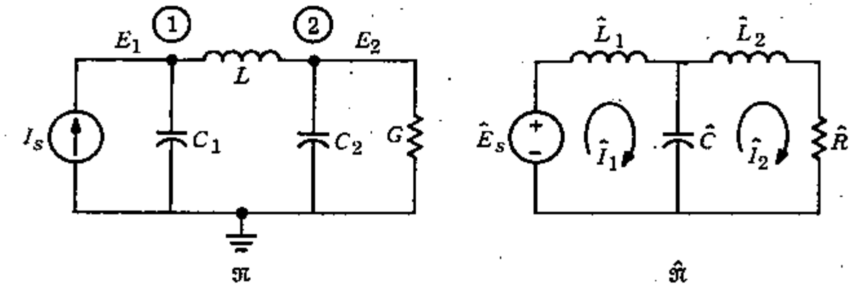


Fig. 4.6 Two networks used to illustrate duality: if $C_1 = \hat{L}_1$, $C_2 = \hat{L}_2$, $L = \hat{C}$, $G = \hat{R}$, and $I_s = \hat{E}_s$, then they are said to be dual networks.

$$-\frac{1}{j\omega L}E_1 + \left(j\omega C_2 + G + \frac{1}{j\omega L}\right)E_2 = 0 \tag{4.2b}$$

Note that I_s is the *phasor* that represents the sinusoidal current of the source.

The second network $\hat{\mathfrak{N}}$ has two meshes. We represent the two sinusoidal mesh currents by the phasors $\hat{I}_1$ and $\hat{I}_2$. Again, by inspection, we obtain the following mesh equations:

$$\left(j\omega\hat{L}_1 + \frac{1}{j\omega\hat{C}}\right)\hat{I}_1 - \left(\frac{1}{j\omega\hat{C}}\right)\hat{I}_2 = \hat{E}_s \tag{4.3a}$$

$$-\left(\frac{1}{j\omega\hat{C}}\right)\hat{I}_1 + \left(j\omega\hat{L}_2 + \hat{R} + \frac{1}{j\omega\hat{C}}\right)\hat{I}_2 = 0 \tag{4.3b}$$

If the element values of the two circuits are related by

$$C_1 = \hat{L}_1 \qquad L = \hat{C} \qquad C_2 = \hat{L}_2 \qquad G = \hat{R}$$

and if the sources have the same phasor

$$I_s = \hat{E}_s$$

then Eqs. (4.2) and (4.3) are *identical*. Therefore, if we have solved one of the networks, we have solved the other. These two networks are an example of a pair of dual networks. There are some interesting relations between them. $\mathfrak{N}$ has two nodes and a datum node; $\hat{\mathfrak{N}}$ has two meshes and an outer mesh. Both have five branches. To a branch between two nodes of $\mathfrak{N}$ (say, the inductor connecting node 1 and node 2) corresponds a branch of $\hat{\mathfrak{N}}$ which is common to the corresponding meshes (the capacitor common to mesh 1 and mesh 2). In $\mathfrak{N}$, the current source I_s and the capacitor C_1 are in parallel; in $\hat{\mathfrak{N}}$, the voltage source $\hat{E}_s$ and the inductor $\hat{L}$ are in series; etc. We note that the relation between $\mathfrak{N}$ and $\hat{\mathfrak{N}}$ involves both graph-theoretic concepts (meshes and nodes) and the nature of the elements (sources, inductors, capacitors, etc.). For this reason we must proceed in two steps, first considering dual graphs and then defining dual networks.

We are ready to introduce the concept of dual graphs. Again we start with a topological graph $\mathcal{G}$ which is assumed to be connected, unhinged, and planar.† Let $\mathcal{G}$ have $n_t = n + 1$ nodes, b branches, and hence, $l = b - n$ meshes (not counting the outer mesh). A planar topological graph $\hat{\mathcal{G}}$ is said to be a **dual graph** of a topological graph $\mathcal{G}$ if

1. There is a one-to-one correspondence between the *meshes* of $\mathcal{G}$ (including the outer mesh) and the *nodes* of $\hat{\mathcal{G}}$.
2. There is a one-to-one correspondence between the *meshes* of $\hat{\mathcal{G}}$ (including the outer mesh) and the *nodes* of $\mathcal{G}$.
3. There is a one-to-one correspondence between the branches of each graph in such a way that whenever two meshes of one graph have the corresponding branch in common, the corresponding nodes of the other graph have the corresponding branch connecting these nodes.

We shall use the symbol ^ to designate all terms pertaining to a dual graph.

It follows from the definition that $\hat{\mathcal{G}}$ has b branches, $l + 1$ nodes, n meshes, and one outer mesh. It is easily checked that if $\hat{\mathcal{G}}$ is a dual graph of $\mathcal{G}$, then $\mathcal{G}$ is a dual graph of $\hat{\mathcal{G}}$. In other words *duality is a symmetric relation between connected, planar, unhinged topological graphs.*

ALGORITHM Given a connected, planar, unhinged topological graph $\mathcal{G}$, we construct a dual graph $\hat{\mathcal{G}}$ by proceeding as follows:

1. To each mesh of $\mathcal{G}$, including the outer mesh, we associate a node of $\hat{\mathcal{G}}$; thus, we associate node $\hat{1}$ to mesh 1 and draw node $\hat{1}$ inside mesh 1; a similar procedure is followed for nodes $\hat{2}$, $\hat{3}$, . . . , including node $\widehat{l+1}$, which corresponds to the outer mesh.
2. For each branch, say k, of $\mathcal{G}$ which is common to mesh i and mesh j, we associate a branch k of $\hat{\mathcal{G}}$ which is connected to nodes $\hat{i}$ and $\hat{j}$.

By its very construction, the resulting graph $\hat{\mathcal{G}}$ is a dual of $\mathcal{G}$.

Example 2 The given planar graph is shown in Fig. 4.7*a*. There are three meshes, not counting the outer mesh. We insert nodes $\hat{1}$, $\hat{2}$, and $\hat{3}$, with one node inside each mesh as shown in Fig. 4.7*b*. We place node $\hat{4}$ outside the graph $\mathcal{G}$ because $\hat{4}$ will correspond to the outer mesh. To complete

† The concept of dual graph can be introduced for arbitrary planar connected graph. However, for simplicity we rule out the case of hinged graph.

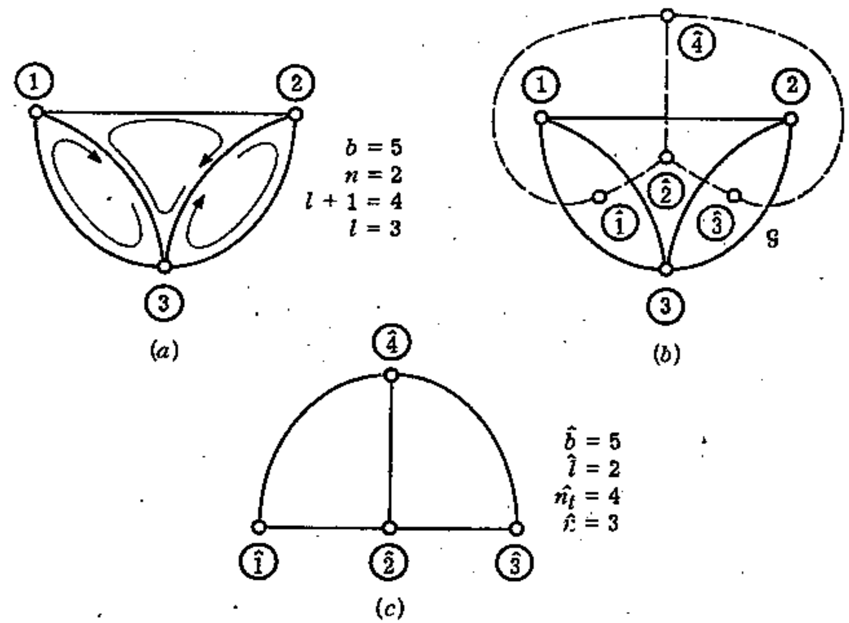


Fig. 4.7 Example 2: Illustrating the construction of a dual graph. (*a*) Given graph; (*b*) construction step; (*c*) dual graph.

the dual graph $\hat{\mathcal{G}}$, we connect two nodes with a branch whenever the corresponding meshes of $\mathcal{G}$ have a branch in common. The dotted lines in Fig. 4.7*b* represent branches of the dual graph $\hat{\mathcal{G}}$. The dual graph $\hat{\mathcal{G}}$ is redrawn in Fig. 4.7*c*.

In case the given graph $\mathcal{G}$ is oriented, i.e., in case each branch has a reference direction, the orientation of the dual graph $\hat{\mathcal{G}}$ can be obtained by adding to the construction above a simple orientation convention. Since the branches in both graphs are oriented, we can imagine the reference directions of the branches to be indicated by vectors which lie along the branch and point in the direction of the reference direction. The reference direction of a branch of the dual graph $\hat{\mathcal{G}}$ is obtained from that of the corresponding branch of the given graph $\mathcal{G}$ by rotating the vector 90° *clockwise.* With this algorithm, given any planar oriented topological graph $\mathcal{G}$, we can obtain in a systematic fashion a dual oriented graph $\hat{\mathcal{G}}$.

Example 3 Consider the oriented graph in Fig. 4.8*a*. Let node 4 be the datum node. We wish to obtain the dual graph $\hat{\mathcal{G}}$ whose outer mesh corresponds to node 4 of $\mathcal{G}$. Following the rules for constructing a dual graph, we insert

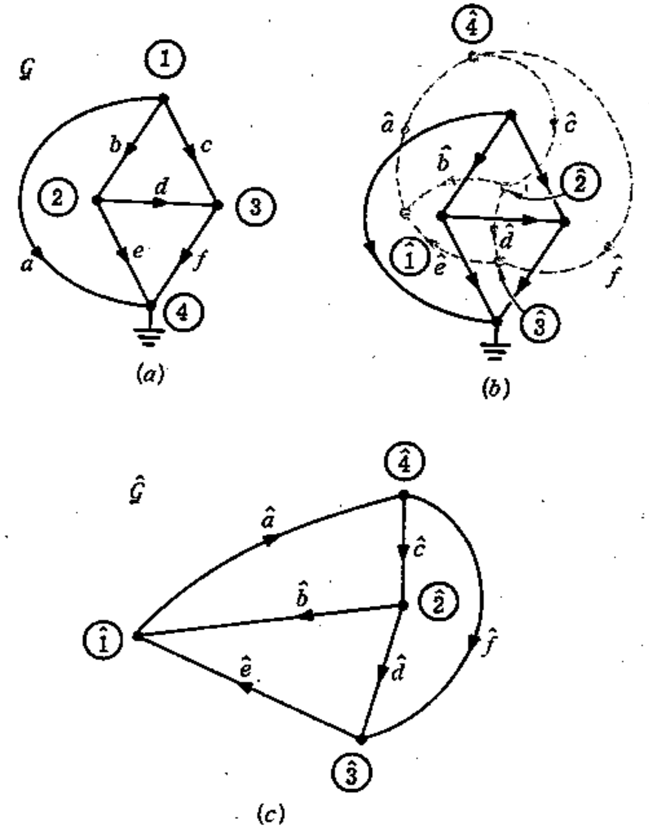


Fig. 4.8 Example 3: Construction of oriented dual graph.

nodes $\hat{1}$, $\hat{2}$, and $\hat{3}$ inside meshes 1, 2, and 3, respectively, of $\mathcal{G}$, leaving node $\hat{4}$ outside the mesh, as shown in Fig. 4.8*b*. To complete the dual graph, branches are drawn in dotted lines, as shown in Fig. 4.8*b*, connecting the nodes of $\hat{\mathcal{G}}$ to correspond to the branches of $\mathcal{G}$. The reference directions of the branches of $\hat{\mathcal{G}}$ are obtained by the method indicated above. The dual graph is redrawn as shown in Fig. 4.8*c*. Care must be taken to ensure that the branches of $\mathcal{G}$ connecting the datum node correspond to the outer mesh of $\hat{\mathcal{G}}$. A moment of thought will lead to the following rule, which gives the appropriate one-to-one correspondence. Since the datum node, node 4 in the example, must stay outside all dotted lines, when placing node $\hat{4}$ of $\hat{\mathcal{G}}$ it is convenient to put it as far away as possible from the datum node 4, as shown in Fig. 4.8*b*.

Remarks

1. In general, a given topological graph $\mathcal{G}$ has many duals. However, if we specify the datum node of $\mathcal{G}$ and specify that it has to correspond to the outer mesh of $\hat{\mathcal{G}}$, then the procedure described above defines a unique dual graph. The branches which are connected to the datum node in $\mathcal{G}$ have corresponding branches which form the outer mesh in $\hat{\mathcal{G}}$.
2. The correspondence between the graph $\mathcal{G}$ and its dual $\hat{\mathcal{G}}$ involves branch versus branch, node versus mesh, and datum node versus outer mesh. Furthermore, the incidence matrix $\mathbf{A}_a$ of the given graph $\mathcal{G}$ is equal to the matrix $\mathbf{M}_a$ of the dual graph $\hat{\mathcal{G}}$.

Exercise Construct a dual graph $\hat{\mathcal{G}}$ of the oriented planar graph $\mathcal{G}$ given in Fig. 4.5. Write the KCL equations of the dual graph for all the nodes; that is,

$$\hat{\mathbf{A}}_a\hat{\mathbf{j}} = \mathbf{0}$$

Show that the set of equations is identical to the KVL equations of the given graph for all the meshes (including the outer mesh); that is,

$$\mathbf{M}_a\mathbf{v} = \mathbf{0}$$

4.3 Dual Networks

In this discussion we restrict ourselves to networks having the following properties: their graphs are *connected, planar,* and *unhinged;* and all their elements are one-port elements. In other words, we *exclude* coupled inductors, transformers, and dependent sources; we *include* independent voltage or current sources, inductors, resistors, and capacitors. It is fundamental to observe that the elements do not have to be linear and/or time-invariant.

We say that a network $\hat{\mathfrak{N}}$ is **the dual of the network** $\mathfrak{N}$ if (1) the topological graph $\hat{\mathcal{G}}$ of $\hat{\mathfrak{N}}$ is a dual of the topological graph $\mathcal{G}$ of $\mathfrak{N}$, and (2) the branch equation of a branch of $\hat{\mathfrak{N}}$ is obtained from its corresponding equation of $\mathfrak{N}$ by performing the following substitutions:

$$v \to \hat{j} \qquad q \to \hat{\phi}$$
$$j \to \hat{v} \qquad \phi \to \hat{q} \tag{4.4}$$

where v, j, q, and ϕ are the branch voltage, current, charge, and flux variables, respectively, for $\mathfrak{N}$, and $\hat{v}$, $\hat{j}$, $\hat{q}$, and $\hat{\phi}$ are the corresponding branch variables for $\hat{\mathfrak{N}}$.

Requirement 2 of the definition means that a resistor of $\mathfrak{N}$ corresponds to a resistor of $\hat{\mathfrak{N}}$. Furthermore, a linear resistor of $\hat{\mathfrak{N}}$ with a resistance of K ohms corresponds to a linear resistor of $\hat{\mathfrak{N}}$ with a conductance of K mhos. Indeed, the branch equation of the resistor of $\mathfrak{N}$ is $v = Kj$; hence, that of the corresponding resistor of $\hat{\mathfrak{N}}$ is $\hat{j} = K\hat{v}$. Similarly, an inductor of $\mathfrak{N}$ corresponds to a capacitor of $\hat{\mathfrak{N}}$. Furthermore, a nonlinear time-varying inductor of $\mathfrak{N}$ which is characterized by $\phi = f(i,t)$, where $f(\cdot,\cdot)$ is a given function of two variables, will correspond to a nonlinear time-varying capacitor of $\hat{\mathfrak{N}}$ which is characterized by $\hat{q} = f(\hat{v},t)$. A voltage source whose voltage is a function $f(\cdot)$ will correspond to a current source whose current is the same function $f(\cdot)$. Also, the dual of a short circuit is an open circuit; a short circuit is characterized by $v = 0$, hence its dual is characterized by $\hat{j} = 0$, the equation for an open circuit.

It is easily checked that if $\hat{\mathfrak{N}}$ is a dual network of $\mathfrak{N}$, then $\mathfrak{N}$ is a dual network of $\hat{\mathfrak{N}}$; in other words, *duality is a symmetric relation between networks.*

Example 4 Consider the nonlinear time-varying network $\mathfrak{N}$ shown in Fig. 4.9*a*. The inductor is nonlinear; its characteristic is $\phi_1 = \tanh j_1$. The capacitor is time-varying and linear; its characteristic is $q_3 = (1 + \epsilon^{-t^2})v_3$. The output resistor is linear and time-varying with resistance $2 + \cos t$; its characteristic is $v_4 = (2 + \cos t)j_4$. The branch orientations are indicated in the figure. Let the mesh currents be i_1 and i_2; then,

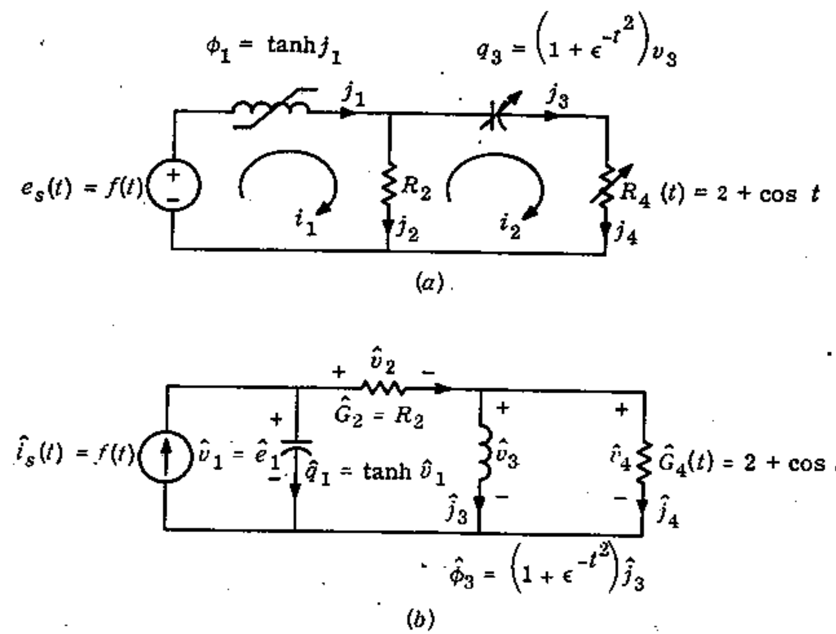


Fig. 4.9 Example 4: Illustrating dual networks.

$j_1 = i_1 \qquad j_2 = i_1 - i_2 \qquad j_3 = j_4 = i_2$

The mesh equations read

$$e_s(t) = f(t) = \frac{1}{\cosh^2 i_1}\frac{di_1}{dt} + R_2(i_1 - i_2) \tag{4.5a}$$

$$0 = R_2(i_2 - i_1) + \frac{q_3(0)}{1 + \epsilon^{-t^2}} + \frac{1}{1 + \epsilon^{-t^2}}\int_0^t i_2(t')\,dt' + (2 + \cos t)i_2(t) \tag{4.5b}$$

The dual network $\hat{\mathfrak{N}}$ is easily found. First, the dual graph $\hat{\mathcal{G}}$ is drawn, including the orientation; then each branch is filled with the appropriate dual element as prescribed by requirement 2. The result is shown in Fig. 4.9*b*. Let the node-to-datum voltages be $\hat{e}_1$ and $\hat{e}_2$. Then the branch voltages are related to the node voltages by $\hat{v}_1 = \hat{e}_1$, $\hat{v}_2 = \hat{e}_1 - \hat{e}_2$, and $\hat{v}_3 = \hat{v}_4 = \hat{e}_2$. The node equations give

$$\hat{i}_s(t) = f(t) = \frac{1}{\cosh^2 \hat{e}_1}\frac{d\hat{e}_1}{dt} + \hat{G}_2(\hat{e}_1 - \hat{e}_2) \tag{4.6a}$$

$$0 = \hat{G}_2(\hat{e}_2 - \hat{e}_1) + \frac{\hat{\phi}(0)}{1 + \epsilon^{-t^2}} + \frac{1}{1 + \epsilon^{-t^2}}\int_0^t \hat{e}_2(t')\,dt' + (2 + \cos t)e_2(t) \tag{4.6b}$$

where the conductance $\hat{G}_2$ is equal to the resistance R_2. Observe that Eqs. (4.6) are identical with Eqs. (4.5) except for the names of the variables.

General property of dual networks The importance of duality cannot be overemphasized. Its power is exhibited by the following general assertion. *Consider an arbitrary planar network $\mathfrak{N}$ and its dual $\hat{\mathfrak{N}}$. Let S be any true statement concerning the behavior of $\mathfrak{N}$. Let $\hat{S}$ be the statement obtained from S by replacing every graph-theoretic word or phrase (node, mesh, loop, etc.) by its dual and every electrical quantity (voltage, current, impedance, etc.) by its dual. Then $\hat{S}$ is a true statement concerning the behavior of $\hat{\mathfrak{N}}$.* In Table 10.1 we give a tabulation of pairs of dual terms. Some of these will be illustrated in later chapters.

Exercise 1 Consider the linear time-invariant *RLC* network $\mathfrak{N}$ (without coupled inductors) shown in Fig. 4.10. Assume that its graph is planar and unhinged. Suppose it is driven by a sinusoidal current source and is in the steady state. Consider the dual network $\hat{\mathfrak{N}}$. Show that the driving-point impedance Z_{in} of $\mathfrak{N}$ is for each ω equal to the driving-point admittance Y_{in} of $\hat{\mathfrak{N}}$.

Exercise 2 Consider the ladder network $\mathfrak{N}$ shown in Fig. 4.11. The functions $f_1(j\omega)$, $f_2(j\omega)$, . . . , $f_5(j\omega)$ specify the *impedances* of the corresponding element of $\mathfrak{N}$; the function f specifies the waveform of the source. Show that the dual of $\mathfrak{N}$ can be obtained (1) by replacing the current source of $f(t)$ amp

Table 10.1 Dual Terms

Types of properties	S	$\hat{S}$
Graph-theoretic properties	Node	Mesh
	Cut set	Loop
	Datum node	Outer mesh
	Tree branch*	Link*
	Fundamental cut set*	Fundamental loop*
	Branches in series	Branches in parallel
	Reduced incidence matrix	Mesh matrix*
	Fundamental cut-set matrix*	Fundamental loop matrix*
Graph-theoretic and electric properties	Node-to-datum voltages	Mesh currents
	Tree-branch voltages*	Link currents*
	KCL	KVL
Electric properties	Voltage	Current
	Charge	Flux
	Resistor	Resistor
	Inductor	Capacitor
	Resistance	Conductance
	Inductance	Capacitance
	Current source	Voltage source
	Short circuit	Open circuit
	Admittance	Impedance
	Node admittance matrix	Mesh impedance matrix*

* The asterisk is used to indicate terms that will be encountered in this and the next chapter.

by a voltage source of $f(t)$ volts, (2) by replacing each series element of $\mathfrak{N}$ of impedance $f_i(j\omega)$ by a shunt element of admittance $f_i(j\omega)$, and (3) by replacing each shunt element of $\mathfrak{N}$ of impedance $f_i(j\omega)$ by a series element of admittance $f_i(j\omega)$.

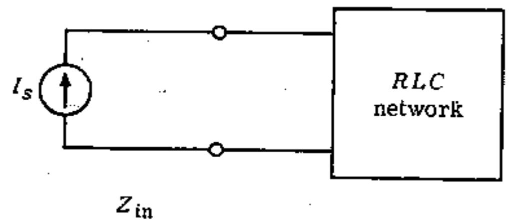


Fig. 4.10 Exercise 1: illustrating the dual of a driving-point impedance.

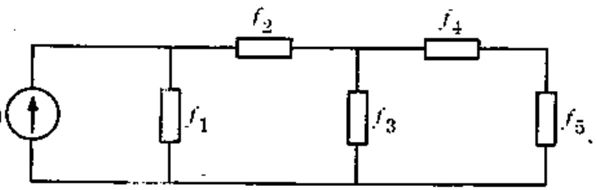


Fig. 4.11 A ladder network $\mathfrak{N}$.

5 Two Basic Facts of Mesh Analysis

Let us consider any network $\mathfrak{N}$ whose graph is connected, planar, and unhinged. We assume that it has n_t nodes and b branches; consequently it has $l = b - n_t + 1$ meshes, not counting the outer mesh. We label the meshes $1, 2, \ldots, l$ and use the clockwise reference directions. The meshes are the duals of the nodes, whereas the outer mesh is the dual of the datum node. We shall employ the concept of duality to develop the two basic facts of mesh analysis. Again it should be emphasized that the two facts are independent of the nature of the network elements. Thus, the network can be linear or nonlinear, time-invariant or time-varying.

5.1 Implications of KVL

Let us apply KVL to meshes $1, 2, \ldots, l$ (omitting the outer mesh). As seen in the example below (Fig. 5.1) each expression is a linear homogeneous algebraic equation in the branch voltages. Thus, we have a system of l linear homogeneous algebraic equations in the b unknowns $v_1, v_2, \ldots, v_b$. The first basic fact of mesh analysis can be stated as follows:

The l linear homogeneous algebraic equations in $v_1, v_2, \ldots, v_l$ obtained by applying KVL to each mesh (except the outer mesh) constitute a set of l linearly independent equations.

The proof of this statement can be given in the same manner as the proof of the comparable statement in node analysis. We shall ask the reader to go through corresponding steps. It is, however, quite easy to use duality to prove the statement above: Denote by $\hat{\mathfrak{N}}$ the network dual to $\mathfrak{N}$. Apply KCL to all nodes of $\hat{\mathfrak{N}}$ except the datum node. If the basic fact above were false, the first basic fact of node analysis would also be false. Since the latter has been proved independently, it follows that the first basic fact of mesh analysis is true.

Analytically KVL may be expressed by the use of the **mesh matrix**

(5.1) $\mathbf{Mv} = \mathbf{0}$ (KVL)

where $\mathbf{M} = (m_{ij})$ is an $l \times b$ matrix defined below by Eq. (5.2). When we write that the ith component of $\mathbf{Mv}$ is zero, we merely assert that the sum of all branch voltages around the ith mesh is zero. Since this ith component is of the form

$$\sum_{k=1}^{b} m_{ik}v_k = 0$$

we must have for $i = 1, 2, \ldots, l$ and $k = 1, 2, \ldots, b$,

$$(5.2) \quad m_{ik} = \begin{cases} 1 & \text{if branch } k \text{ is in mesh } i \text{ and if their reference directions } \textit{coincide} \\ -1 & \text{if branch } k \text{ is in mesh } i \text{ and if their reference directions } \textit{do not coincide} \\ 0 & \text{if branch } k \textit{ does not belong} \text{ to mesh } i\,. \end{cases}$$

The basic fact established above implies that *the mesh matrix* $\mathbf{M}$ *has a rank equal to* l.

Note that the mesh matrix $\mathbf{M}$ is obtained from the matrix $\mathbf{M}_a$ by deleting the row of $\mathbf{M}_a$ which corresponds to the outer mesh.

Example 1 Consider the oriented graph of Fig. 5.1, which is the dual graph of Fig. 2.3. There are three nodes and five branches; thus, $l = 5 - 3 + 1 = 3$. The three meshes are labeled as shown. The branch voltage vector is

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

The mesh matrix, obtained from Eq. (5.2), is

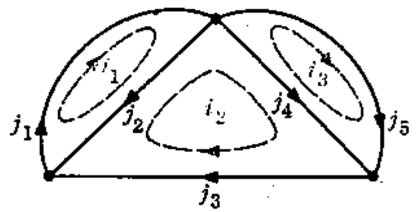


Fig. 5.1 An oriented graph which is the dual graph of that of Fig. 2.3.

$$\begin{array}{cc} & \text{Mesh} \\ \mathbf{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \end{array} \\ \text{Branch} \quad \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 & 4 & 5 \end{array} & \end{array}$$

The mesh equation is therefore

$$\mathbf{Mv} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} v_1 + v_2 &= 0 \\ -v_2 + v_3 + v_4 &= 0 \\ -v_4 + v_5 &= 0 \end{aligned}$$

which are clearly the three mesh equations obtained from KVL for meshes 1, 2, and 3. The three equations are clearly linearly independent, since each equation includes a variable which is not present in the other two.

Exercise Let $\mathbf{M}$ be the mesh matrix of an oriented graph $\mathcal{G}$. Let $\hat{\mathbf{A}}$ be the reduced incidence matrix of the dual graph $\hat{\mathcal{G}}$. Show that $\hat{\mathbf{A}} = \mathbf{M}$. Verify that the dual of definition (2.3) of a_{ik} gives definition (5.2) of m_{ik}.

5.2 Implications of KCL

Let us call $i_1, i_2, \ldots, i_l$ the mesh currents. For convenience let us assign to each one a *clockwise* reference direction. First, let us observe that the mesh currents $i_1, i_2, \ldots, i_l$ are linearly independent as far as KCL is concerned. Since each mesh current runs around a loop, if mesh current i_k crosses an arbitrarily chosen cut set in the positive direction, it also crosses the cut set in the negative direction and, hence, cancels out from the KCL equation applied to that cut set. In other words, if we write KCL for any cut set and express the branch currents in terms of mesh currents, everything cancels out. Thus, KCL has nothing to say about the mesh currents, which makes them linearly independent as far as KCL is concerned.

The next step is to show that the branch currents can be calculated in terms of the mesh currents by the equation

(5.3) $\mathbf{j} = \mathbf{M}^T\mathbf{i}$ (KCL)

where $\mathbf{M}^T$ is the transpose of the mesh matrix $\mathbf{M}$. Equation (5.3) means that every branch current can be expressed as a linear combination of mesh currents, and that the matrix which specifies these linear combinations is the transpose of the mesh matrix defined previously. We shall ask the reader to apply duality to the proof that $\mathbf{v} = \mathbf{A}^T\mathbf{e}$ in order to justify Eq. (5.3).

Example 2 Consider the network whose graph is shown in Fig. 5.1. It is obvious that we can relate the branch currents and mesh currents as follows:

$j_1 = i_1$

$j_2 = i_1 - i_2$

$j_3 = i_3$

$j_4 = i_2 - i_3$

$j_5 = i_3$

or

$$\mathbf{j} = \mathbf{M}^T\mathbf{i} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

Exercise 1 Express Kirchhoff's laws in the form of Eq. (5.1) and (5.3) for the graph of Fig. 4.7*a*.

Exercise 2 Suppose that a network has a graph consisting of a large square divided into 25 equal squares (five on a side). Suppose that you have a large supply of zero-impedance ammeters. What is the minimum number of ammeters required to measure all mesh currents? Where would you put them?

Exercise 3 Use definition (5.2) of m_{ik} to prove Eq. (5.3).

Summary Equation (5.1)

$\mathbf{Mv} = \mathbf{0}$ (KVL)

and Eq. (5.3)

$\mathbf{j} = \mathbf{M}^T\mathbf{i}$ (KCL)

respectively, are the two basic equations of mesh analysis. Since the two equations are obtained from the network graph (planar, connected, and unhinged) and the two Kirchhoff laws, they are independent of the nature of the elements of the network. Equation (5.1) expresses KVL and consists of a set of l linearly independent equations in terms of the b branch voltages $v_1, v_2, \ldots, v_b$. Equation (5.3) expresses KCL and relates the b branch currents $j_1, j_2, \ldots, j_b$ to the l mesh currents $i_1, i_2, \ldots, i_l$. To solve for the l network variables $i_1, i_2, \ldots, i_l$, we need to know the branch characterization of the network, i.e., the b branch equations which relate the branch voltages to the branch currents. Only in these branch equations does the nature of network elements come into the analysis. In the next section we shall treat exclusively linear time-invariant networks. Nonlinear and time-varying networks will be considered later.

6 Mesh Analysis of Linear Time-invariant Networks

The mesh analysis of a linear time-invariant network requires a sequence of steps which is the dual of that required for the node analysis of the dual network $\hat{\mathfrak{N}}$. This will allow us to treat this material briefly.

6.1 Sinusoidal Steady-state Analysis

Since the analysis of resistive networks is a special case of the sinusoidal steady-state analysis, we treat only the latter.

Let $\mathfrak{N}$ be a linear time-invariant network with b branches and n_t nodes. Let its graph $\mathcal{G}$ be connected, planar, and unhinged. Let the sources be sinusoidal and have all the same frequency ω. Call $\mathbf{J}_s$ and $\mathbf{V}_s$ the b-vectors whose kth components are the phasors representing the sinusoidal sources in the kth branch. Similarly, $\mathbf{V}$ and $\mathbf{J}$ are the b-vectors whose kth components are the phasors representing the branch voltage v_k and the branch current j_k. Call $\mathbf{I}$ the l-vector whose components are the phasors representing the mesh currents $i_1, i_2, \ldots, i_l$. Kirchhoff's laws give

(6.1) $\mathbf{MV} = \mathbf{0}$ (KVL)

(6.2) $\mathbf{J} = \mathbf{M}^T\mathbf{I}$ (KCL)

The branch equations are

(6.3) $\mathbf{V} = \mathbf{Z}_b(j\omega)\mathbf{J} - \mathbf{Z}_b(j\omega)\mathbf{J}_s + \mathbf{V}_s$

The $b \times b$ matrix $\mathbf{Z}_b(j\omega)$ is called the **branch-impedance matrix**. The substitution gives

$$(\mathbf{M}\mathbf{Z}_b(j\omega)\mathbf{M}^T)\mathbf{I} = \mathbf{M}\mathbf{Z}_b(j\omega)\mathbf{J}_s - \mathbf{V}_s$$

or

$$\boxed{\mathbf{Z}_m(j\omega)\mathbf{I} = \mathbf{E}_s} \tag{6.4}$$

where $\mathbf{Z}_m(j\omega)$ is an $l \times l$ matrix called the **mesh impedance matrix,** given by

$$\mathbf{Z}_m(j\omega) = \mathbf{M}\mathbf{Z}_b(j\omega)\mathbf{M}^T \tag{6.5}$$

and $\mathbf{E}_s$, the **mesh voltage source vector,** is the l vector given by

$$\mathbf{E}_s = \mathbf{M}\mathbf{Z}_b(j\omega)\mathbf{J}_s - \mathbf{M}\mathbf{V}_s \tag{6.6}$$

Equations (6.4) are called the **mesh equations** of $\mathfrak{N}$; they constitute a system of l linear algebraic equations (with complex coefficients) in l unknowns, the phasors representing the mesh currents $I_1, I_2, \ldots, I_l$. The solution of (6.4) specifies all mesh currents. Then the branch currents are obtained by (6.2), and the branch voltages by (6.3).

Example 1 Consider the linear time-invariant network $\mathfrak{N}$ shown in Fig. 6.1. The phasor V_{s1} represents the sinusoidal voltage of the source; $v_{s1}(t) = |V_{s1}| \cos(\omega t + \measuredangle V_{s1})$. By inspection,

$$\mathbf{M} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \tag{6.7}$$

Let the inductance matrix of the branches 3, 4, and 5 be

$$\mathbf{L} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 5 \end{bmatrix} \tag{6.8}$$

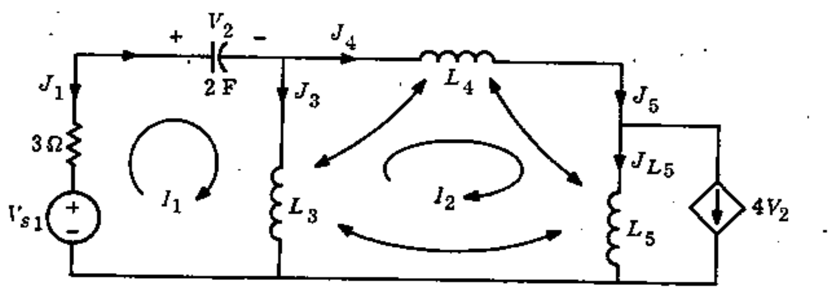


Fig. 6.1 Network analyzed in Example 1.

hence

$$\begin{bmatrix} V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 3j\omega & j\omega & -j\omega \\ j\omega & 4j\omega & 2j\omega \\ -j\omega & 2j\omega & 5j\omega \end{bmatrix} \begin{bmatrix} J_3 \\ J_4 \\ J_{L5} \end{bmatrix} \tag{6.9}$$

If we note that the current J_5 is related to J_{L5}, the current into the inductor, by

$$J_{L5} = J_5 - 4V_2 = J_5 - \frac{2}{j\omega} J_2 \tag{6.10}$$

we obtain the following branch equations:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2j\omega} & 0 & 0 & 0 \\ 0 & 2 & 3j\omega & j\omega & -j\omega \\ 0 & -4 & j\omega & 4j\omega & 2j\omega \\ 0 & -10 & -j\omega & 2j\omega & 5j\omega \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} + \begin{bmatrix} V_{s1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using (6.5) and (6.6), we obtain the following mesh equations:

$$\begin{bmatrix} 5 + 3j\omega + \frac{2}{j\omega} & -3j\omega \\ -16 - 3j\omega & 16j\omega \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_{s1} \\ 0 \end{bmatrix}$$

Properties of the mesh impedance matrix

1. If the network $\mathfrak{N}$ has no coupling elements, $\mathbf{Z}_b(j\omega)$ is *diagonal* and $\mathbf{Z}_m(j\omega)$ is *symmetric;* that is, $\mathbf{Z}_m(j\omega) = \mathbf{Z}_m^T(j\omega)$ [this follows from (6.5)].
2. Again, if $\mathfrak{N}$ has no coupling elements, the mesh impedance $\mathbf{Z}_m(j\omega)$ can be written by inspection as follows:
   a. Call z_{ii} the diagonal element of $\mathbf{Z}_m$ in the ith row and ith column. z_{ii} is the sum of all the impedances of branches in mesh i and is called the **self-impedance of mesh** i.
   b. Call z_{ik} the (i,k)th element of $\mathbf{Z}_m$. z_{ik} is the negative of the sum of all the impedances of the branches which are in common with meshes i and k† and is called the **mutual impedance between mesh** i **and mesh** k.
3. If, using the Thévenin equivalent, we convert all sources into voltage sources, then e_{sk} is the algebraic sum of all the source voltages in mesh k.

† The fact that z_{ik} is the negative of the sum of the impedances common to meshes i and k is a consequence of the convention that *all* mesh currents are given clockwise reference directions.

4. In the case of resistive networks, if all resistances are *positive*, then $\det(\mathbf{Z}_m) > 0$. Cramer's rule then guarantees that whatever the independent sources may be, the mesh equations (6.4) have a *unique* solution.

In case $\mathfrak{N}$ has coupling elements, the only general conclusions are that $\mathbf{Z}_b(j\omega)$ is no longer diagonal and $\mathbf{Z}_m(j\omega)$ is usually no longer symmetric.

Exercise 1 Prove statements $2a$ and $2b$. [Hint: Use (6.5) and (5.2)].

Exercise 2 By inspection, write the mesh equations of the network shown in Fig. 6.1, assuming that all mutual inductances and the dependent current sources are set to zero.

Exercise 3 Find a network with coupling elements which has a symmetric mesh impedance matrix. (Hint: Use symmetry.)

6.2 Integrodifferential Equations

Let us illustrate the general procedure for writing the integrodifferential mesh equations. We choose a simple case so that we can concentrate on the handling of initial conditions. The method, however, is completely general.

Consider the linear time-invariant network shown in Fig. 6.2. We are given the element values R_3, C_4, and α and the inductance matrix

$$\begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}$$

Note that M is positive in view of the reference directions of j_1 and j_2, which enter the inductors through the dotted terminals. In addition, we need the initial charge on the capacitor or, equivalently, $v_4(0)$, and the initial inductor currents $j_1(0)$ and $j_2(0)$. Finally, we need the input waveform $v_{s3}(\cdot)$. To obtain the mesh equations, we proceed as follows:

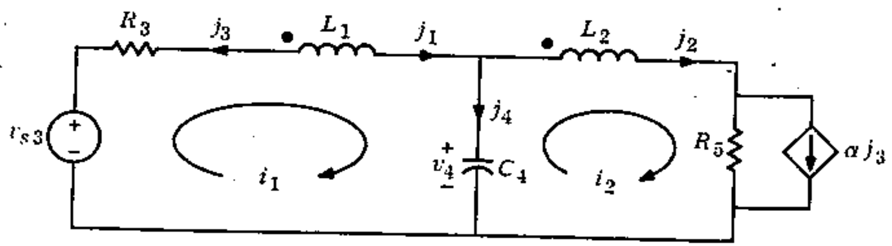


Fig. 6.2 Network used to illustrate the writing of integrodifferential equations.

Step 1 We write KVL; thus, $\mathbf{Mv} = \mathbf{0}$, and

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.11}$$

Step 2 We write KCL; thus, $\mathbf{j} = \mathbf{M}^T\mathbf{i}$, and

$$\begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{6.12}$$

Step 3 We write the branch equations in matrix form as follows:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} L_1D & MD & 0 & 0 & 0 \\ MD & L_2D & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{C_4D} & 0 \\ 0 & 0 & -\alpha R_5 & 0 & R_5 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_{s3} \\ v_4(0) \\ 0 \end{bmatrix} \tag{6.13}$$

We may think of this equation as being in the form

$$\mathbf{v} = \mathbf{Z}_b(D)\mathbf{j} + \mathbf{v}_s$$

where, in this case, $\mathbf{Z}_b$ is a matrix whose elements involve the operators D and D^{-1}.

Step 4 We use (6.12) to eliminate the j_k's from (6.13); then we use (6.11) to eliminate the v_k's from the resulting equation. Rearranging terms, we obtain

$$\mathbf{M}\mathbf{Z}_b(D)\mathbf{M}^T\mathbf{i} = -\mathbf{M}\mathbf{v}_s \tag{6.14}$$

or

$$\mathbf{Z}_m(D)\mathbf{i} = \mathbf{e}_s$$

where $\mathbf{Z}_m(D) \triangleq \mathbf{M}\mathbf{Z}_b(D)\mathbf{M}^T$ is recognized as the *mesh impedance matrix operator*. In this example, the matrix equation reads

$$\begin{bmatrix} L_1D + R_3 + \dfrac{1}{C_4D} & MD - \dfrac{1}{C_4D} \\ MD + \alpha R_5 - \dfrac{1}{C_4D} & L_2D + R_5 + \dfrac{1}{C_4D} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} v_{s3} - v_4(0) \\ v_4(0) \end{bmatrix} \tag{6.15}$$

or, in scalar form,

$$\begin{aligned} &L_1\frac{di_1}{dt} + R_3 i_1 + \frac{1}{C_4}\int_0^t i_1(t')\,dt' + M\frac{di_2}{dt} - \frac{1}{C_4}\int_0^t i_2(t')\,dt' \\ &\qquad = v_{s3}(t) - v_4(0) \\ &M\frac{di_1}{dt} + \alpha R_5 i_1 - \frac{1}{C_4}\int_0^t i_1(t')\,dt' + L_2\frac{di_2}{dt} + R_5 i_2(t) + \frac{1}{C_4}\int_0^t i_2(t')\,dt' \\ &\qquad = v_4(0) \end{aligned} \tag{6.16}$$

This system of Eqs. (6.16) is a system of two integrodifferential equations in two unknowns, $j_1(\cdot)$ and $j_2(\cdot)$. The required initial conditions are $j_1(0)$, $j_2(0)$, and $v_4(0)$.

Remark Any system of integrodifferential equations such as (6.16) can always be put in the form of a system of differential equations by the introduction of appropriate variables. Define the waveforms $q_1(\cdot)$ and $q_2(\cdot)$ by

$$q_1(t) = \int_0^t i_1(t')\,dt' \qquad q_2(t) = \int_0^t i_2(t')\,dt' \tag{6.17}$$

Physically, $q_1(t)$ is the net amount of charge (in coulombs) that has passed through R_3 or L_1 in the interval $[0,t]$. In terms of $q_1(\cdot)$ and $q_2(\cdot)$, the integrodifferential system of (6.16) becomes a system of *differential* equations,

$$L_1\ddot{q}_1 + R_3\dot{q}_1 + \frac{1}{C_4}q_1 + M\ddot{q}_2 - \frac{1}{C_4}q_2 = v_{s3}(t) - v_4(0)$$

$$M\ddot{q}_1 + R_5\alpha\dot{q}_1 - \frac{1}{C_4}q_1 + L_2\ddot{q}_2 + R_5\dot{q}_2 + \frac{1}{C_4}q_2 = v_4(0)$$

with the initial conditions

$$q_1(0) = 0 \qquad q_2(0) = 0 \qquad \text{see (6.17)}$$
$$\dot{q}_1 = j_1(0) \qquad \dot{q}_2(0) = j_2(0)$$

Summary

- In the node analysis, the n node-to-datum voltages $e_1, e_2, \ldots, e_n$ are used as network variables. By applying KCL to all the nodes except the datum

node, we obtain n linearly independent equations in terms of branch currents. For linear time-invariant networks, taking the branch equations into account, the n equations can be written explicitly in terms of the n node-to-datum voltages. In general, the resulting network equation gives in matrix form

(*a*) $\mathbf{Y}_n(D)\mathbf{e} = \mathbf{i}_s$

Once $\mathbf{e}$ is determined, the b branch voltages can be obtained immediately from

(*b*) $\mathbf{v} = \mathbf{A}^T\mathbf{e}$

The b branch currents are then obtainable from the branch equations. The writing of the node matrix equation in (*a*) can be done formally using the step-by-step procedure and matrix multiplications. However, for simple circuits which do not contain complicated coupling elements, the node admittance matrix $\mathbf{Y}_n(D)$ as well as the source vector $\mathbf{i}_s$ can be written by inspection. Frequently it is advisable to convert all voltage sources into current sources before starting node analysis.

- In mesh analysis the l mesh currents $i_1, i_2, \ldots, i_l$ are used as network variables, and l linearly independent equations in terms of branch voltages are obtained by applying KVL to all the meshes except the outer mesh. For linear time-invariant networks, taking the branch equations into account, the l equations can be written explicitly in terms of the l mesh currents. In general, the resulting mesh equation gives in matrix form

(*c*) $\mathbf{Z}_m(D)\mathbf{i} = \mathbf{e}_s$

Once $\mathbf{i}$ is determined, the b branch currents can be obtained from

(*d*) $\mathbf{j} = \mathbf{M}^T\mathbf{i}$

The b branch voltages are then obtainable from the branch equations. Again, a step-by-step procedure can be used to write the mesh matrix equation of (*c*); or in case the network does not include complicated coupling elements, $\mathbf{Z}_m(D)$ and $\mathbf{e}_s$ can be obtained by inspection. Frequently, it is advisable to convert all current sources into voltage sources before starting mesh analysis.

- Whereas node analysis is completely general, mesh analysis is restricted to planar networks.
- The two methods are dual to each other in the case of planar networks. The natural question to ask is which is a better method. The answer depends on the given network. In node analysis there are n network variables to be determined, whereas in mesh analysis there are l network variables. Thus, it is reasonable to say that if for a given graph the number of nodes $n + 1$ is much smaller than the number of meshes l, the node

analysis is preferable. If it is the other way around, then the mesh analysis is more advantageous. However, other factors must also be considered. One crucial point deals with the number and kind of sources in the network. If all given sources (dependent and independent) are current sources, node analysis probably is more convenient since one can often write the node equations by inspection. On the other hand, if all given sources are voltage sources, mesh analysis may be easier to use. Experience will help you to decide one way or the other.

- Two connected, unhinged, planar topological graphs are said to be dual to each other if (1) there is a one-to-one correspondence between the meshes of one (including the outer mesh) and the nodes of the other, and vice-versa, and (2) there is a one-to-one correspondence between the branches of each graph such that whenever two meshes of one graph have a branch in common, the corresponding nodes of the other graph have the corresponding branch connecting these nodes.

- Two networks are said to be dual to each other if (1) the topological graph of one is the dual of the topological graph of the other, and (2) the branch equations of one are obtainable from the corresponding branch equations of the other by performing the following substitutions:

$v \to \hat{j} \qquad q \to \hat{\phi}$

$j \to \hat{v} \qquad \phi \to \hat{q}$

- The main facts of node and mesh analysis are tabulated in Table 10.2 in a form that makes the duality of the two methods readily apparent.

Table 10.2 **Summary of Node and Mesh Analyses**

	Node analysis	Mesh analysis
Network variables	$\mathbf{e}$, node-to-datum voltages	$\mathbf{i}$, mesh currents
Basic facts	$\mathbf{Aj} = \mathbf{0}$ (KCL) $\mathbf{v} = \mathbf{A}^T\mathbf{e}$ (KVL)	$\mathbf{Mv} = \mathbf{0}$ (KVL) $\mathbf{j} = \mathbf{M}^T\mathbf{i}$ (KCL)
Linear time-invariant resistive networks	Branch equations	
	$\mathbf{j} = \mathbf{Gv} + \mathbf{j}_s - \mathbf{Gv}_s$	$\mathbf{v} = \mathbf{Rj} + \mathbf{v}_s - \mathbf{Rj}_s$
	Network equations	
	$\mathbf{Y}_n\mathbf{e} = \mathbf{i}_s$	$\mathbf{Z}_m\mathbf{i} = \mathbf{e}_s$
	$\mathbf{Y}_n \triangleq \mathbf{AGA}^T$ $\mathbf{i}_s \triangleq \mathbf{AGv}_s - \mathbf{Aj}_s$	$\mathbf{Z}_m \triangleq \mathbf{MRM}^T$ $\mathbf{e}_s \triangleq \mathbf{MRj}_s - \mathbf{Mv}_s$

Problems

Rank and inverse

1. *a.* Give the rank of the following matrices:

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 0 & 2 & -2 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 & 1 & 2 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -2 & -2 & 2 \\ -1 & 1 & 0 & -2 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 4 & 7 & 9 \\ 4 & 6 & 2 & 0 \\ 1 & 3 & 2 & 7 \end{bmatrix}$$

b. Find the inverse of the following matrices:

$$\mathbf{B}_1 = \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix} \qquad \mathbf{B}_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ -\frac{1}{2} & -\frac{2}{3} \end{bmatrix} \qquad \mathbf{B}_3 = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

Incidence matrices

2. Are the following matrices possible reduced incidence matrices? In each case justify your answer.

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 \end{bmatrix} \quad [1 \quad 0]$$

Node analysis

3. Consider the linear time-invariant network shown in Fig. P10.3.

a. Using the reference directions shown, obtain the oriented graph of the network.

b. Write the expression for KCL and KVL in matrix form ($\mathbf{Aj} = \mathbf{0}$, and $\mathbf{v} = \mathbf{A}^T\mathbf{e}$).

c. Assuming the sinusoidal steady state at frequency ω and the source phasor J_s for the current source, write the node equations in matrix form.

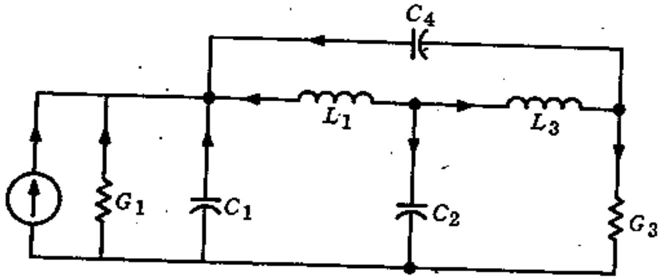


Fig. P10.3

d. Assuming zero initial conditions, obtain the integrodifferential equations of node analysis for the network shown.

Node analysis

4. Consider the linear time-invariant network shown in Fig. P10.3.

a. Obtain by inspection the node equations for the sinusoidal steady state.

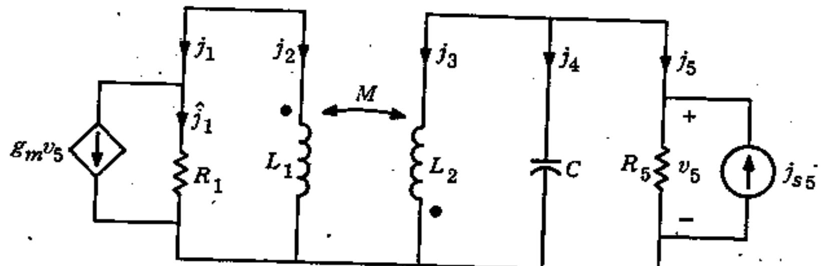
b. Obtain by inspection the node integrodifferential equations for the case in which all initial conditions are zero.

Node analysis

5. Consider the linear time-invariant network shown in Fig. P10.5.

a. Obtain the node integrodifferential equations by the systematic method.

b. Assuming the sinusoidal steady state at frequency ω, obtain the node equations.



$R_1 = R_5 = 10\ \Omega$ $g_m = 1$ mho $L_1 = 10$ H $L_2 = M = 1$ H $C = 1$ F

Fig. P10.5

Node analysis

6. Repeat Prob. 5 with $M = 0$, using the shortcut method.

Node analysis

7. Let the linear time-invariant network shown in Fig. P10.7 be in the sinusoidal steady state. An oriented graph of this network is shown in Fig. P10.7. Use node analysis to find all branch voltages and currents, given $e_s(t) = \cos 10^6 t$.

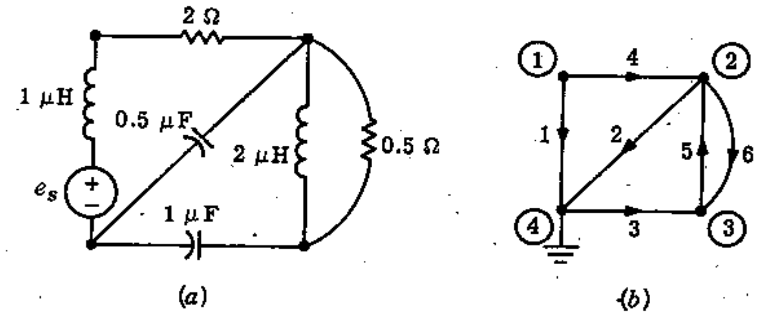


Fig. P10.7

Node analysis

8. Consider the linear time-invariant network shown in Fig. P10.8.

a. Assume it is in the sinusoidal steady state and write its node equations by inspection.

b. Assume all initial conditions to be zero, and write the integrodifferential node equation.

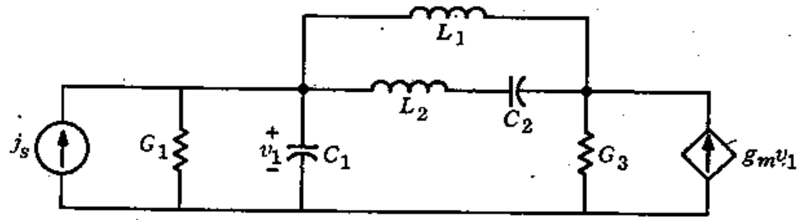


Fig. P10.8

Duality

9. Find the dual of the general resistive branch shown in Fig. P10.9. By transforming the voltage source in the dual branch, show that the dual

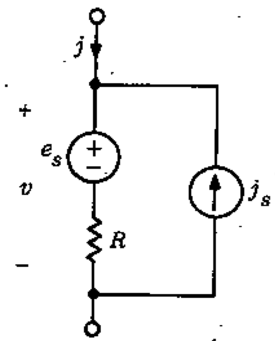


Fig. P10.9

takes the same form as the original with $\hat{e}_s = -j_s$ and $\hat{j}_s = -e_s$. (Hint: If a voltage source appears in series with a current source, the voltage source may be shorted out.)

Dual graphs

10. For each of the topological graphs shown in Fig. P10.10, give two different dual topological graphs.

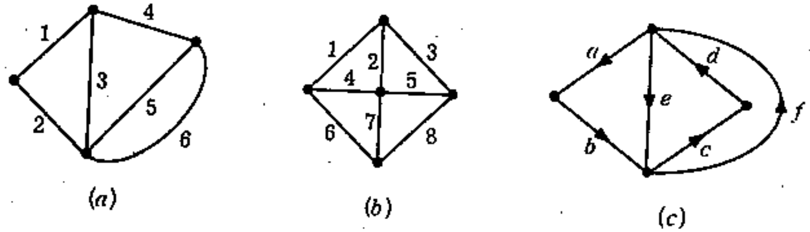


Fig. P10.10

ual networks

11. Given the nonlinear network $\mathfrak{N}$ shown in Fig. P10.11, find a dual network $\hat{\mathfrak{N}}$. (Be sure to specify $\hat{\mathfrak{N}}$ completely.) The branch equations for $\mathfrak{N}$ are

$v_R = 2j_R + 3j_R^2$ $\qquad j_L = 10^{-2}\phi_L + \tanh \phi_L$

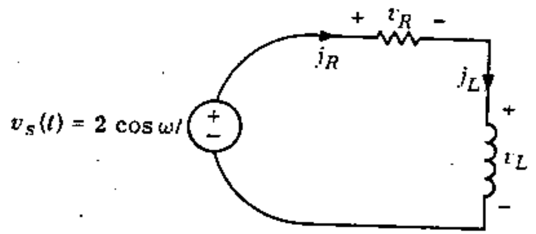
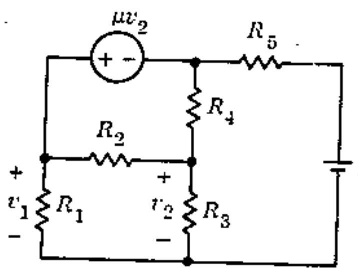


Fig. P10.11

h analysis

12. The circuit shown in Fig. P10.12 is made of linear time-invariant elements. Calculate the voltage v_1.



$E = 5$ volts

$R_1 = R_2 = R_4 = R_5 = 1\ \Omega$

$R_3 = 2\ \Omega$

$\mu = 2$

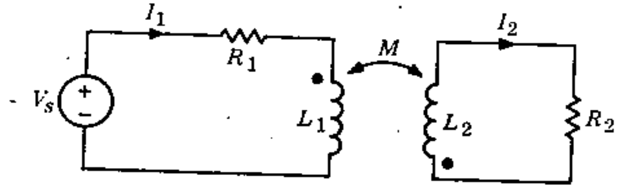
g. P10.12

Mesh analysis

13. The network shown in Fig. P10.13 is in the sinusoidal steady state with angular frequency $\omega = 2$ rad/sec.

a. Find the phasors I_1 and I_2.

b. What is the driving-point impedance seen by the voltage source?



$L_1 = M = 1$ H $\qquad L_2 = 2$ H $\qquad R_1 = R_2 = 1\ \Omega$

Fig. P10.13

Mesh analysis

14. The linear time-invariant circuit shown in Fig. P10.14 represents the Maxwell bridge, an instrument used to measure the inductance L_x and resistance R_x of a linear time-invariant physical inductor. The measurement is made with the circuit in sinusoidal steady state by adjusting R_1 and C until $v_A = v_B$ (then the bridge is said to be "balanced"). It is important to note that when balance is achieved, the current through the detector D is zero regardless of the value of the detector impedance.

Show that when the bridge is balanced,

$$L_x = R_2R_3C \qquad R_x = \frac{R_2R_3}{R_1}$$

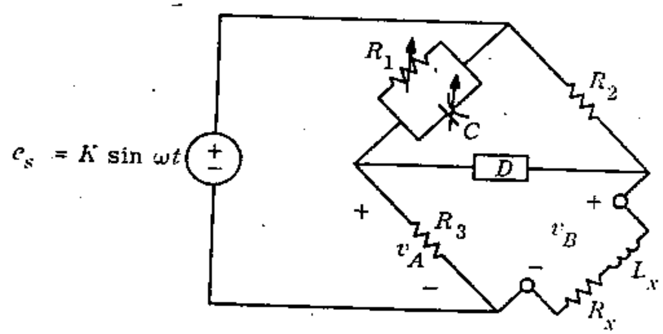


Fig. P10.14

Mesh analysis

15. Consider the linear time-invariant network shown in Fig. P10.3.

a. Obtain the mesh matrix for this network.

b. Write KVL and KCL in matrix form

c. Assuming the sinusoidal steady state at frequency ω, use (a) and (b) to obtain the mesh equations.

d. Assume the following initial conditions:

$j_{L_1}(0) = 1 \quad j_{L_2}(0) = 0 \qquad v_{C_1}(0) = 0 \qquad v_{C_2}(0) = 1$

Obtain the integrodifferential mesh equations.

esh analysis

16. Repeat (c) and (d) of Prob. 15 by inspection.

esh analysis

17. Consider the linear time-invariant network of Fig. P10.5.

a. Obtain the mesh matrix for this network.

b. Assuming the network is in sinusoidal steady state, write the mesh equations.

c. Obtain, by the systematic method, the integrodifferential mesh equations for zero initial conditions.

esh analysis

18. Repeat the previous problem using the shortcut method.

esh analysis

19. Write the mesh equations of the circuit shown in Fig. P10.19 using the charge q_1 and q_2 as variables. Indicate the necessary initial conditions. At $t = 0$ the current in the inductor is I_0, and the voltage across the capacitor is V_0.

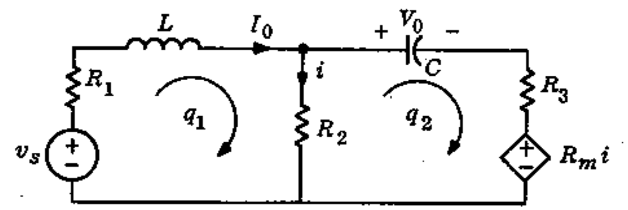
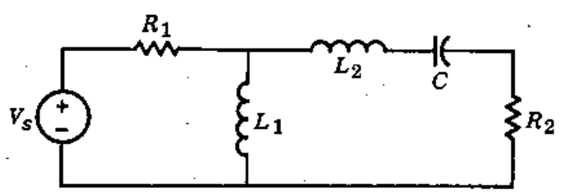


Fig. P10.19

Node and esh analysis

20. For the linear time-invariant network shown in Fig. P10.20, which is assumed to be in the sinusoidal steady state,



$L_1 = L_2 = 2\,\text{H}$
$R_1 = R_2 = 3\,\Omega$
$C = 4\,\text{F}$

Fig. P10.20

a. Draw the network graph; call it $\mathcal{G}$.

b. Draw the dual graph; call it $\hat{\mathcal{G}}$.

c. Obtain the dual network; call it $\hat{\mathcal{N}}$.

d. Write the mesh equations for the given network and the node equations of $\hat{\mathcal{N}}$.

e. Compare the equations in (d).

Dual networks

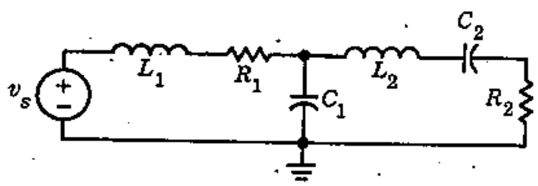
21. Consider the linear time-invariant network $\mathcal{N}$ shown in Fig. P10.21.

a. Find a dual network of $\mathcal{N}$; call if $\hat{\mathcal{N}}$.

b. Assuming that the sinusoidal steady state has been reached, write by inspection the mesh equations of $\mathcal{N}$ in terms of voltage and current phasors; that is, $\mathbf{Z}_m\mathbf{I} = \mathbf{E}_s$, where $\mathbf{Z}_m$ is the mesh impedance matrix, $\mathbf{I}$ is the phasor which represents the mesh current vector, and $\mathbf{E}_s$ is the phasor which represents the mesh voltage source vector.

c. Assuming that the sinusoidal steady state has been reached, write by inspection the node equations of $\hat{\mathcal{N}}$ in terms of voltage and current phasors; that is, $\mathbf{Y}_n\mathbf{E} = \mathbf{I}_s$, where $\mathbf{Y}_n$ is the node admittance matrix, $\mathbf{E}$ is the phasor which represents the node-to-datum voltage vector, and $\mathbf{I}_s$ is the phasor which represents the node current source vector.

d. Solve for the mesh currents of $\mathcal{N}$ and the node-to-datum voltages of $\hat{\mathcal{N}}$. Give your results in the form of real functions of time.



$v_s = 5 \sin 3t$ volts
$R_1 = 2\,\Omega$
$R_2 = 1\,\Omega$
$L_1 = 3\,\text{H}$
$L_2 = 2\,\text{H}$
$C_1 = 1\,\text{F}$
$C_2 = 2\,\text{F}$

Fig. P10.21

11 Loop and Cut-set Analysis

In the previous chapter we have learned to perform systematically the node analysis of *any* linear time-invariant network. We have also learned to perform the mesh analysis for any such network provided its graph is planar. In this chapter we briefly discuss two generalizations, or perhaps variations, of these methods, namely, the cut-set analysis and the loop analysis. There are two reasons for studying loop and cut-set analysis: first, these methods are useful because they are much more flexible than mesh and node analysis, and, second, they use concepts and teach us points of view that are indispensable for writing state equations.

In Sec. 1, we introduce some new graph-theoretic concepts and prove a fundamental theorem. In Sec. 2, we study loop analysis, and in Sec. 3 we study cut-set analysis. Section 4 is devoted to comments on these methods. In Sec. 5 we establish a basic relation between the loop matrix **B** and the cut-set matrix **Q**.

1 Fundamental Theorem of Graph Theory

In order to develop this theorem we need to indicate precisely what we mean by a *tree*. Let $\mathcal{G}$ be a connected graph and T a subgraph of $\mathcal{G}$. We say that T is a **tree of the connected graph** $\mathcal{G}$ if (1) T is a *connected* subgraph, (2) it contains *all the nodes* of $\mathcal{G}$, and (3) it contains *no loops*.

Given a connected graph $\mathcal{G}$ and a tree T, the branches of T are called **tree branches**, and the branches of $\mathcal{G}$ not in T are called **links**. (Some authors call them cotree branches, or chords.)

A graph has usually many trees. In Fig. 1.1 we show a few trees of a connected graph $\mathcal{G}$. To help you understand the definition, in Fig. 1.2 we show a few subgraphs (of the same graph $\mathcal{G}$) which are *not* trees of $\mathcal{G}$. To emphasize the fact that complicated graphs have many trees, remember that if a graph has n_t nodes and has a single branch connecting every pair of nodes, then it has $n_t^{n_t-2}$ trees. For such graphs, when $n_t = 5$, there are 125 trees; when $n_t = 10$, there are 10^8 trees.

Exercise Draw all possible trees for the graph shown in Fig. 1.3.

The following fundamental theorem relates the properties of loops, cut