EE555 Second Midterm Exam 19 Dec 2023

First name:__________________________

Last name: Sol'n______________________

Student ID:__________________________

Signature:__________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the following second-order system

$$\begin{aligned} \dot{x}_1 &= x_1 + (1 + x_1^2)x_2 \\ \dot{x}_2 &= u \end{aligned}$$

where u is the control input. Using backstepping, design a state feedback law $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2)$) is globally asymptotically stable.

$\dot{x}_1 = x_1 + (1+x_1^2)x_2 \Big|_{x_2 = -x_1} = x_1 - (1+x_1^2)x_1 = -x_1^3$

$\Rightarrow \dot{x}_1 = -x_1^3$ (GAS with $V(x_1) = \frac{1}{2}x_1^2$)

Let $z = x_2 + x_1$ (0) Then

$\dot{x}_1 = x_1 + (1+x_1^2)(x_2 + x_1 - x_1)$

$= -x_1^3 + (1+x_1^2)z$ (1)

Also,

$\dot{z} = \dot{x}_1 + \dot{x}_2 = x_1 + (1+x_1^2)x_2 + u =: v$ (2)

(1) & (2) implies

$$\left.\begin{aligned} \dot{x}_1 &= -x_1^3 + (1+x_1^2)z \\ \dot{z} &= v \end{aligned}\right\}$$

Using $V_c(x_1, z) = V(x_1) + \frac{1}{2}z^2 = \frac{1}{2}x_1^2 + \frac{1}{2}z^2$

we can write

$\dot{V}_c = x_1\dot{x}_1 + z\dot{z} = -x_1^4 + x_1(1+x_1^2)z + zv$

$= -x_1^4 + [x_1(1+x_1^2) + v]z$

Choosing $v = -x_1(1+x_1^2) - z$ (3) we have

$\dot{V}_c = -x_1^4 - z^2 < 0$

Combining (0), (2), & (3) we have

$u = -2x_1 - x_2 - (1+x_1^2)(x_1+x_2)$

Hence

$$\boxed{\phi(x_1, x_2) = -x_1 - (x_1+x_2)(2+x_1^2)}$$

is a stabilizing feedback law.

Q2.

Consider the following system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= B^T P x \end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ make a controllable pair (A, B). The matrix $P \in \mathbb{R}^{n \times n}$ is symmetric positive definite and satisfies $A^T P + PA = 0$.

(a) Is this system passive? Explain.

(b) Is this system zero state observable? Explain.

(c) Find (if possible) an output feedback $u = \phi(y)$ such that the origin of the resulting closed-loop system is globally asymptotically stable.

a) Let $V(x) = \frac{1}{2} x^T P x$. Then

$$\begin{aligned} \dot{V} = x^T P \dot{x} &= x^T P [Ax + Bu] \\ &= \underbrace{x^T P A x}_{=0} + x^T P B u \\ &= y^T u \end{aligned}$$

$\Rightarrow$ $\dot{V} = y^T u$ $\Rightarrow$ System is (lossless) passive.

b) ZSO is equivalent to observability for LTI systems. Because (A, B) is controllable the controllability matrix is full rank. That is,

$$\operatorname{rank} [B \;\; AB \;\cdots\; A^{n-1} B] = n \qquad (1)$$

Let us construct the observability matrix W

$$W = \begin{bmatrix} B^T P \\ B^T P A \\ B^T P A^2 \\ \vdots \\ B^T P A^{n-1} \end{bmatrix} = \begin{bmatrix} B^T P \\ -B^T A^T P \\ B^T A^{2T} P \\ \vdots \\ \mp B^T A^{nT} P \end{bmatrix} = \underbrace{\begin{bmatrix} B^T \\ -B^T A^T \\ B^T A^{2T} \\ \vdots \\ \mp B^T A^{(n-1)T} \end{bmatrix}}_{\operatorname{rank} = n \text{ (due to (1))}} \underbrace{P}_{\operatorname{rank} = n}$$

Hence $\operatorname{rank} W = n$ and the system is observable.

c) Since the system is passive with a pos. def. radially unb. storage function and ZSO we can choose, for instance, $\boxed{u = -y}$ to stabilize the origin. (Thm 14.4)

Q3.

Consider the following second-order system.

$$\begin{aligned}\dot{x}_1 &= -x_1(1-\sin x_2)\\ \dot{x}_2 &= -x_2+u\end{aligned}$$

(a) Show that the origin of the unforced ($u=0$) system is globally asymptotically stable.

(b) Show that under bounded input the state remains bounded.

(c) Show that the system is not input-to-state stable. *Hint: Consider* $u(t)\equiv\frac{\pi}{2}$ *with* $x_2(0)=\frac{\pi}{2}$.

a) Let $v(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ (pos. def. rad. unb.)

$\Rightarrow \dot{v} = -x_1^2(1-\sin x_2) - x_2^2 < 0$ (neg. def.)

$\Rightarrow x=0$ is GAS.

b) Let $v_1(x_1) = \frac{1}{2}x_1^2$

$\Rightarrow \dot{v}_1 = x_1\dot{x}_1 = -x_1^2\underbrace{(1-\sin x_2)}_{\geq 0} \Rightarrow \dot{v}_1 \leq 0$

$\Rightarrow \frac{1}{2}x_1(t)^2 \leq \frac{1}{2}x_1(0)^2 \Rightarrow |x_1(t)| \leq |x_1(0)|$

As for the second subsystem $\dot{x}_2 = -x_2 + u$, it is LTI with Hurwitz A matrix ($A=-1$) hence it is ISS. Therefore:

$\rightarrow x_1(t)$ bounded (regardless of u) (1)

$\rightarrow x_2(t)$ bounded for bounded u (2)

(1) & (2) $\Rightarrow$ $x(t)$ bounded under bounded u.

c) Suppose not. That is, there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma\Big\{\sup_{\tau\in[0,t]} \|u(t)\|\Big\} \quad (3)$$

Let now $x_2(0) = \frac{\pi}{2}$ & $u(t) \equiv \frac{\pi}{2}$. We then have

$$\left.\begin{aligned}\dot{x}_1 = 0 &\Rightarrow x_1(t) = x_1(0)\\ \dot{x}_2 = 0 &\Rightarrow x_2(t) = \tfrac{\pi}{2}\end{aligned}\right\} \text{ for all } t$$

Then (3) implies

$$\begin{aligned}\|x_1(0)\| &= \|x_1(t)\|\\ &\leq \|x(t)\|\\ &\leq \beta(\|x(0)\|, t) + \gamma\left(\tfrac{\pi}{2}\right) \quad (4)\end{aligned}$$

Now (4) should hold for any $x_1(0)$. But pick, for instance, $x_1(0) = 2\gamma(\frac{\pi}{2})$. Then we have

$$2\gamma\left(\tfrac{\pi}{2}\right) \leq \beta(\|x(0)\|, t) + \gamma\left(\tfrac{\pi}{2}\right)$$

Since the right-hand side converges to $\gamma(\frac{\pi}{2})$ as $t\to\infty$ this poses a contradiction. Hence the system cannot be ISS.

Q4.

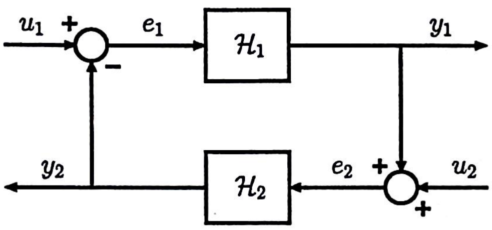
Consider the below systems where a and b are positive constants.

$$\mathcal{H}_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a x_1^2 x_2 - x_1^3 + x_1 e_1 \\ y_1 = x_1 x_2 \end{cases} \qquad \mathcal{H}_2 : \begin{cases} \dot{x}_3 = -b x_3^3 - x_4 + e_2 \\ \dot{x}_4 = x_3^3 \\ y_2 = x_3^3 \end{cases}$$

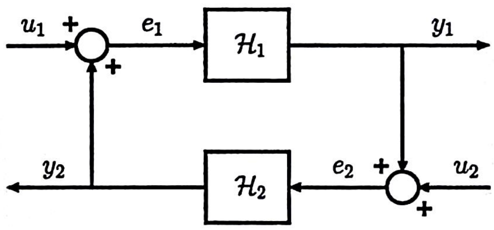
(a) Show that the system $\mathcal{H}_1$ is $\mathcal{L}_2$ stable. *Hint: Use* $V_1 = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

(b) Show that the system $\mathcal{H}_2$ is $\mathcal{L}_2$ stable. *Hint: Use* $V_2 = \frac{1}{4}x_3^4 + \frac{1}{2}x_4^2$.

(c) For each of the below feedback connections find conditions on a and b so that the closed-loop system is $\mathcal{L}_2$ stable from the input $u = [u_1\ u_2]^T$ to the output $y = [y_1\ y_2]^T$.



(I)



(II)

a) $\dot{V}_1 = x_1^3 \dot{x}_1 + x_2 \dot{x}_2 = x_1^3 x_2 - a x_1^2 x_2^2 - x_2 x_1^3 + x_2 x_1 e_1$

$= -a(x_1 x_2)^2 + (x_1 x_2) e_1$

$= -a y_1^2 + y_1 e_1$

$\Rightarrow$ the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain $\gamma_1 \leq \frac{1}{a}$. (Lem 6.5)

b) $\dot{V}_2 = x_3^3 \dot{x}_3 + x_4 \dot{x}_4 = -b x_3^6 - x_3^3 x_4 + x_3^3 e_2 + x_4 x_3^3$

$= -b x_3^6 + x_3^3 e_2$

$= -b y_2^2 + y_2 e_2$

$\Rightarrow$ the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain $\gamma_2 \leq \frac{1}{b}$.

c) Connection I : closed-loop system is finite-gain $\mathcal{L}_2$ stable for all $\boxed{a, b > 0}$ (Lem 6.8)

Connection II : closed-loop system is finite-gain $\mathcal{L}_2$ stable if $\gamma_1 \gamma_2 < 1$. Hence

$\frac{1}{a} \cdot \frac{1}{b} < 1 \Rightarrow \boxed{ab > 1}$

(Small gain thm.)