EE555 Second Midterm Exam 21 Dec 2022

First name:______________________

Last name: KEY

Student ID:______________________

Signature:______________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the following second-order system

$$\begin{aligned}\dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= (1+x_2^2)u\end{aligned}$$

where u is the control input. Using backstepping, design a state feedback law $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2)$) is globally asymptotically stable.

$\dot{x}_1 = x_1 x_2 \Big|_{x_2 = -x_1^2} = -x_1^3$ GAS with $V_1(x_1) = \frac{1}{2}x_1^2$

Let $z := x_2 + x_1^2$

$\Rightarrow \dot{x}_1 = x_1 x_2 - x_1^3 + x_1^3 = -x_1^3 + x_1 z$

& $\dot{z} = \dot{x}_2 + 2x_1\dot{x}_1 = (1+x_2^2)u + 2x_1(x_1x_2)$

$= (1+x_2^2)u + 2x_1^2 x_2$

Let $v := 2x_1^2 x_2 + (1+x_2^2)u$ (1)

$\Rightarrow \dot{x}_1 = -x_1^3 + x_1 z$

$\dot{z} = v$

Let $V_c = V_1(x_1) + \frac{1}{2}z^2$

$\Rightarrow \dot{V}_c = x_1\dot{x}_1 + z\dot{z} = x_1(-x_1^3 + x_1 z) + zv$

$= -x_1^4 + z(x_1^2 + v)$ Chose $v = -x_1^2 - z$

$= -x_1^4 - z^2$

$< 0 \Rightarrow$ the origin GAS

Now, $v = -x_1^2 - z = -x_1^2 - (x_2 + x_1^2)$

$(1) \Rightarrow -2x_1^2 - x_2 = 2x_1^2 x_2 + (1+x_2^2)u$

$$\boxed{\Rightarrow u = -\frac{2x_1^2(1+x_2) + x_2}{1+x_2^2}}$$

Q2.

Consider the following system, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices.

$$\dot{x} = [A - \|x\|^2 I]\, x + Bu \quad =: f(x)$$

For each of the below claims either provide a proof (if true) or find a counterexample (if false).

(a) If A is Hurwitz then the system is ISS.

(b) If the system is ISS then A is Hurwitz.

(a) $\boxed{\text{TRUE}}$

A Hurwitz $\Rightarrow$ we can find a symmetric pos. def. matrix $P \in \mathbb{R}^{n\times n}$ satisfying

$$A^T P + PA = -I$$

Let $V(x) = x^T P x$. Then

$$\langle \nabla V(x), f(x)\rangle = 2x^T P\left([A - \|x\|^2 I]x + Bu\right)$$

$$= 2x^T PAx - 2\|x\|^2 x^T P x + 2x^T P B u$$

$$= -\|x\|^2 - 2\|x\|^2 x^T P x + 2x^T P B u$$

$$\leq -\|x\|^2 + 2\|P\|\cdot\|B\|\cdot\|x\|\cdot\|u\|$$

$$= -\frac{1}{2}\|x\|^2 - \frac{1}{2}\|x\|\left\{\|x\| - 4\|P\|\cdot\|B\|\cdot\|u\|\right\}$$

$\Rightarrow \dot{V}(x) \leq -\frac{1}{2}\|x\|^2$ when $\|x\| \geq 4\|P\|\cdot\|B\|\cdot\|u\|$

$\Rightarrow$ the system ISS. (Thm 4.19)

(b) $\boxed{\text{FALSE}}$

Let $n=1$, $A=0$, and $B=1$. Note that A is not Hurwitz.

$$\Rightarrow \quad \dot{x} = -x^3 + u$$

Let $V(x) = \frac{1}{2}x^2$

$$\Rightarrow \quad \dot{V} = -x^4 + xu$$

$$\leq -x^4 + |x|\cdot|u|$$

$$= -\frac{1}{2}x^4 - \frac{1}{2}|x|\left\{|x|^3 - 2|u|\right\}$$

$\Rightarrow \dot{V}(x) \leq -\frac{1}{2}x^4$ when $|x| \geq 2^{1/3}|u|^{1/3}$

$\Rightarrow$ the system is ISS (despite the fact that A is not Hurwitz).

Q3.

(a) Is the below system $\mathcal{L}_2$ stable? Explain. *Hint: Use* $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1+x_1^2)x_2 - x_1^3 + x_1 u \\ y &= x_1 x_2 \end{aligned}$$

(b) Is the below system[1] $\mathcal{L}_\infty$ stable? Explain. *Hint: Use* $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \int_0^{x_1} \mathrm{sat}(z)dz$.

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1 - \mathrm{sat}(x_1) - x_2 + u \\ y &= x_2 =: h(x) \end{aligned}$$

(a) $\dot{V} = x_1^3 \dot{x}_1 + x_2 \dot{x}_2$

$= x_1^3 x_2 + x_2 \{ -(1+x_1^2)x_2 - x_1^3 + x_1 u \}$

$= -(1+x_1^2)x_2^2 + x_1 x_2 u$

$\leq -(x_1 x_2)^2 + x_1 x_2 u = -y^2 + uy$

$\Rightarrow uy \geq \dot{V} + y^2$

$\Rightarrow$ The system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain ≤ 1. (Lemma 6.5)

(b) $\dot{V} = (x_1 + \mathrm{sat}(x_1))\dot{x}_1 + x_2 \dot{x}_2$

$= (x_1 + \mathrm{sat}(x_1))(-x_1 + x_2)$
$\quad + x_2(-x_1 - \mathrm{sat}(x_1) - x_2 + u)$

$= -x_1^2 - x_1 \mathrm{sat}(x_1) - x_2^2 + x_2 u$

$\leq -x_1^2 - x_2^2 + x_2 u$

$\leq -\|x\|^2 + \|x\| \cdot \|u\|$

$= -\frac{1}{2}\|x\|^2 - \frac{1}{2}\|x\|\{\|x\| - 2\|u\|\}$

$\Rightarrow \dot{V}(x) \leq -\frac{1}{2}\|x\|^2$ when $\|x\| \geq 2\|u\|$

$\Rightarrow$ the system is ISS

Also, $\|h(x)\| \leq \|x\|$

Hence the system is $\mathcal{L}_\infty$ stable. (Thm 5.3)

[1] $\mathrm{sat}(\alpha) = \begin{cases} \min\{\alpha, 1\} & \text{for } \alpha \geq 0 \\ \max\{\alpha, -1\} & \text{for } \alpha < 0 \end{cases}$

Q4.

(a) Consider the following second-order system.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(1+x_1^2)x_2 - x_1^3 + x_1 u\\ y &= x_1x_2\end{aligned}$$

i) Is this system strictly passive? Explain.

ii) Is this system output strictly passive? Explain.

iii) Is this system zero-state observable? Explain.

(b) Investigate the stability properties of the origin of the fourth-order system below.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(1+x_1^2)x_2 - x_1^3 - x_1x_4\\ \dot{x}_3 &= -x_3 + x_4\\ \dot{x}_4 &= -x_3 - \text{sat}(x_3) - x_4 + x_1x_2\end{aligned}$$

(a) In Q3 we obtained

$$uy \ge \dot{V} + y^2$$

Hence

i) The system is not strictly passive.

ii) The system is output strictly passive.

iii) Let $u \equiv 0$ & $y \equiv 0$

$\Rightarrow$ Either $x_1 \equiv 0$ or $x_2 \equiv 0$.

Let $x_1 \equiv 0$

$x_1 \equiv 0 \Rightarrow \dot{x}_1 \equiv 0 \Rightarrow x_2 \equiv 0 \Rightarrow x \equiv 0$

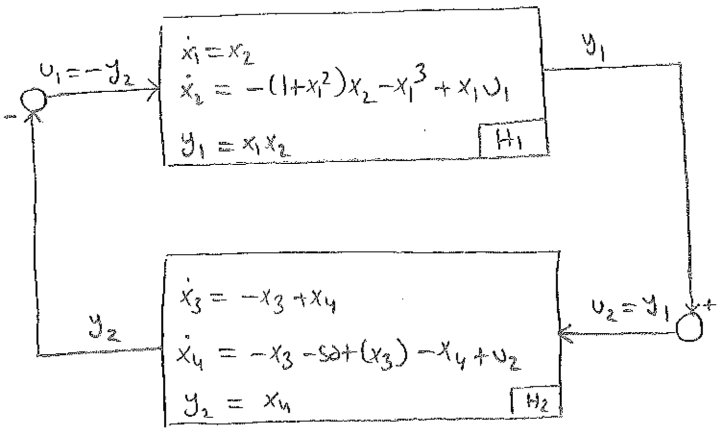
Let $x_2 \equiv 0$

$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow -x_1^3 \equiv 0 \Rightarrow x_1 \equiv 0 \Rightarrow x \equiv 0$

Therefore $u \equiv 0$ & $y \equiv 0 \Rightarrow x \equiv 0$

$\Rightarrow$ The system is ZSO.

(b) Note that this 4^{th} order system is the result of the following feedback connection:



In Q3 we obtained for H_2:

$$\dot{V} \le -x_3^2 - x_4^2 + x_4u_2 = -\|x\|^2 + y_2u_2$$

$\Rightarrow$ H_2 is SP.

Hence

$\{H_1 \text{ OSP} + \text{ZSO}\} + \{H_2 \text{ SP}\} \Rightarrow$ GAS

(Thm 6.3)

(both storage func. are rad. unb.)