First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

(a) Consider the following second-order system

$$\begin{aligned}\dot{x}_1 &= -x_1 + \tilde{u}\\ \dot{x}_2 &= -x_1^3 - x_2^2\tilde{u}\end{aligned}$$

where $\tilde{u}$ is the control input. Show that the feedback law $\tilde{u} = x_2$ asymptotically stabilizes the origin of the closed-loop system. *Hint: Use* $V(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

(b) Consider the following third-order system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_3\\ \dot{x}_2 &= -x_1^3 - x_2^2 x_3\\ \dot{x}_3 &= u\end{aligned}$$

where u is the control input. Using backstepping, design a state feedback law $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2, x_3)$) is globally asymptotically stable.

Q2.

(a) Consider the following second-order system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + \tilde{u} \\ \dot{x}_2 &= -x_1^3 - x_2^3\end{aligned}$$

where $\tilde{u}$ is the control input. Show that this system is input-to-state stable. *Hint: Use* $V(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ *and recall the inequality* $2(x_1^4 + x_2^4) \geq \|x\|^4$.

(b) Consider the following third-order system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + x_3 \\ \dot{x}_2 &= -x_1^3 - x_2^3 \\ \dot{x}_3 &= u\end{aligned}$$

where u is the control input. Find a state feedback law $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2, x_3)$) is globally asymptotically stable.

Q3.

(a) Consider the following second-order system

$$\begin{aligned} \dot{x}_1 &= -2x_1^3 - x_2 + u \\ \dot{x}_2 &= x_1^3 \\ y &= h(x) \end{aligned}$$

i) Choose a suitable output function h so that this system is passive through the storage function $V(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

ii) Is this system zero-state observable for the h you chose above? Explain.

iii) Is this system finite-gain $\mathcal{L}_2$ stable for the h you chose above? If yes, find an upper bound on the $\mathcal{L}_2$ gain.

(b) Investigate the stability properties of the origin of the fourth-order system below.

$$\begin{aligned} \dot{x}_1 &= -2x_1^3 - x_2 - x_3^3 \\ \dot{x}_2 &= x_1^3 \\ \dot{x}_3 &= -2x_3^3 - x_4 + x_1^3 \\ \dot{x}_4 &= x_3^3 \end{aligned}$$

Q4.

For each of the below claims either provide a proof (if true) or find a counterexample (if false).

(a) If an LTI system is BIBO stable then it is also $\mathcal{L}_2$ stable.

(b) If an LTI system is $\mathcal{L}_2$ stable then it is also input-to-state stable.