M E T U
Department of Mathematics

	COMPLEX CALCULUS MidTerm 2	
Code : *Math 353* Acad. Year : *2016-2017* Semester : *Spring* Instructor : *Fırat Arıkan* Date : *May.12.2017* Time : *12:40* Duration : *90 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 5 QUESTIONS ON 4 PAGES TOTAL 70 POINTS	

1	2	3	4	5		
S	O	L	U	T	IONS	SHOW YOUR WORK

Q1 (4+6 pts) (a) State the Liouville Theorem.

Hypothesis:

IF $f(z)$ is entire, and $|f(z)|$ is bounded on $\mathbb{C}$,

Conclusion:

THEN $f(z) \equiv$ constant on $\mathbb{C}$.

(b) Suppose that f, g are entire functions satisfying $\lim_{|z|\to\infty} |f(z) - g(z)| = 1$. Prove that there exists a constant $c \in \mathbb{C}$ such that

$$f(z) - g(z) = c, \quad \forall z \in \mathbb{C}.$$

Soln b) By assumption for $\varepsilon = 1$, $\exists r \in \mathbb{R}_+$ s.t. $|z| \geq r \Rightarrow$

$\big|\, |f(z)-g(z)| - 1 \,\big| < 1$

(1)

$\Rightarrow |f(z)-g(z)| - 1 < 1 \Rightarrow$ $|f(z)-g(z)| < 2$ on $\{|z| \geq r\}$.

Also f, g entire $\Rightarrow |f|, |g|$ takes their max (MMP) values M_f, M_g on $\{|z| \leq r\}$

(2)

$\Rightarrow |f(z)-g(z)| \leq |f(z)| + |g(z)| = M_f + M_g$ on $\{|z| \leq r\}$

Let $M = \max\{2, M_f + M_g\}$.

Then (1) & (2) $\Rightarrow$ $|f(z)-g(z)| < M \quad \forall z \in \mathbb{C}$

$\therefore$ $f(z) - g(z) \equiv$ constant by Liouville Thm on $\mathbb{C}$

$\square$.

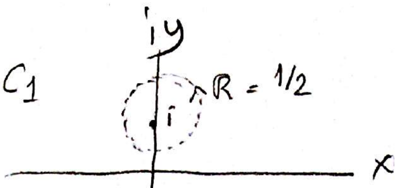
Q2 (8+8+8+8 pts) Evaluate $\oint_C \frac{z+1}{z^2+1}\,dz$ for each curve C given below:

(In each part, write your supporting reasons explicitly!!!)

(a) $C = C_1$: $2|z-i| = 1$ with positive orientation.

$\oint_C \frac{z+1}{(z-i)(z+i)}\,dz$ $\quad f(z) = \frac{z+1}{z+i}$ is analytic on $\mathbb{C}\setminus\{-i\}$, is analytic on C_1

C_1

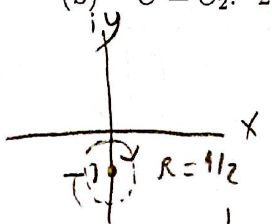


C_1 is closed clockwise direction

by Couchy Integral formula

$$\oint_{C_1} \frac{f(z)}{(z-i)}\,dz = 2\pi i f(i) = \cancel{2}\pi \cancel{i}\cdot\frac{(i+1)}{\cancel{2i}} = i\pi + \pi$$

(b) $C = C_2$: $2|z+i| = 1$ with negative orientation.



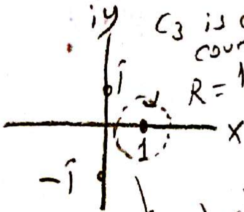
C_2 is closed counter clock wise direction

$f(z) = \frac{z+1}{z-i}$ is analytic on $\mathbb{C}\setminus\{i\}$, is analytic on C_2

$$\oint_C \frac{f(z)\,dz}{z+i} = -\oint_{-C_2} \frac{f(z)\,dz}{z+i} = -2\pi i f(-i) = -\cancel{2}\pi \cancel{i}\,\frac{(1-i)}{\cancel{-2i}} = \pi - i\pi$$

by Cauchy Integral formula

(c) $C = C_3$: $2|z-1| = 1$ with negative orientation.



C_3 is closed counter clockwise direction

i and -i is not in circle

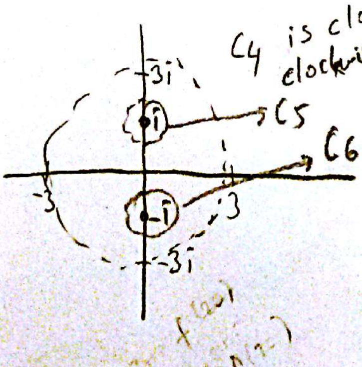
$$\oint_C f(z)\,dz = \oint_C \frac{z+1}{(z-i)(z+i)}\,dz = 0 \text{ (by Cauchy Thm)}$$

f is analytic on $\mathbb{C}-\{i\}$, $\mathbb{C}-\{-i\}$

f is analytic in C_3

so

(d) $C = C_4$: $|z| = 3$ with positive orientation.



C_4 is closed clockwise direction

$$\oint_C \frac{z+1}{(z-i)(z+i)}\,dz = \oint_{C_5} \frac{f_1(z)\,dz}{z-i} + \oint_{C_6} \frac{f_2(z)\,dz}{z+i}$$

$$= 2\pi i\cdot f_1(i) + 2\pi i\cdot f_2(-i)$$

$f_1(z) = \frac{z+1}{z+i}$ is analytic on $\mathbb{C}-\{-i\}$, is analytic in C_5

$f_2(z) = \frac{z+1}{z-i}$ is analytic on $\mathbb{C}-\{i\}$, in C_6

$$\cancel{2}\pi \cancel{i}\cdot\frac{i+1}{\cancel{2i}} + \cancel{2}\pi \cancel{i}\cdot\frac{1-i}{\cancel{-2i}}$$

$$\pi i + \pi + \pi + \pi i = 2\pi i$$

by Cauchy Integral formula

Q3 (4+4+4 pts) (a) State the Maximum Modulus Principle: (MMP)

IF $A \subset \mathbb{C}$ is open, connected and bounded, $f: cl(A) \to \mathbb{C}$ is analytic on A and continuous on $cl(A)$,

THEN $|f(z)|$ takes its max. value on $cl(A)$ at some point $z_0 \in Bdry(A)$.

ALSO IF the max value $|f(z_0)|$ is attained also at an interior point $z_1 \in int(A)$,

THEN f must be constant on $cl(A)$.

(b) Let $f(z)$ be a nowhere-zero analytic function defined on a domain

$$D = \{z \in \mathbb{C} : |z + 2 - i| \leq 1\}.$$

Show that $|f(z)|$ attains its absolute minimum at a point z_0 on the boundary ∂D of D.

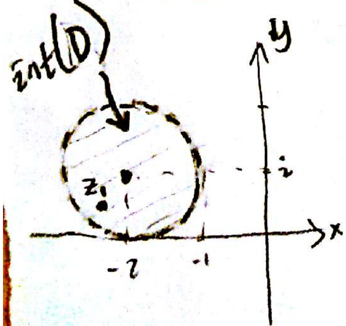
Consider $g(z) = \dfrac{1}{f(z)}$. Since $f(z) \neq 0 \ \forall z \in D$ & f is analytic on D, $g(z)$ is well-defined and analytic on D. So, by MMP,

$\exists z_0 \in \partial D$ s.t. $|g(z_0)| = \max\{|g(z)| \mid z \in D\}$

$\therefore \ |f(z_0)| = \left|\dfrac{1}{g(z_0)}\right| = \min\{|f(z)| \mid z \in D\}$

(That is, $|f(z)|$ attains its abs. min value at a point $z_0 \in \partial D$.) $\square$.

(c) For $f(z)$ in part (b), if we further assume that $\underbrace{f(-5/2+i/2) = f(z) \text{ for all } z \text{ satisfying } |z-(-2+i)| = 1}_{(*)}$, then show that f must be constant on D.



Note that the point $z_1 = -\frac{5}{2} + \frac{i}{2}$ satisfies

$$|z_1 - (-2+i)| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + \left(-\tfrac{i}{2}\right)^2} = \sqrt{\tfrac{1}{4}+\tfrac{1}{4}} = \sqrt{\tfrac{1}{2}} < 1$$

$\Rightarrow \ z_1 \in int(D)$. So, the given condition $(*)$ implies that $g(z_1) = g(z_0) \Rightarrow |g(z)|$ takes its max value at an interior point $z_1 \in int(D)$.

$\Rightarrow \ g \equiv$ constant on D by MMP. $\therefore \ f = \frac{1}{g} \equiv$ constant on D $\blacksquare$.

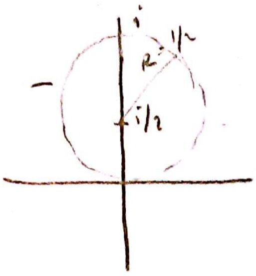
Q4 (8 pts) Find the center, radius and circle of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{(2z-i)^{k+2}}{k+1}.$$

Apply Ratio test; $\lim_{n\to\infty} \left| \dfrac{\frac{(2z-i)^{n+3}}{n+2}}{\frac{(2z-i)^{n+2}}{n+1}} \right| = \lim_{n\to\infty} \left| (2z-i) \cdot \frac{n+1}{n+2} \right| = \lim_{n\to\infty} \left(\frac{n+1}{n+2} |2z-i| \right)$

$= |2z-i| < 1$ (for being convergence.) $\Rightarrow 2\left|z-\frac{i}{2}\right| < 1$

$\Rightarrow \boxed{\left|z-\frac{i}{2}\right| < \frac{1}{2}}$

Circle of conv. 

8

Circle of convergence.

Hence radius of conv. $\boxed{R = \frac{1}{2}}$

Thus $\boxed{z_0 = \frac{i}{2}}$ - center of conv.

Q5 (8 pts) Prove that $\sum_{k=1}^{\infty} \frac{i^k}{k}$ is conditionally convergent.

Soln:

$$\boxed{\bigstar = \sum_{k=1}^{\infty} \frac{i^k}{k}}$$

First, $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent by p-test ($p=1$) $\Rightarrow \bigstar$ is not abs. convergent.

Now, Call $a_k = \frac{i^k}{k}$ & let $\{S_n\} :=$ The sequence of partial sums of $\bigstar$.

Consider odd & even terms (seperately):

$a_1 + a_3 + a_5 + a_7 + \dots + a_{2n+1} = \sum_{k=0}^{n} a_{2k+1} =: A_n$

$a_2 + a_4 + a_6 + a_8 + \dots + a_{2n} = \sum_{k=1}^{n} a_{2k} =: B_n$

Note: odd & even indexed terms are alternating seperately

$a_1 = i$

$a_2 = -\frac{1}{2}$

$a_3 = -\frac{i}{3}$

$a_4 = \frac{1}{4}$

$a_5 = \frac{i}{5}$

$a_6 = -\frac{1}{6}$

$a_7 = -\frac{i}{7}$

$a_8 = \frac{1}{8}$

Observe that:

$S_{2n+1} = A_n + B_n$

$S_{2n} = A_{n-1} + B_n$

Also

$a_{2k+1} = \frac{(-1)^k i}{2k+1}$

$a_{2k} = \frac{(-1)^k}{2k}$

$\Downarrow$

$\{A_n\}$ is convergent by AST
$\{B_n\}$ is convergent by AST
AST: Alternating Series Test

$\Downarrow$

Both $\{S_{2n+1}\}$ and $\{S_{2n}\}$ are convergent

$\Downarrow$

$\{S_n\}$ is convergent

$\therefore \bigstar = \sum_{k=1}^{\infty} \frac{i^k}{k} = \lim_{n\to\infty} \overbrace{\sum_{k=1}^{n} \frac{i^k}{k}}^{S_n}$ is convergent $\square$

Scanned by CamScanner