MATH 420 "Point Set Topology"

Set #3

SOLUTIONS

Section 17:

P4: If U is open in X and A is closed in X, then $X - U$ is closed in X and $X - A$ is open in X. Thus, $U - A = U \cap (X - A)$ is the intersection of two open sets, and hence is open. Also $A - U = A \cap (X - U)$ is the intersection of two closed sets, and hence is closed.

P12: Let A be a subspace of X, X Hausdorff. Given $a, b \in A$ distinct points, since X is Hausdorff we can find U, V open in X, $U \cap V = \emptyset$, $a \in U$, $b \in V$. Then $U' = U \cap A$, $V' = V \cap A$ are open in A, $a \in U'$, $b \in V'$ and $U' \cap V' = \emptyset$. Thus, A is Hausdorff.

P13: X Hausdorff $\Rightarrow \Delta$ closed in $X \times X$: Let (a, b) be in $X \times X - \Delta$, then $a \neq b$ in X. Since X is Hausdorff, we can find U nbhd of a and V nbhd of b such that $U \cap V = \emptyset$. Then $(a, b) \in U \times V$ open in $X \times X$. Also note that if $(z, z) \in U \times V$, then $z \in U \cap V$, a contradiction to U, V being disjoint so $U \times V \subset X \times X - \Delta$. Thus, we see that $X \times X - \Delta$ is open. Hence, Δ is closed.

Δ closed in $X \times X \Rightarrow X$ Hausdorff: Δ closed in $X \times X$ implies that $X \times X - \Delta$ is open. Now if $a, b \in X$ are distinct, then $(a, b) \in X \times X - \Delta$. But since $X \times X - \Delta$ is open we can find basis set $U \times V$ such that $(a, b) \in U \times V \subset X \times X - \Delta$. Since $U \times V$ does not intersect Δ we conclude that U and V are disjoint open sets of X. Since $a \in U$ and $b \in V$, we conclude that X is indeed Hausdorff.

P16: (a) *Standard topology:* closure(K)=$K \cup \{0\}$.
K-topology: closure(K)=K. (To see this, note that

$$\mathbb{R} - K = [(-100, 100) - K] \cup (99, \infty) \cup (-\infty, -99)]$$

and these three pieces are open in the K-topology so $\mathbb{R} - K$ is open in the K-topology, and so K is closed in the K-topology.)
Finite complement topology: closure(K)=$\mathbb{R}$. This is because if $a \in \mathbb{R}$, then a nbhd of a in the finite complement topology is of the form $\mathbb{R} - F$, F a finite set. Since K is infinite we see $\mathbb{R} - F$ must intersect K and so every nbhd of a intersects K, so $a \in$ closure(K).
Upper limit topology: closure(K)=K.

$$\mathbb{R} - K = (-\infty, 0] \cup \bigcup_{n \in \mathbb{Z}_+} A_n \cup (1, \infty)$$

where $A_n = (1/(n+1), 1/n)$. Each of these pieces is open in the upper limit topology (which is finer than the standard topology) and so $\mathbb{R} - K$ is open in the upper limit topology and so K is closed in this topology.
Ray topology, basis $(-\infty, a)$*:* closure(K)=$\{x \in \mathbb{R} | x \geq 0\}$. If $b \in \mathbb{R}, b \geq 0$ and $b \in (-\infty, a)$, then $a > b$ and so $a > 0$. Thus, $(-\infty, a)$ intersects K. So $b \in$ closure(K). On the other hand, if $b < 0$, then $b \in (-\infty, b/2)$ which is disjoint from K, so b is not in closure(K).

(b) Note that Hausdorff $\Rightarrow T_1$. Standard Topology: Hausdorff.
K-topology: Hausdorff (finer than standard).
Finite complement topology: T_1 but not Hausdorff. This is since $\{x\}$ has complement $\mathbb{R} - \{x\}$ which is open in finite complement topology since $|\{x\}| = 1$ is finite. Thus, $\{x\}$ is closed in this topology. On the other hand, given a, b distinct, U nbhd of a, V nbhd of B in this topology, then $U = \mathbb{R} - F$, $V = \mathbb{R} - F'$ where F, F' are finite. Since $F \cup F'$ is finite and $\mathbb{R}$ is infinite there must be $x \in \mathbb{R} - (F \cup F')$, and this x will be in both U and V, so U and V intersect. Thus, this topology is not Hausdorff.
Upper Limit Topology: Hausdorff (finer than standard.)
Ray topology: Neither. It is not T_1 since if $\{x\}$ is a point, then $\mathbb{R} - \{x\} = (-\infty, x) \cup (x, \infty)$, and if $y \in (x, \infty)$, no basis element around y lies inside $\mathbb{R} - \{x\}$. Thus, $\mathbb{R} - \{x\}$ is not open in this topology and so $\{x\}$ is not closed in this topology. If a space is not T_1, it cannot be Hausdorff. (contrapositive of the statement at the top of this question's solution)

P18: Closure(A)=$A \cup \{((0,1))\}$. Closure(B)=$B \cup \{((1,0))\}$.
Closure(C)=$C \cup \{((x,1)) | 0 \leq x < 1\} \cup \{((1,0))\}$.
Closure(D)=$D \cup \{((x,1)) | 0 \leq x < 1\} \cup \{((x,0)) | 0 < x \leq 1\}$.
Closure(E)=$E \cup \{((1/2,1)), ((1/2,0))\}$.

P20: **(a)** Int(A)=$\emptyset$, Boundary(A)=A.
(b) Int(B)=B, Boundary(B)=(y-axis) $\cup$ (nonnegative x-axis)
(c) Int(C)=$\{((x,y)) | x > 0\}$, Boundary(C)=(y-axis) $\cup$ (negative x-axis).
(d) Int(D)=$\emptyset$, Boundary(D)=$\mathbb{R}^2$.
(e) Int(E)=$\{((x,y)) | 0 < x^2 - y^2 < 1\}$,
Boundary(E)=$\{((x,y)) | x^2 - y^2 = 1 \text{ or } 0\}$.
(f) Int(F)=$\{((x,y)) | x \neq 0, y < 1/x\}$,
Boundary(F)=(y-axis) $\cup$ $\{((x,y)) | x \neq 0, y = 1/x\}$.