First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Consider the following second-order system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= (1 + x_2^2) u \end{aligned}$$

where u is the control input. Using backstepping, design a state feedback law $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2)$) is globally asymptotically stable.

Q2.

Consider the following system, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices.

$$\dot{x} = [\,A - \|x\|^2 I\,]\,x + Bu$$

For each of the below claims either provide a proof (if true) or find a counterexample (if false).

(a) If A is Hurwitz then the system is ISS.

(b) If the system is ISS then A is Hurwitz.

Q3.

(a) Is the below system $\mathcal{L}_2$ stable? Explain. *Hint: Use* $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1 + x_1^2)x_2 - x_1^3 + x_1 u \\ y &= x_1 x_2 \end{aligned}$$

(b) Is the below system[1] $\mathcal{L}_\infty$ stable? Explain. *Hint: Use* $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \int_0^{x_1} \text{sat}(z)dz$.

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1 - \text{sat}(x_1) - x_2 + u \\ y &= x_2 \end{aligned}$$

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[1] $\text{sat}(\alpha) = \begin{cases} \min\{\alpha, 1\} & \text{for} \quad \alpha \geq 0 \\ \max\{\alpha, -1\} & \text{for} \quad \alpha < 0 \end{cases}$

Q4.

(a) Consider the following second-order system.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1 + x_1^2)x_2 - x_1^3 + x_1 u \\ y &= x_1 x_2\end{aligned}$$

i) Is this system strictly passive? Explain.

ii) Is this system output strictly passive? Explain.

iii) Is this system zero-state observable? Explain.

(b) Investigate the stability properties of the origin of the fourth-order system below.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1 + x_1^2)x_2 - x_1^3 - x_1 x_4 \\ \dot{x}_3 &= -x_3 + x_4 \\ \dot{x}_4 &= -x_3 - \mathrm{sat}(x_3) - x_4 + x_1 x_2\end{aligned}$$