EE555 Second Midterm Exam 4 Jan 2022

First name:____________________

Last name: KEY____________________

Student ID:____________________

Signature:____________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.

(a) Consider the following second-order system

$$\begin{aligned} \dot{x}_1 &= -x_1 + \tilde{u} \\ \dot{x}_2 &= -x_1^3 - x_2^2 \tilde{u} \end{aligned}$$

where $\tilde{u}$ is the control input. Show that the feedback law $\tilde{u} = x_2$ asymptotically stabilizes the origin of the closed-loop system. *Hint: Use* $V(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

(b) Consider the following third-order system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_3 \\ \dot{x}_2 &= -x_1^3 - x_2^2 x_3 \\ \dot{x}_3 &= u \end{aligned}$$

where u is the control input. Using backstepping, design a state feedback law $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2, x_3)$) is globally asymptotically stable.

a) Under the feedback $\tilde{u} = x_2$ we have

$\dot{x}_1 = -x_1 + x_2$

$\dot{x}_2 = -x_1^3 - x_2^3$

Then

$\dot{V} = x_1^3 \dot{x}_1 + x_2 \dot{x}_2 = x_1^3\{-x_1 + \cancel{x_2}\} + x_2\{-\cancel{x_1^3} - x_2^3\}$

$= -x_1^4 - x_2^4 < 0$

Hence

$\left.\begin{aligned} &V : \text{pos. def.} \\ &\dot{V} : \text{neg. def} \end{aligned}\right\} \Rightarrow$ the origin is asy. stable. $\blacksquare$

b) Let: $z = x_3 - x_2$. We can write

$\dot{x}_1 = -x_1 + x_2 + (x_3 - x_2) = -x_1 + x_2 + z$

$\dot{x}_2 = -x_1^3 - x_2^3 - x_2^2(x_3 - x_2) = -x_1^3 - x_2^3 - x_2^2 z$

$\dot{z} = \dot{x}_3 - \dot{x}_2 = u + x_1^3 + x_2^2 x_3 =: v$

Using the composite Lyap. func. $V_c = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \frac{1}{2}z^2$

$\dot{V}_c = x_1^3 \dot{x}_1 + x_2 \dot{x}_2 + z\dot{z}$

$= x_1^3\{-x_1 + x_2 + z\} + x_2\{-x_1^3 - x_2^3 - x_2^2 z\} + zv$

$= -x_1^4 - x_2^4 + x_1^3 z - x_2^3 z + zv$

$= -x_1^4 - x_2^4 + \{x_1^3 - x_2^3 + v\} z$

Choosing $v = -x_1^3 + x_2^3 - z$ we obtain

$\dot{V}_c = -x_1^4 - x_2^4 - z^2 < 0$

Hence

$\left.\begin{aligned} &V_c > 0 \text{ \& rad. unbounded} \\ &\dot{V}_c < 0 \end{aligned}\right\}$ the origin is GAS.

Finally,

$u = v - x_1^3 - x_2^2 x_3$

$= -x_1^3 + x_2^3 - (x_3 - x_2) - x_1^3 - x_2^2 x_3$

$= -2x_1^3 + x_2^3 - x_2^2 x_3 - x_3 + x_2$

That is, the feedback

$$\boxed{\phi(x) = -2x_1^3 + x_2^3 - x_2^2 x_3 - x_3 + x_2}$$

asy. stabilizes the origin.

Q2.

(a) Consider the following second-order system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + \tilde{u}\\ \dot{x}_2 &= -x_1^3 - x_2^3\end{aligned}$$

where $\tilde{u}$ is the control input. Show that this system is input-to-state stable. *Hint: Use* $V(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ *and recall the inequality* $2(x_1^4 + x_2^4) \geq \|x\|^4$.

(b) Consider the following third-order system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + x_3\\ \dot{x}_2 &= -x_1^3 - x_2^3\\ \dot{x}_3 &= u\end{aligned}$$

where u is the control input. Find a state feedback law $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2, x_3)$) is globally asymptotically stable.

a) $\dot V = x_1^3 \dot x_1 + x_2 \dot x_2$

$= x_1^3\{-x_1 + \cancel{x_2} + \tilde u\} + x_2\{\cancel{-x_1^3} - x_2^3\}$

$= -x_1^4 - x_2^4 + x_1^3 \tilde u$

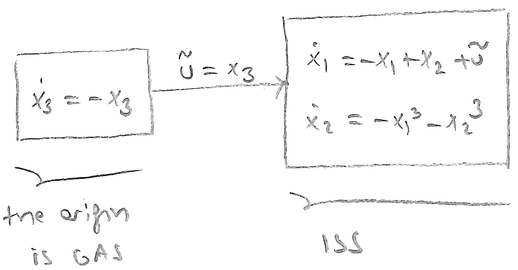
$\le -\frac{1}{2}\|x\|^4 + \|x\|^3\|\tilde u\|$

$= -\frac{1}{4}\|x\|^4 - \frac{1}{4}\|x\|^3\{\|x\| - 4\|\tilde u\|\}$

$\Rightarrow \quad \dot V \le -\frac{1}{4}\|x\|^4 \quad$ for $\|x\| > 4\|\tilde u\|$

The sys. is ISS by Thm. 4.19. ∎

b) Let $u = -x_3$. Then the origin of the subsystem $\dot x_3 = -x_3$ is GAS. Now we can write



By Lem. 4.7 therefore the origin of the third-order sys. is GAS. That is,

$\boxed{\phi(x) = -x_3}$ does the job.

Q3.

(a) Consider the following second-order system

$$
\begin{aligned}
\dot{x}_1 &= -2x_1^3 - x_2 + u \\
\dot{x}_2 &= x_1^3 \\
y &= h(x)
\end{aligned}
$$

i) Choose a suitable output function h so that this system is passive through the storage function $V(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

ii) Is this system zero-state observable for the h you chose above? Explain.

iii) Is this system finite-gain $\mathcal{L}_2$ stable for the h you chose above? If yes, find an upper bound on the $\mathcal{L}_2$ gain.

(b) Investigate the stability properties of the origin of the fourth-order system below.

$$
\begin{aligned}
\dot{x}_1 &= -2x_1^3 - x_2 - x_3^3 \\
\dot{x}_2 &= x_1^3 \\
\dot{x}_3 &= -2x_3^3 - x_4 + x_1^3 \\
\dot{x}_4 &= x_3^3
\end{aligned}
$$

a) i) $\dot V = x_1^3 \dot x_1 + x_2 \dot x_2$

$= x_1^3\{-2x_1^3 - x_2 + u\} + x_2 x_1^3$

$= -2x_1^6 + x_1^3 u$

Choose $y = x_1^3$. Then

$\dot V = -2y^2 + yu \qquad (1)$

$\Rightarrow$ the sys. is output strictly passive.

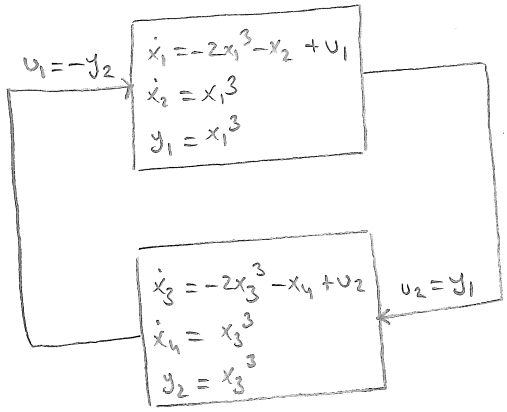
ii) Let $u \equiv 0$ & $y \equiv 0$. Then

$x_1 \equiv 0 \Rightarrow \dot x_1 \equiv 0 \Rightarrow x_2 \equiv 0$.

YES, the sys. is ZSO.

iii) By (1) and Lem 6.5 the sys. is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain $\le \frac{1}{2}$

b) We can write



Both subsystems (being identical) are ZSO & OSP. Furthermore, the storage functions are rad. unbounded. Hence, the origin of the fourth-order system is GAS by Thm 6.3.

Q4.

For each of the below claims either provide a proof (if true) or find a counterexample (if false).

(a) If an LTI system is BIBO stable then it is also $\mathcal{L}_2$ stable.

(b) If an LTI system is $\mathcal{L}_2$ stable then it is also input-to-state stable.

a) FALSE. Consider

$$\dot{x}_1 = -x_1 + u$$
$$\dot{x}_2 = x_2$$
$$y = x_1 + x_2$$

The TF of this sys. is $G(s) = \frac{1}{s+1}$. Hence the system is BIBO stable. Now, $\mathcal{L}_2$ stability implies

$$\|y\|_{\mathcal{L}_2} \leq \beta \quad \text{when} \quad u \equiv 0 \qquad (1)$$

where β is fixed (and determined by $x(0)$)

However for our system $u \equiv 0$ and $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ yields $y(t) = e^{t}$. Hence we cannot find β satisfying (1).

b) FALSE. Consider

$$\dot{x}_1 = -x_1 + u$$
$$\dot{x}_2 = x_2$$
$$y = x_1$$

This sys. is $\mathcal{L}_2$ stable. However, the origin is unstable. Hence the sys. cannot be ISS.