EE555 Second Midterm Exam 17 Dec 2025

First name:

Last name: KEY

Student ID:

Signature:

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the following second-order system

$$\dot{x}_1 = x_1^3 + x_1 x_2$$
$$\dot{x}_2 = u$$

where u is the control input. Using backstepping, design a state feedback law $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2)$) is globally asymptotically stable.

$$\dot{x}_1 = x_1^3 + x_1 x_2 \Big|_{x_2 = -2x_1^2} \Rightarrow \dot{x}_1 = -x_1^3 \quad (1)$$

The orig. $x_1 = 0$ of sys. (1) is AS with $V(x_1) = \frac{1}{2}x_1^2$

Let $\eta := x_2 + 2x_1^2$. We can write

$$\dot{x}_1 = x_1^3 + x_1 x_2 = x_1^3 + x_1\{\eta - 2x_1^2\}$$
$$= -x_1^3 + x_1\eta$$

And $\dot{\eta} = \dot{x}_2 + 4x_1\dot{x}_1 = u + 4x_1(x_1^3 + x_1 x_2)$

$$= u + 4x_1^4 + 4x_1^2 x_2$$

Let $v := u + 4x_1^4 + 4x_1^2 x_2$. Then we have

$$\dot{x}_1 = -x_1^3 + x_1\eta$$
$$\dot{\eta} = v$$

Let $V_c = V(x_1) + \frac{1}{2}\eta^2 = \frac{1}{2}x_1^2 + \frac{1}{2}\eta^2$. Then

$$\dot{V}_c = x_1\dot{x}_1 + \eta\dot{\eta} = -x_1^4 + x_1^2\eta + \eta v$$
$$= -x_1^4 + \eta[x_1^2 + v]$$

choose $v = -x_1^2 - \eta$

$$= -x_1^4 - \eta^2$$
$$< 0$$

$V_c > 0$ and rad. unb. & $\dot{V}_c < 0 \Rightarrow \begin{bmatrix} x_1 \\ \eta \end{bmatrix} = 0$ is GAS

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ is GAS

Finally, $v = -x_1^2 - \eta = -x_1^2 - [x_2 + 2x_1^2]$

& $u = v - 4x_1^4 - 4x_1^2 x_2$

$$\Rightarrow \boxed{\phi(x_1, x_2) = -3x_1^2 - (1 + 4x_1^2)x_2 - 4x_1^4}$$

Q2.

Let $V_1,\, V_2 : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, positive definite, and radially unbounded; and let $f_1,\, f_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Suppose there exist $\alpha_1,\, \alpha_2 > 0$ such that the inequalities

$$\begin{aligned} \langle \nabla V_1(x_1),\, f_1(x_1,\, u_1)\rangle &\le -\|x_1\| + \alpha_1\|u_1\|\,, \\ \langle \nabla V_2(x_2),\, f_2(x_2,\, u_2)\rangle &\le -\|x_2\| + \alpha_2\|u_2\| \end{aligned}$$

hold for all $x_1,\, x_2,\, u_1,\, u_2 \in \mathbb{R}^n$.

(10) (a) Show that the system $\dot{x}_1 = f_1(x_1,\, u_1)$ is ISS.

(15) (b) Show that the cascade system $\left\{ \begin{array}{rcl} \dot{x}_1 &=& f_1(x_1,\, x_2) \\ \dot{x}_2 &=& f_2(x_2,\, u) \end{array} \right\}$ is ISS.

a) $\dot{V}_1 \le -\|x_1\| + \alpha_1\|u_1\|$

$= -\frac{1}{2}\|x_1\| - \frac{1}{2}\left(\|x_1\| - 2\alpha_1\|u_1\|\right)$

$\Rightarrow \dot{V}_1 \le -\frac{1}{2}\|x_1\|$ when $\|x_1\| \ge 2\alpha_1\|u_1\|$

$\Rightarrow$ The system is ISS by Thm 4.19.

b) Let $V(x_1, x_2) = V_1(x_1) + (1+\alpha_1)V_2(x_2)$

Note that V is pos. def. & rad. unb.

We can write

$\dot{V} = \dot{V}_1 + (1+\alpha_1)\dot{V}_2$

$\le -\|x_1\| + \alpha_1\|x_2\| + (1+\alpha_1)\{-\|x_2\| + \alpha_2\|u\|\}$

$= -(\|x_1\| + \|x_2\|) + (1+\alpha_1)\alpha_2\|u\|$ & $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\le -\|x\| + (1+\alpha_1)\alpha_2\|u\|$

$= -\frac{1}{2}\|x\| - \frac{1}{2}\{\|x\| - 2\alpha_2(\alpha_1+1)\|u\|\}$

$\Rightarrow \dot{V} \le -\frac{1}{2}\|x\|$ when $\|x\| \ge 2\alpha_2(\alpha_1+1)\|u\|$

$\Rightarrow$ The system is ISS by Thm 4.19.

Q3.

(7) **(a)** Is the below system $\mathcal{L}_2$ stable? Explain. *Hint: Use* $V(x) = \frac{1}{2}x^T x$.

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - x_2 y + x_2 u \\ \dot{x}_3 &= x_3 y - x_3 u \\ y &= x_2^2 - x_3^2 \end{aligned}$$

(b) Consider the following system.

$$\begin{aligned} \dot{x} &= -\frac{x}{1+x^2} + u \\ y &= \frac{x}{1+x^2} \end{aligned}$$

(6) i) Show that this system is strictly passive. What is the storage function?

(6) ii) Is this system $\mathcal{L}_\infty$ stable? Explain.

(6) iii) Is this system ISS? Explain.

a) $\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3$

$= \cancel{-x_1x_2} + \cancel{x_2x_1} - x_2^2 y + x_2^2 u + x_3^2 y - x_3^2 u$

$= -y(x_2^2 - x_3^2) + u(x_2^2 - x_3^2)$ $\quad y = x_2^2 - x_3^2$

$= -y^2 + uy$

$\Rightarrow$ The system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain ≤ 1 (Lem 6.5)

b) i) $V(x) = \int_0^x \frac{\alpha}{1+\alpha^2} d\alpha$

Note that V is pos. def.

$\Rightarrow \dot{V} = \frac{x}{1+x^2} \cdot \dot{x}$

$= -\left(\frac{x}{1+x^2}\right)^2 + \frac{x}{1+x^2} \cdot u$

$= -\left(\frac{x}{1+x^2}\right)^2 + uy =: -\Psi(x) + uy$

$\Rightarrow$ The sys. is SP with stor. func. $V(x)$

ii) YES. Note that $\frac{|x|}{1+x^2} < 1$ for all x. Hence we can write

$\|y\|_{\mathcal{L}_\infty} \le \alpha \cdot \|u\|_{\mathcal{L}_\infty} + 1$ for any $\alpha > 0$.

$\Rightarrow$ The sys. is finite-gain $\mathcal{L}_\infty$ stable.

iii) NO. To see that suppose otherwise. Then we can find $\beta \in \mathcal{KL}$ & $\gamma \in \mathcal{K}$ such that

$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma\left(\sup_{0\le \tau \le t} \|u(\tau)\|\right)$ (1)

Now, let $x(0) = 3\gamma(1)$ and $u(t) \equiv \frac{x(0)}{1+x(0)^2} < 1$

Then $\dot{x} = 0 \Rightarrow x(t) \equiv x(0) = 3\gamma(1)$. Then

(1) $\Rightarrow$ $3\gamma(1) \le \beta(3\gamma(1), t) + \gamma(1)$ (2)

Choose T large enough such that $\beta(3\gamma(1), T) < \gamma(1)$

Then

(2) $\Rightarrow$ $3\gamma(1) \le 2\gamma(1)$ $\Rightarrow$ contradiction. $\blacksquare$

Q4.

(a) Consider the following system.

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - x_2 \mathrm{sat}(y) + x_2 u \\ \dot{x}_3 &= x_3 \mathrm{sat}(y) - x_3 u \\ y &= x_2^2 - x_3^2 \end{aligned}$$

(8) i) Is this system output strictly passive? Explain.

(8) ii) Is this system zero-state observable? Explain.

(9) (b) Is the origin of the below system asymptotically stable? Is it GAS? Explain.

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - x_2 \mathrm{sat}(x_2^2 - x_3^2) - \frac{x_2 x_4}{1+x_4^2} \\ \dot{x}_3 &= x_3 \mathrm{sat}(x_2^2 - x_3^2) + \frac{x_3 x_4}{1+x_4^2} \\ \dot{x}_4 &= -\frac{x_4}{1+x_4^2} + x_2^2 - x_3^2 \end{aligned}$$

a) i) Let $V_1 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2$

$\Rightarrow \dot{V}_1 = -y\,\mathrm{sat}(y) + uy$

Since $y\,\mathrm{sat}(y) > 0$ for all $y \neq 0$, the system is OSP.

ii) Let $u \equiv 0$ & $y \equiv 0$. Then the dynamics read

$\left.\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 \end{aligned}\right\} \ddot{x}_2 + x_2 = 0 \Rightarrow x_2(t) = A\cos(t+\phi)$ (1)

$\dot{x}_3 = 0 \Rightarrow x_3(t) \equiv x_3(0)$

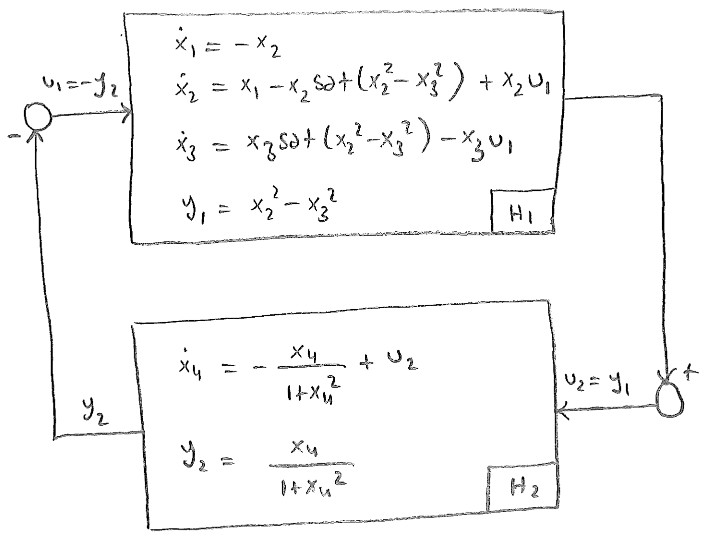
$y \equiv 0 \Rightarrow x_2^2 \equiv x_3^2 \Rightarrow |x_2(t)| \equiv |x_3(0)|$ (2)

(1) & (2) $\Rightarrow A = 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow x_3(t) \equiv 0$

Also, $x_1(t) \equiv 0$ because $x_1 = \dot{x}_2 \equiv 0$

Hence, the system is ZSO.

b) We can write



$V_1 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 \Rightarrow$ rad. unbounded

$V_2 = \int_0^{x_4} \frac{\alpha}{1+\alpha^2}\, d\alpha = \frac{1}{2}\ln(1+x_4^2) \Rightarrow$ rad. unb.

Hence

$\{H_1: \text{OSP} + \text{ZSO}\} + \{H_2: \text{SP}\} \Rightarrow$ GAS (Thm 6.3)

Both storage functions rad. unb.