First name:____________________

Last name:____________________

Student ID:____________________

Signature:____________________

Read before you start:

- There are five questions.
- The examination is open-book.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Total**

Q1.

Consider the following second-order system

$$\begin{aligned}\dot{x}_1 &= x_1^2 x_2 \\ \dot{x}_2 &= \sin(x_1) + u\end{aligned}$$

where u is the control input. Using backstepping, design a state feedback law $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that the origin of the closed-loop system (obtained by letting $u = \phi(x_1, x_2)$) is globally asymptotically stable.

Sol'n. Observe that the origin of $\dot{x}_1 = -x_1^3$ is GAS which can be established by the Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$. We can write $\dot{x}_1 = x_1^2 x_2 = -x_1^3 + x_1^2(x_1 + x_2) = -x_1^3 + x_1^2 z$ where we introduce the new state variable $z := x_1 + x_2$. Then $\dot{z} = \dot{x}_1 + \dot{x}_2 = x_1^2 x_2 + \sin(x_1) + u = v$ where we introduce the new input $v := x_1^2 x_2 + \sin(x_1) + u$. Hence the dynamics of the (x_1, z) system read

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_1^2 z \\ \dot{z} &= v\,.\end{aligned}$$

Construct the Lyapunov function $V(x_1, z) = V_1(x_1) + \frac{1}{2}z^2$. We can write

$$\begin{aligned}\dot{V} &= x_1 \dot{x}_1 + z\dot{z} \\ &= x_1(-x_1^3 + x_1^2 z) + zv \\ &= -x_1^4 + (x_1^3 + v)z\,.\end{aligned}$$

Choosing $v = -x_1^3 - kz$ for any constant $k > 0$ we obtain $\dot{V} = -x_1^4 - kz^2 < 0$. That is, the feedback $v = -x_1^3 - kz$ stabilizes the origin. Finally, we write

$$\begin{aligned}u &= v - x_1^2 x_2 - \sin(x_1) \\ &= -x_1^3 - kz - x_1^2 x_2 - \sin(x_1) \\ &= -x_1^3 - k(x_1 + x_2) - x_1^2 x_2 - \sin(x_1)\,.\end{aligned}$$

Hence the feedback law $\phi(x_1, x_2) = -x_1^3 - k(x_1 + x_2) - x_1^2 x_2 - \sin(x_1)$ does what we desire.

Q2.

(a) Using input-to-state stability (ISS) arguments, show that the origin of the below third-order system is globally asymptotically stable.

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2 + x_3^2 \\ \dot{x}_2 &= -x_2^3 + x_3^4 \\ \dot{x}_3 &= -x_3^5\end{aligned}$$

(b) Determine whether the below second-order system is ISS or not.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{x_1}{1+|x_1|} - x_2 + u\end{aligned}$$

Sol'n. (a) Let $f_1 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x_1, u_1) = -x_1 + \|u_1\|^2$. Consider the system $\dot{x}_1 = f_1(x_1, u_1)$ where u_1 is the input. The positive definite function $V_1(x_1) = \frac{1}{2}x_1^2$ allows us to write

$$\dot{V}_1 = -x_1^2 + x_1\|u_1\|^2 \leq -\frac{1}{2}x_1^2 - |x_1|\left(\frac{1}{2}|x_1| - \|u_1\|^2\right) .$$

Hence we have $\dot{V}_1 \leq -\frac{1}{2}x_1^2$ whenever $|x_1| \geq 2\|u_1\|^2$. Then we can assert by Thm. 4.19 that the system $\dot{x}_1 = f_1(x_1, u_1)$ is ISS. Let now $f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined as $f_2(x_2, u_2) = -x_2^3 + u_2^4$. The system $\dot{x}_2 = f(x_2, u_2)$ can be shown to be also ISS using the above arguments. Finally, we can establish using the Lyapunov function $V_3(x_3) = \frac{1}{2}x_3^2$ that the origin of the the scalar system $\dot{x}_3 = -x_3^5$ is GAS. Now, let us put all the pieces together. Using the cascade structure of the original system we can first claim that the origin $(x_2, x_3) = (0, 0)$ of the subsystem

$$\begin{aligned}\dot{x}_2 &= -x_2^3 + x_3^4 = f_2(x_2, u_2)\Big|_{u_2=x_3} \\ \dot{x}_3 &= -x_3^5\end{aligned}$$

is GAS by Lemma 4.7. Then invoking Lemma 4.7 for the second time we can assert that the origin of the original system is GAS because we can write

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2 + x_3^2 = f_1(x_1, u_1)\Big|_{u_1=[x_2\ x_3]^T} \\ \dot{x}_2 &= -x_2^3 + x_3^4 \\ \dot{x}_3 &= -x_3^5\end{aligned}$$

(b) For the initial condition $(x_1(0), x_2(0)) = (0, 1)$ and the bounded input $u(t) = x_2(0) + \frac{x_1(t)}{1+|x_1(t)|}$ we can write

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0\end{aligned}$$

which yields the unbounded solution $(x_1(t), x_2(t)) = (t, 1)$. That is, a bounded input can generate an unbounded solution. Hence the system cannot be ISS.

Q3.

Consider the following systems $\mathcal{H}_1$ and $\mathcal{H}_2$

$$\mathcal{H}_1 : \begin{cases} \dot{x} &= [S - \alpha C^T C]x + C^T e_1 \\ y_1 &= Cx\,, \end{cases} \qquad \mathcal{H}_2 : \begin{cases} \dot{z}_1 &= z_2^3 \\ \dot{z}_2 &= -\gamma z_1^3 - \beta z_2^3 + e_2 \\ y_2 &= z_2^3\,. \end{cases}$$

The system $\mathcal{H}_1$ is of order n, where $e_1 \in \mathbb{R}$ is the input, $y_1 \in \mathbb{R}$ is the output, $S \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix, $C \in \mathbb{R}^{1\times n}$, and (the scalar) α is a positive constant. The system $\mathcal{H}_2$ is second order, whose input and output are e_2 and y_2, respectively. The constants β and γ are positive.

(a) Choosing a suitable storage function, show that the system $\mathcal{H}_1$ is output strictly passive. Find an upper bound on the $\mathcal{L}_2$ gain of the system.

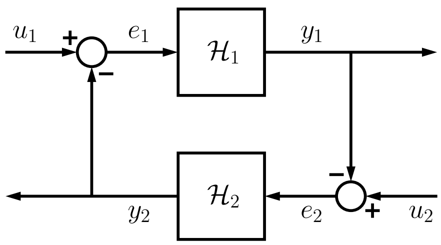
(b) Choosing a suitable storage function, show that the system $\mathcal{H}_2$ is also output strictly passive. Find an upper bound on the $\mathcal{L}_2$ gain of the system.

Sol'n. (a) Let $V_1(x) = \frac{1}{2}x^T x$ be our storage function. Since $x^T S x = \frac{1}{2}x^T[S + S^T]x = 0$ we can write $\dot{V}_1 = x^T\dot{x} = x^T[S - \alpha C^T C]x + x^T C^T e_1 = -\alpha x^T C^T C x + x^T C^T e_1 = -\alpha y_1^2 + y_1 e_1$. Hence the system is OSP. And by Lemma 6.5 its $\mathcal{L}_2$ gain is no larger than $\frac{1}{\alpha}$.

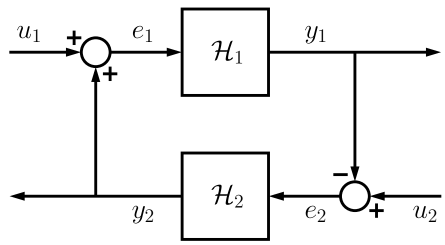
(b) Let $V_2(z) = \frac{1}{4}(\gamma z_1^4 + z_2^4)$ be our storage function. We have $\dot{V}_2 = \gamma z_1^3 \dot{z}_1 + z_2^3 \dot{z}_2 = \gamma z_1^3 z_2^3 + z_2^3(-\gamma z_1^3 - \beta z_2^3 + e_2) = -\beta y_2^2 + y_2 e_2$. Hence the system is OSP. And by Lemma 6.5 its $\mathcal{L}_2$ gain is no larger than $\frac{1}{\beta}$.

Q4.

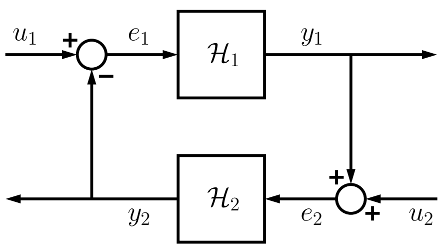
Consider the systems $\mathcal{H}_1$ and $\mathcal{H}_2$ of the previous problem. For each of the below feedback connections find (unrestrictive) conditions on the parameters α, β, γ under which the closed-loop system with input $u = [u_1\ u_2]^T$ and output $y = [y_1\ y_2]^T$ is finite-gain $\mathcal{L}_2$ stable.



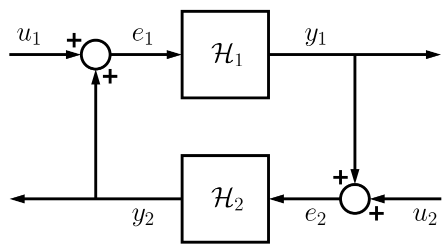
(a)



(b)



(c)



(d)

Sol'n. From the previous problem we know that both subsystems $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite gain $\mathcal{L}_2$ stable. And the small gain theorem tells us that their feedback connection will inherit the finite-gain $\mathcal{L}_2$ stability property if the product of the gains is strictly less than unity. Therefore if $\frac{1}{\alpha}\frac{1}{\beta} < 1$ or, equivalently, if $\alpha\beta > 1$ the closed-loop system will be finite-gain $\mathcal{L}_2$ stable for all cases (a), (b), (c), and (d) since the small gain theorem does not care about the sign of feedback (see its proof). For the negative feedback connections (b) and (c) we can further use Lemma 6.8 and make the stronger claim that the closed-loop system will always be finite-gain $\mathcal{L}_2$ stable (even for $\alpha\beta \leq 1$) because we have earlier obtained $e_1 y_1 \geq \dot{V}_1 + \alpha y_1^2$ and $e_2 y_2 \geq \dot{V}_2 + \beta y_2^2$.

Q5.

Consider the feedback connection (c) of the previous problem under the condition that the external inputs are zero: $u_1 = u_2 = 0$. Let $\eta = [x^T \; z_1 \; z_2]^T \in \mathbb{R}^{n+2}$ denote the state of the closed-loop system. Investigate the stability properties of the origin $\eta = 0$ in terms of the parameters S, C, α, β, γ.

Sol'n. Combining the storage functions we can construct the compound Lyapunov function $V(\eta) = V_1(x) + V_2(z) = \frac{1}{2}x^T x + \frac{1}{4}(\gamma z_1^4 + z_2^4)$ which yields $\dot{V} = -\alpha y_1^2 - \beta y_2^2 \leq 0$. Since V is positive definite the stability of the origin is therefore guaranteed for all parameter values.

To study GAS we can use Thm. 6.3. We have already established that both subsystems $\mathcal{H}_1$ and $\mathcal{H}_2$ are OSP. Let us find conditions under which they are also zero state observable. Let us consider $\mathcal{H}_1$ under the condition $e_1 \equiv 0$ and $y_1 \equiv 0$. We can write

$$y_1 \equiv 0 \implies Cx \equiv 0 \implies \dot{x} = Sx\,.$$

Hence if (and only if) the pair $(C,\, S)$ is observable then we can claim

$$Cx \equiv 0 \implies x \equiv 0\,.$$

Consequently, $\mathcal{H}_1$ is ZSO if the pair $(C,\, S)$ is observable. Similarly, let us consider $\mathcal{H}_2$ under the condition $e_2 \equiv 0$ and $y_2 \equiv 0$. We can write

$$y_2 \equiv 0 \implies z_2^3 \equiv 0 \implies z_2 \equiv 0 \implies \dot{z}_2 \equiv 0 \implies -\gamma z_1^3 \equiv 0 \implies z_1 \equiv 0\,.$$

This means that $\mathcal{H}_2$ is ZSO for all parameter values. Therefore, under the condition that the pair $(C,\, S)$ is observable, both subsystems are OSP and ZSO, which implies that the origin of the closed-loop system is GAS by Thm. 6.3.