METU
Department of Mathematics

Code : *Math 353* Acad. Year : *2016-2017* Semester : *Spring* Instructor : *Fırat Arıkan* Date : *June.06.2017* Time : *17:00* Duration : *100 minutes*	**COMPLEX CALCULUS** **Final Exam** Last Name : Name : Student No. : Department : Section : Signature : 5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5		
S	O	L	U	T	IONS	**SHOW YOUR WORK**

Q1 (6+6+6+6 pts) Consider the function $f(z) = \left(\dfrac{\cos z - 1}{z}\right)^2$.

(a) Express f as a power series about $z = 0$ on the largest possible domain?

We know $\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$ $\forall z \in \mathbb{C}$

$$\Rightarrow f(z) = \left[\frac{1}{z}\left(\cancel{1} + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} - \cancel{1}\right)\right]^2 = \left[\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k-1}\right]^2 \quad (\forall z \in \mathbb{C}^*)$$

$$= \left(-\frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots\right)\left(-\frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots\right) = \frac{z^2}{(2!)^2} - 2\frac{z^4}{(2!)(4!)} + \left(2\frac{1}{(2!)(6!)} + \frac{1}{(4!)(4!)}\right)z^6 + \cdots$$

(b) Does f have a limit as $z \to 0$? Verify!

By part a), <u>yes</u> f has a limit (equal to 0) as $z \to 0$. Indeed,

$$\lim_{z\to 0} f(z) \underset{\text{part (a)}}{=} \lim_{z \to 0}\left[\frac{z^2}{(2!)^2} - 2\frac{z^4}{(2!)(4!)} + \left(2\frac{1}{(2!)(6!)} + \frac{1}{(4!)(4!)}\right)z^6 + \cdots\right] = 0$$

(c) Extend f continuously to a function $\tilde{f} : \mathbb{C} \to \mathbb{C}$.

Define $\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in \mathbb{C} \setminus \{0\} \\ 0 & \text{if } z = 0 \end{cases}$

<u>Note</u>: f is cont on $\mathbb{C} \setminus \{0\} \Rightarrow$ so is $\tilde{f}$. Also part (b) $\Rightarrow$
$\lim_{z\to 0} \tilde{f}(z) = \lim_{z \to 0} f(z) \underset{(b)}{=} 0 = \tilde{f}(0) \Rightarrow \tilde{f}$ is cont at $z = 0$. $\therefore \tilde{f}$ is cont on $\mathbb{C}$.

(d) Prove that $\tilde{f}$ is, indeed, entire.

Since $\tilde{f}(0) = 0$, the power series in part (a) can be considered as the Taylor Series of $\tilde{f}$, and clearly equal to $\tilde{f}(z)$ $\forall z \in \mathbb{C}$.

Since any power series is infinitely many times (complex) differentiable on the interior of its region of convergence, we conclude that $\tilde{f}$ is analytic on the whole $\mathbb{C}$, that is, it is entire.

SOLUTIONS

Q2 (8+14 pts) (a) State **Residue Theorem** for a function $f(z)$ which is analytic inside and on a closed contour C, except for a single pole of order m which lies inside C:

IF f is analytic inside and on a closed contour C, except for a single pole z_0 of order m which lies inside C,

THEN
$$\oint_C f(z)\,dz = 2\pi i \cdot \mathrm{Res}(f, z_0)$$
(where $\mathrm{Res}(f,z_0)$ is the residue of f at z_0).

(b) Prove the version of the Residue Theorem stated in part (a) in the particular case where $m = 2$. (Hint: You may need to use other important theorems such as Laurent's Theorem, Cauchy's Theorem, Cauchy's Integral Theorem, etc...)

By Laurent's Thm, $f(z)$ is equal to its Laurent Series on some domain D containing C and its interior except z_0. Also, by the assumption $m=2$, the principal part contains only two terms. Namely,

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + \sum_{k=0}^{\infty} a_k (z-z_0)^k. \quad \forall z \in D \setminus \{z_0\}.$$

Consider
$$\oint_C f(z)\,dz = \underbrace{\oint_C \frac{a_{-2}}{(z-z_0)^2}\,dz}_{\mathrm{I}} + \underbrace{\oint_C \frac{a_{-1}}{z-z_0}\,dz}_{\mathrm{II}} + \underbrace{\oint_C \left(\sum_{k=0}^{\infty} a_k (z-z_0)^k\right) dz}_{\mathrm{III}}$$

Observe every term in III is entire (monomials are entire), and so $\oint_C a_k (z-z_0)^k = 0$ $\forall k \geq 0$ by Cauchy's Thm. Therefore, by term-by-term integration thm we have $\mathrm{III} = 0$.

Also, by CIF for derivatives, we have $\mathrm{I} = 2\pi i \left.\frac{d}{dz}(\underset{\text{constant}}{a_{-2}})\right|_{z=z_0} = 2\pi i \cdot 0 = 0$

On the other hand, by the basic computation, $\mathrm{II} = a_{-1} \oint_C \frac{dz}{z-z_0} = a_{-1} \cdot 2\pi i$

$$\therefore \oint_C f(z)\,dz = 0 + 2\pi i \cdot a_{-1} + 0 = 2\pi i \cdot a_{-1} \underset{\text{def'n.}}{=} 2\pi i \cdot \mathrm{Res}(f, z_0)$$
$\square$

SOLUTIONS

Q3 (10+10+10 pts) Let C denote the unit circle with positive orientation.

(In each part, write your supporting reasons explicitly!!!)

(a) Evaluate $I = \oint_C \left[\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots + \frac{100}{z^{100}}\right] dz$

By CIF for derivatives, $\oint_C \frac{k}{z^k} dz = 2\pi i \, \frac{d^{(k)}}{dz}(\underbrace{k}_{\text{Constant}})\Big|_{z=0} = 2\pi i \cdot 0 = 0 \quad \forall k \geq 2$

But when $k=1$, $\oint_C \frac{1}{z} dz = 2\pi i$

$\therefore \quad I = 2\pi i + \underbrace{0 + \cdots + 0}_{100 \text{ of them}} = 2\pi i$

(b) Evaluate $I = \oint_C \frac{706 e^z \cos 3z}{z^2} dz \overset{\text{Residue Thm}}{=} 2\pi i \cdot \operatorname{Res}(f, 0)$ where $f(z) = \frac{706 e^z \cos 3z}{z^2}$

(By a thm, $z=0$ is a pole of order 2 for f)

$\operatorname{Res}(f,0) = \lim_{z \to 0}\left[\frac{1}{(2-1)!} \frac{d}{dz}\left(z^2 \cdot f(z)\right)\right]$

$= \lim_{z\to 0}\left[\frac{d}{dz}\left(706 e^z \cos 3z\right)\right] = \lim_{z \to 0}\left[706 e^z \cos 3z - \overset{\to 0}{706 e^z \cdot 3 \cdot \sin 3z}\right] = 706$

$\therefore \quad I = 2\pi i \cdot 706 = 1412\pi i$

(c) Compute the Laurent series of $f(z) = \frac{\arctan z}{z^3}$ in powers of z, and use it to find $\oint_C f(z)\, dz$.

(Recall: $\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$, for $|t|<1$)

$\arctan z = \int_0^z \frac{1}{1+t^2} dt = \int_0^z \left(\sum_{k=0}^{\infty} (-1)^k \cdot t^{2k}\right) dt$ (for $|t^2|<1 \Leftrightarrow |t|<1$)

$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{t^{2k+1}}{2k+1}\Big|_0^z = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1}$, for $|z|<1$.

$\Rightarrow f(z) = \frac{1}{z^3} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k-2}}{2k+1} = \frac{1}{z^2} - \frac{1}{3} + \frac{z^2}{5} - \frac{z^4}{7} + \cdots$ for $0<|z|<1$

$\Rightarrow \operatorname{Res}(f,0) = a_{-1} = 0$

$\Rightarrow \oint_C f(z)\, dz \underset{\text{Residue Thm}}{=} 2\pi i \cdot \operatorname{Res}(f,0) = 0$

(or by CIF for derivatives $\oint_C f(z) dz = \overset{=0}{\frac{d^2}{dz}(\arctan z)}\Big|_{z=0} \cdot \frac{2\pi i}{2!} = 0$)

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SOLUTIONS

Q4 (8 pts) Find the principle and the analytic parts of the Laurent series of

$$f(z) = \frac{1}{2iz^2 + z^3}$$

which is valid on the region $0 < |z| < 2$.

$$f(z) = \frac{1}{z^2}\,\frac{1}{2i+z} = \frac{1}{z^2}\cdot\frac{1}{2i}\,\frac{1}{1-\left(-\frac{z}{2i}\right)} = \frac{1}{z^2}\cdot\frac{1}{2i}\sum_{k=0}^{\infty}\left(\frac{-z}{2i}\right)^k \quad \left(0<\left|\frac{-z}{2i}\right|<1\right)$$

$$\Updownarrow$$

$$= \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2i}\right)^{k+1} z^{k-2} \quad (0<|z|<2)$$

$\therefore$ The required Laurent series expansion is

$$f(z) = \underbrace{\frac{1/2i}{z^2} - \frac{(1/2i)^2}{z}}_{\text{Principal part}} + \underbrace{\left(\frac{1}{2i}\right)^3 - \left(\frac{1}{2i}\right)^4 z + \left(\frac{1}{2i}\right)^5 z^2 - \cdots}_{\text{Analytic part}} \quad (0<|z|<2)$$

Q5 (8+8 pts) In each part, find a linear fractional transformation $w = T(z)$ which satisfies the given condition(s):

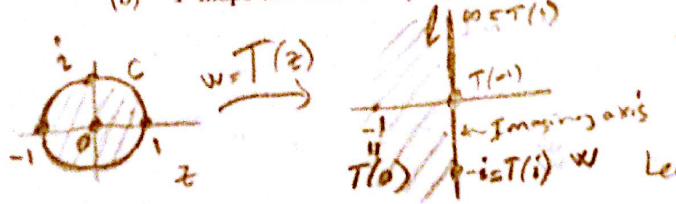
(a) $T(1) = 0$, $T(3) = i - 1$, $T(\infty) = 2i$.

$$T(z) = \frac{a(z-1)}{cz+d} \Longrightarrow T(z) = \frac{2i(z-1)}{z+d}$$

Now, $T(3) = i-1 \Rightarrow \dfrac{2i(3-1)}{3+d} = i-1 \Rightarrow d = -1-2i$

$$\therefore\ T(z) = \frac{2i(z-1)}{z-1-2i}$$

(b) T maps the unit disk $\{z : |z| < 1\}$ to the left half-plane $\{w : \mathrm{Re}(w) < 0\}$.



T must map unit circle C onto the imaginary axis l of w-plane. Let's choose 3 points in C as $-1, 1, i$ and try to find some T s.t. $T(-1) = 0$ & $T(1) = \infty$. Indeed, $T(z) = \dfrac{z+1}{z-1}$ just works. However, $T(C)$ can be any line passing through the origin, but observe $T(i) = -i \Rightarrow T(C)$ is the imaginary axis l.

Finally, $T(0) = -1 \Rightarrow T$ maps interior of C to the left of l.

$\therefore$ $T(z) = \dfrac{z+1}{z-1}$ can be taken as an answer (note: T is not unique!)