First name:__________________________

Last name:__________________________

Student ID:__________________________

Signature:__________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Consider the following second-order system

$$\begin{aligned}\dot{x}_1 &= x_1 + (1 + x_1^2)x_2 \\ \dot{x}_2 &= u\end{aligned}$$

where u is the control input. Using backstepping, design a state feedback law $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2)$) is globally asymptotically stable.

Q2.

Consider the following system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= B^T P x \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ make a controllable pair (A, B). The matrix $P \in \mathbb{R}^{n\times n}$ is symmetric positive definite and satisfies $A^T P + PA = 0$.

(a) Is this system passive? Explain.

(b) Is this system zero state observable? Explain.

(c) Find (if possible) an output feedback $u = \phi(y)$ such that the origin of the resulting closed-loop system is globally asymptotically stable.

Q3.

Consider the following second-order system.

$$\begin{aligned}\dot{x}_1 &= -x_1(1 - \sin x_2)\\ \dot{x}_2 &= -x_2 + u\end{aligned}$$

(a) Show that the origin of the unforced ($u = 0$) system is globally asymptotically stable.

(b) Show that under bounded input the state remains bounded.

(c) Show that the system is not input-to-state stable. *Hint: Consider* $u(t) \equiv \frac{\pi}{2}$ *with* $x_2(0) = \frac{\pi}{2}$.

Q4.

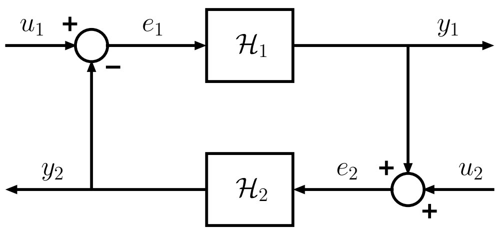
Consider the below systems where a and b are positive constants.

$$\mathcal{H}_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_1^2 x_2 - x_1^3 + x_1 e_1 \\ y_1 = x_1 x_2 \end{cases} \qquad \mathcal{H}_2 : \begin{cases} \dot{x}_3 = -bx_3^3 - x_4 + e_2 \\ \dot{x}_4 = x_3^3 \\ y_2 = x_3^3 \end{cases}$$

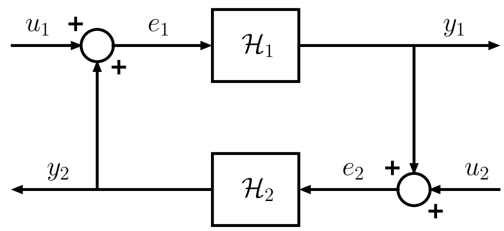
(a) Show that the system $\mathcal{H}_1$ is $\mathcal{L}_2$ stable. *Hint: Use* $V_1 = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

(b) Show that the system $\mathcal{H}_2$ is $\mathcal{L}_2$ stable. *Hint: Use* $V_2 = \frac{1}{4}x_3^4 + \frac{1}{2}x_4^2$.

(c) For each of the below feedback connections find conditions on a and b so that the closed-loop system is $\mathcal{L}_2$ stable from the input $u = [u_1\ u_2]^T$ to the output $y = [y_1\ y_2]^T$.



(I)



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