First name: ____________________

Last name: KEY

Student ID: ____________________

Signature: ____________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Please EXPLAIN all your answers.

Q1	Q2	Q3	Q4	Total

Q1.

It is known that the origin of $\dot{\eta} = [F + GH]\eta$ is asymptotically stable, where $F \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times 1}$, and $H \in \mathbb{R}^{1 \times n}$. In particular, we have two symmetric positive definite matrices P, $Q \in \mathbb{R}^{n \times n}$ satisfying the Lyapunov equation

$$[F + GH]^T P + P[F + GH] = -Q\,.$$

Consider now the system

$$\begin{aligned} \dot{\eta} &= F\eta + G\xi \\ \dot{\xi} &= u\,. \end{aligned}$$

Using backstepping, design a feedback law $u = \psi(\eta, \xi)$ under which the origin of the closed-loop system is asymptotically stable.

Define $z = \xi - H\eta$ and write (η, z) system

$$\dot{\eta} = F\eta + G(z + H\eta) = [F + GH]\eta + Gz$$

$$\dot{z} = \dot{\xi} - H\dot{\eta} = \underbrace{u - HF\eta - HG\xi}_{=:v}$$

$$\Rightarrow \quad \dot{\eta} = [F + GH]\eta + Gz$$

$$\dot{z} = v$$

Let $V_c = \eta^T P \eta + \frac{1}{2} z^2$

$$\Rightarrow \dot{V}_c = \dot{\eta}^T P \eta + \eta^T P \dot{\eta} + z\dot{z}$$

$$= \eta^T \left([F + GH]^T P + P[F + GH] \right) \eta + 2\eta^T PGz + zv$$

$$= -\eta^T Q \eta + (2\eta^T PG + v) z$$

Choose $v = -2\eta^T PG - kz \qquad (k > 0)$

$$\Rightarrow \dot{V}_c = -\eta^T Q \eta - kz^2 \quad \text{neg. definite}$$

$$\Rightarrow (\eta, z) \to 0 \Rightarrow (\eta, \xi) \to 0$$

$$\Rightarrow u = HF\eta + HG\xi + v$$

$$= HF\eta + HG\xi - 2\eta^T PG - k(\xi - H\eta)$$

That is, we can let

$$\boxed{\psi(\eta, \xi) = [HF - 2G^T P + kH]\eta + [HG - k]\xi}$$

Q2.

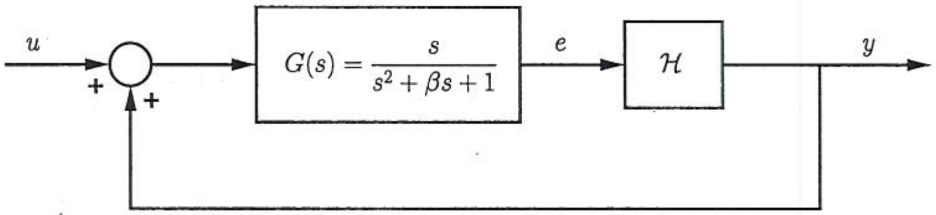
Consider the second order system

$$\mathcal{H}: \begin{cases} \dot{x}_1 &= -\alpha x_1^3 - x_2 + e \\ \dot{x}_2 &= x_1^3 \\ y &= x_1^3 \end{cases}$$

with $\alpha > 0$, where e is the input and y the output.

a) Show that the system $\mathcal{H}$ is finite-gain $\mathcal{L}_2$ stable. Find an upper bound on the gain. *Hint: You may want to work with* $V(x) = 0.25x_1^4 + 0.5x_2^2$.

b) Find a condition (in terms of α and β) under which the below closed-loop system (with input u and output y) is finite-gain $\mathcal{L}_2$ stable. Find an upper bound on the gain.



a) $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$

$\Rightarrow \dot{V} = x_1^3\dot{x}_1 + x_2\dot{x}_2 = -\alpha x_1^6 - \cancel{x_1^3 x_2} + x_1^3 e + \cancel{x_2 x_1^3}$

$= -\alpha y^2 + ye$

$\Rightarrow$ system finite-gain $\mathcal{L}_2$ stable with gain $\gamma_H \le \frac{1}{\alpha}$

b) The linear system $G(s) = \frac{s}{s^2+\beta s+1}$ is $\mathcal{L}_2$ stable if $\beta > 0$. The gain equals $\gamma_G = \frac{1}{\beta}$. Now:

$\|y\|_{\mathcal{L}_2} \le \gamma_H \|e\|_{\mathcal{L}_2} + C_H$ (C_H : some constant)

$\le \gamma_H \{\gamma_G \|u+y\|_{\mathcal{L}_2} + C_G\} + C_H$

$\le \gamma_H\gamma_G \|u\|_{\mathcal{L}_2} + \gamma_H\gamma_G\|y\|_{\mathcal{L}_2} + \gamma_H C_G + C_H$

Suppose $\gamma_H\gamma_G < 1$

Then $\|y\|_{\mathcal{L}_2} \le \frac{\gamma_H\gamma_G}{1-\gamma_H\gamma_G}\|u\|_{\mathcal{L}_2} + \frac{\gamma_H C_G + C_H}{1-\gamma_H\gamma_G}$

i.e. closed loop finite-gain $\mathcal{L}_2$ stable with

gain $\gamma_{CL} \le \frac{\gamma_H\gamma_G}{1-\gamma_H\gamma_G}$

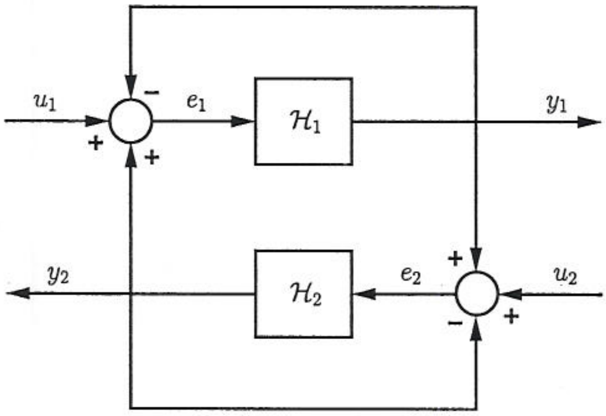
The condition $\gamma_H\gamma_G < 1$ is satisfied if

$\frac{1}{\alpha}\cdot\frac{1}{\beta} < 1 \Rightarrow \boxed{\alpha\beta > 1}$

and $\gamma_{CL} \le \frac{\frac{1}{\alpha}\cdot\frac{1}{\beta}}{1-\frac{1}{\alpha}\cdot\frac{1}{\beta}} = \boxed{\frac{1}{\alpha\beta-1}}$

Q3.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two single-input single-output systems. Both $\mathcal{H}_1$ and $\mathcal{H}_2$ are output strictly passive and zero state observable. Consider the following feedback connection with input $u = [u_1\ u_2]^T$ and output $y = [y_1\ y_2]^T$.



a) Is the closed-loop system (from u to y) output strictly passive?

b) Is the closed-loop system zero state observable?

c) Is the origin of the unforced ($u = 0$) closed-loop system asymptotically stable?

$$H_1 \begin{cases} \dot{x}_1 = f_1(x_1, e_1) \\ y_1 = h_1(x_1, e_1) \end{cases} \quad \& \quad \dot{V}_1 \leq -y_1\rho_1(y_1) + e_1 y_1$$

$$H_2 \begin{cases} \dot{x}_2 = f_2(x_2, e_2) \\ y_2 = h_2(x_2, e_2) \end{cases} \quad \& \quad \dot{V}_2 \leq -y_2\rho_2(y_2) + e_2 y_2$$

a) YES. Let $V = V_1 + V_2$

Note $e_1 = u_1 + y_2 - y_1$ & $e_2 = u_2 + y_1 - y_2$

$$\Rightarrow \dot{V} = \dot{V}_1 + \dot{V}_2$$

$$\leq -y_1\rho_1(y_1) - y_2\rho_2(y_2) + (u_1 + y_2 - y_1)y_1 + (u_2 + y_1 - y_2)y_2$$

$$= \underbrace{-y_1\rho_1(y_1) - y_2\rho_2(y_2)}_{-\underbrace{[y_1\ y_2]}_{y^T}\underbrace{\begin{bmatrix}\rho_1(y_1)\\ \rho_2(y_2)\end{bmatrix}}_{\rho(y)}} + \underbrace{u_1y_1 + u_2y_2}_{u^Ty} - (y_1 - y_2)^2$$

$$\Rightarrow \dot{V} \leq -y^T\rho(y) + u^Ty \quad \Rightarrow \text{OSP}$$

b) YES. Set $u_1 = u_2 = 0$ & $y_1 = y_2 = 0$.

$\Rightarrow e_1 = 0$ & $e_2 = 0$

$e_1 = 0$ & $y_1 = 0 \Rightarrow x_1 = 0$ because H_1 ZSO

$e_2 = 0$ & $y_2 = 0 \Rightarrow x_2 = 0$ because H_2 ZSO

Hence $u = 0$ & $y = 0 \Rightarrow \begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0 \Rightarrow$ ZSO

c) YES. Because the closed-loop system is both OSP & ZSO.

Q4.

Determine whether each of the below systems is input-to-state stable or not. *Hint: If needed, you may want to work with $V(x) = \|x\|^2$ and use the inequality $2(x_1^4 + x_2^4) \geq \|x\|^4$.*

$$\text{a)} \begin{cases} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= -x_1 - x_2^3 + u \end{cases} \qquad \text{b)} \begin{cases} \dot{x}_1 &= -x_1^3 - x_2 + u \\ \dot{x}_2 &= -x_1 - x_2^3 \end{cases}$$

a) $\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

$= -2x_1^4 + \cancel{2x_1x_2} - \cancel{2x_2x_1} - 2x_2^4 + 2x_2u$

$\leq -\|x\|^4 + 2\|x\| \cdot \|u\|$

$= -\frac{1}{2}\|x\|^4 - \frac{1}{2}\|x\|(\|x\|^3 - 4\|u\|)$

$\Rightarrow \dot{V} \leq -\frac{1}{2}\|x\|^4$ for $\|x\| \geq 4^{1/3}\|u\|^{1/3}$

$\Rightarrow \boxed{\text{ISS}}$

b) Set $u=0$. The linearization at the origin:

$\dot{\eta} = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{A}\eta$

$|sI - A| = s^2 - 1 = (s-1)(s+1) \Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$

$\lambda_1 > 0 \Rightarrow$ the origin $x=0$ is NOT AS

NOT AS $\Rightarrow \boxed{\text{NOT ISS}}$