First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Consider the following second-order system

$$\begin{aligned} \dot{x}_1 &= x_1^3 + x_1 x_2 \\ \dot{x}_2 &= u \end{aligned}$$

where u is the control input. Using backstepping, design a state feedback law $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2)$) is globally asymptotically stable.

Q2.

Let $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, positive definite, and radially unbounded; and let $f_1, f_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Suppose there exist $\alpha_1, \alpha_2 > 0$ such that the inequalities

$$\begin{aligned} \langle \nabla V_1(x_1),\, f_1(x_1,\, u_1)\rangle &\leq -\|x_1\| + \alpha_1\|u_1\|\,, \\ \langle \nabla V_2(x_2),\, f_2(x_2,\, u_2)\rangle &\leq -\|x_2\| + \alpha_2\|u_2\| \end{aligned}$$

hold for all $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$.

(a) Show that the system $\dot{x}_1 = f_1(x_1,\, u_1)$ is ISS.

(b) Show that the cascade system $\left\{ \begin{array}{rcl} \dot{x}_1 & = & f_1(x_1,\, x_2) \\ \dot{x}_2 & = & f_2(x_2,\, u) \end{array} \right\}$ is ISS.

Q3.

(a) Is the below system $\mathcal{L}_2$ stable? Explain. *Hint: Use* $V(x) = \frac{1}{2}x^T x$.

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - x_2 y + x_2 u \\ \dot{x}_3 &= x_3 y - x_3 u \\ y &= x_2^2 - x_3^2\end{aligned}$$

(b) Consider the following system.

$$\begin{aligned}\dot{x} &= -\frac{x}{1+x^2} + u \\ y &= \frac{x}{1+x^2}\end{aligned}$$

i) Show that this system is strictly passive. What is the storage function?

ii) Is this system $\mathcal{L}_\infty$ stable? Explain.

iii) Is this system ISS? Explain.

Q4.

(a) Consider the following system.

$$\begin{aligned}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1 - x_2 \text{sat}(y) + x_2 u \\
\dot{x}_3 &= x_3 \text{sat}(y) - x_3 u \\
y &= x_2^2 - x_3^2
\end{aligned}$$

i) Is this system output strictly passive? Explain.

ii) Is this system zero-state observable? Explain.

(b) Is the origin of the below system asymptotically stable? Is it GAS? Explain.

$$\begin{aligned}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1 - x_2 \text{sat}(x_2^2 - x_3^2) - \frac{x_2 x_4}{1 + x_4^2} \\
\dot{x}_3 &= x_3 \text{sat}(x_2^2 - x_3^2) + \frac{x_3 x_4}{1 + x_4^2} \\
\dot{x}_4 &= -\frac{x_4}{1 + x_4^2} + x_2^2 - x_3^2
\end{aligned}$$