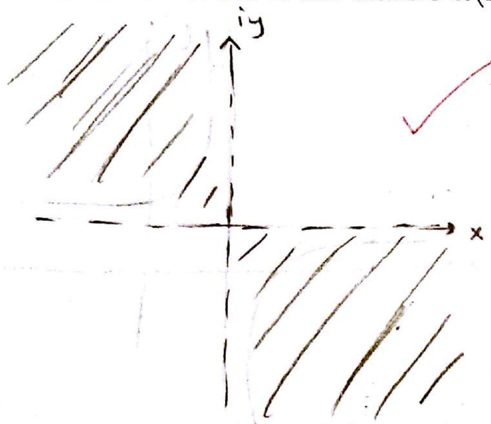
eight

Q1 (6 +6 pts) (a) Sketch the set $A = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z^2) < 0\} \subset \mathbb{C}$. Is A open? closed? bounded? What is the closure $cl(A)$ of A?

Let $z = x+iy$ Then $A = \{z \in \mathbb{C} \,|\, 2xy < 0\}$

$z^2 = x^2 - y^2 + 2xyi$ $A = \{z \in \mathbb{C} \,|\, xy < 0\}$



A is open since $\mathrm{int}(A) = A$.

A is not closed since $cl(A) \neq A$.

$cl(A) = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z^2) \leq 0\}$

A is not bounded because the regions are infinite regions and infinite region can not be bounded

(b) Find all fourth roots z_0, z_1, z_2, z_3 of $81i$.

Let $81i = 3^4 i = z^4 \Rightarrow z^4 = 3^4 \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

We know that $z_k = \sqrt[4]{3^4} \cdot \left(\cos\left(\frac{\pi}{2\cdot 4} + \frac{2\pi k}{4}\right) + i\sin\left(\frac{\pi}{24} + \frac{2\pi k}{4}\right)\right)$

$z_0 = 3 \cdot \left[\cos\left(\frac{\pi}{8}\right) + i \cdot \sin\left(\frac{\pi}{8}\right)\right] = 3e^{i\frac{\pi}{8}}$

$z_1 = 3 \left[\cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)\right] = 3 \cdot e^{i\frac{5\pi}{8}}$

$z_2 = 3 \left[\cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)\right] = 3 \cdot e^{i\frac{9\pi}{8}}$

$z_3 = 3 \left[\cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{7}\right)\right] = 3 \cdot e^{i\frac{13\pi}{8}}$

Q2 (6 pts) Compute the limit if it exists. If not, explain why it does not.

$$\lim_{z\to 1+2i} \frac{z^2 + 3 - 4i}{z - 1 - 2i}$$

$\lim_{z\to 1+2i} \frac{(z-1-2i)(z+1+2i)}{(z-1-2i)} = \lim_{z\to 1+2i} z+1+2i = 1+2i+1+2i = \underline{\underline{2+4i}}$

limit exists and it equal

Q3 (5+5 pts) (a) Find all values of $\log(\overbrace{-\sqrt{2}-\sqrt{2}i}^{z})$.

$|z| = \sqrt{2+2} = 2$

$\arg(z) = \frac{5\pi}{4}$

$\log(-\sqrt{2}-\sqrt{2}i) = \log(2) + i\cdot\arg(z)$

$= \log(2) + i\cdot\left(\frac{5\pi}{4} + \underbrace{2\pi n}_{\text{since we want to get all values}}\right) = \log(2) + i\pi\left(\frac{5}{4} + 2n\right)$ where $n \in \mathbb{Z}$

(b) Compute the principal value of $(\sqrt{3}-i)^i$

$a^b = e^{b\log a}$.

$(\sqrt{3}-i)^i = e^{i\cdot \mathrm{Log}(\sqrt{3}-i)} = e^{i[\log(2) - \frac{\pi i}{6}]} = e^{i\log(2) + \frac{\pi}{6}} = e^{i\log(2)}\cdot e^{\pi/6}$

$\left[\begin{aligned} \mathrm{Log}(\sqrt{3}-i) &= \log|\sqrt{3}-i| + i\cdot \mathrm{Arg}(\sqrt{3}-i) \\ &= \log(2) + i\cdot\left(-\frac{\pi}{6}\right)\end{aligned}\right]$

$= e^{\pi/6}\left(\cos(\log(2)) + i\cdot\sin(\log(2))\right)$

Q4 (6+6 pts) (a) Find (explicitly sketch) the image $f(A)$ of the set

$$A = \{z = x + iy \mid 0 < y^2 - x^2 < 3\}$$

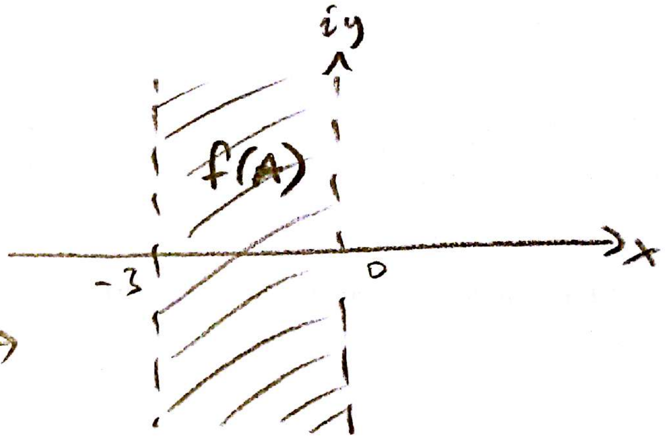
under the mapping $w = f(z) = z^2$.

$$f(z) = z^2 = \underbrace{x^2 - y^2}_{u} + \underbrace{2xy}_{v}\, i$$

$$\underbrace{A:\ 0 < y^2 - x^2 < 3}$$

$$\downarrow f$$

$$\boxed{f(A):\ -3 < u < 0 \quad \text{no condition on } v}$$



(b) Find (explicitly sketch) the image $g(B)$ of the set

$$B = \{z = x + iy \mid 0 \le x < \infty,\ 0 \le y \le \pi/6\}$$

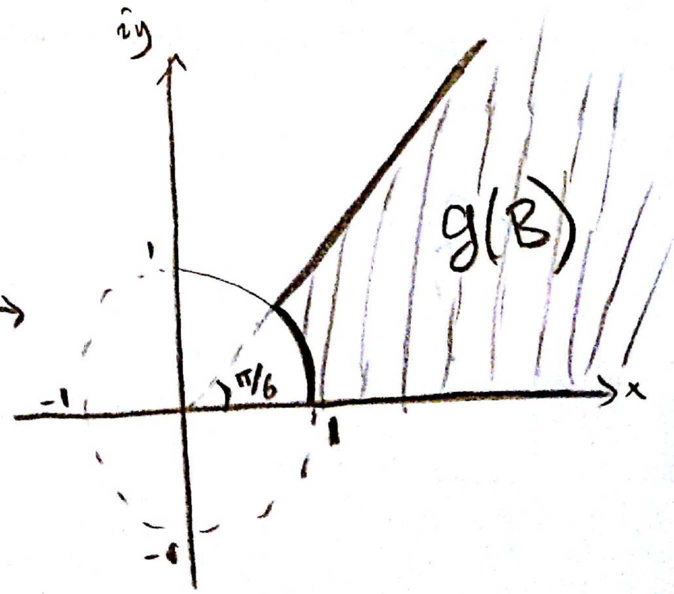
under the mapping $w = g(z) = e^z$.

$$e^z = e^{x+iy} = e^x e^{iy} \left(= r \cdot e^{i\theta}\right)$$

$$\underbrace{B:\ 0 \le x < \infty,\ 0 \le y \le \pi/6}$$

$$\downarrow g$$

$$\boxed{g(B):\ 1 \le \underset{e^x}{\underset{\parallel}{r}} < \infty,\ 0 \le \underset{y}{\underset{\parallel}{\theta}} \le \pi/6}$$



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Q5 (5+5 pts) Consider the function $f(z) = \underbrace{2xy}_{u} + \underbrace{(x^2 + y^2)}_{v} i$ for $z = x + yi \in \mathbb{C}$.

(a) Where is f complex differentiable? Explain?

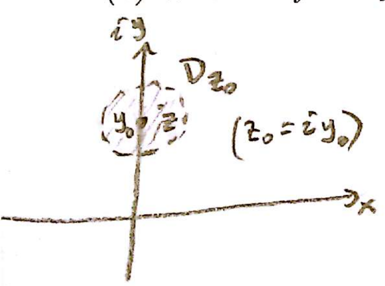
To be $\mathbb{C}$-diff'lable at $z = x + iy$, C-R Eqn's must hold at z:

$u_x = 2y = v_y$ holds everywhere.

But $u_y = 2x$, $v_x = 2x \Rightarrow u_y = -v_x$ holds only if $2x = -2x \Leftrightarrow x = 0$

$\therefore$ f is $\mathbb{C}$-differentiably on the imaginary line $\{x + iy \mid x = 0\}$.

(b) Where is f analytic? Explain?



To be analytic at z_0 C-R Eqn's must hold on an open disk D_{z_0} centered at z_0. But this can not be true for the given f: Even for a point $z_0 = iy_0$ on the imaginary axis where C-R Eqn's hold, any nbhd D_{z_0} of z_0 contains a point z which is not contained in the axis.

$\therefore$ f is analytic nowhere!

Q6 (10 pts) Determine the entire function $f(z) = u + iv$ satisfying

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$$\text{Re}(f(z)) = u(x,y) = (e^x - e^{-x})\cos(y) \quad = e^x\cos(y) - e^{-x}\cos y$$

and $f(0) = 0$. Also compute $f'(z)$.

$f(z) = (e^x - e^{-x})\cos(y) + i \cdot v(x,y)$

Since f is entire, C-R equations hold. $\boxed{\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}}$

$\underbrace{u_x = e^x\cos y + e^{-x}\cos y = v_y}$

Integrate w.r.t y.

$\Rightarrow \boxed{v(x,y) = e^x.\sin y + e^{-x}\sin y + h(x).}$

Diff. w.r.t x

$u_y = \boxed{-e^x\sin y + e^{-x}\sin y = -v_x}$

$\Rightarrow v_x = e^x.\sin y - e^{-x}\sin y + h'(x)$ So $h'(x) = 0 \Rightarrow h(x) = c$ (constant)

$\therefore v(x,y) = e^x.\sin y + e^{-x}.\sin y + c$

$\therefore f(z) = e^x\cos y - e^{-x}\cos y + i(e^x.\sin y + e^{-x}\sin y + c)$

$f((0,0)) = 1 - 1 + i(0 + 0 + c) = i.c = 0 \Rightarrow \boxed{c = 0}$

So $\boxed{f(z) = e^x\cos y - e^{-x}\cos y + i(e^x\sin y + e^{-x}\sin y)}$

$\boxed{f'(z) = e^x\cos y + e^{-x}\cos y + i(e^x\sin y - e^{-x}\sin y)}$

Q7 (5+5 pts) (a) Suppose that $f(z)$ is analytic on a domain D. Show that if $|f(z)|$ is constant on the whole D, then so is $f(z)$.

$$\left.\begin{array}{l} f = u + iv \\ |f(z)| = c \end{array}\right\} \Rightarrow u^2 + v^2 = c^2 \Rightarrow \begin{cases} 2u \cdot u_x + 2v \cdot v_x = 0 \\ 2u \cdot u_y + 2v \cdot v_y = 0 \end{cases}$$

$$\left.\begin{array}{l} \langle u, v\rangle \cdot \langle u_x, v_x\rangle = 0 \\ \langle u, v\rangle \cdot \langle u_y, v_y\rangle = 0 \end{array}\right\} \Rightarrow \langle u_x, v_x\rangle \,//\, \langle u_y, v_y\rangle \Rightarrow \begin{array}{l} u_x = k u_y \\ v_x = k v_y \end{array} \quad \text{for some } k \neq 0$$

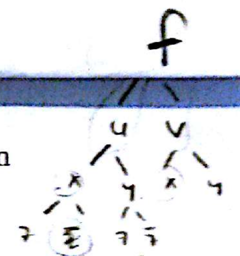
Also, since f is analytic, C-R Eqn's hold $\Rightarrow \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}$.

So, $$\left.\begin{cases} u_x = k u_y = -k v_x = -k^2 v_y = -k^2 u_x \Rightarrow (1+k^2) u_x = 0 \Rightarrow u_x = 0 \\ u_y = -v_x = -k v_y = -k u_x = -k^2 u_y \Rightarrow (1+k^2) u_y = 0 \Rightarrow u_y = 0 \end{cases}\right\} \Rightarrow u \text{ constant.}$$

Similarly, $\left.\begin{array}{l} v_x = 0 \\ v_y = 0 \end{array}\right\} \Rightarrow v$ constant

$\therefore$ $f = u + iv$ = constant. $\square$.

(b) For any complex function $f(z) = u + vi$, prove that u, v satisfy Cauchy-Riemann Equations on a domain D if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ on D. let $z = x + iy$



C-R eqns $\Rightarrow$ $\boxed{\begin{array}{l} u_x = v_y \\ v_x = -u_y \end{array}}$

Think $x = \frac{z + \bar{z}}{2}$, $y = \frac{z - \bar{z}}{2i}$ as a func. of z and $\bar{z}$.

$\cancel{\frac{x + iy - x + iy}{2i}}$

Look $$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$= \left(\underset{1}{\frac{\partial f}{\partial u}} \cdot \frac{\partial u}{\partial x} + \underset{i}{\frac{\partial f}{\partial v}} \cdot \frac{\partial v}{\partial x} \right) \cdot \frac{\partial x}{\partial \bar{z}} + \left(\underset{1}{\frac{\partial f}{\partial u}} \frac{\partial u}{\partial y} + \underset{i}{\frac{\partial f}{\partial v}} \cdot \frac{\partial v}{\partial y} \right) \frac{\partial y}{\partial \bar{z}}$$

$$= (u_x + i v_x) \frac{1}{2} + (u_y + i v_y) \left(-\frac{1}{2i}\right)$$

$$= \frac{1}{2} u_x + \frac{1}{2} i v_x - \frac{1}{2i} u_y - \frac{1}{2} v_y = 0$$

5

$u_x = v_y$
$v_x = -u_y$

$$= \frac{1}{2}(u_x - v_y) + \frac{1}{2} i v_x - \frac{1}{2i} u_y = 0$$

$\underset{\text{iff}}{\Longleftrightarrow}$ $\boxed{u_x = v_y}$ & $\boxed{v_x = -u_y} = 0$..

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