First name:________________

Last name: KEY

Student ID:________________

Signature:________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1. 20%

Consider the system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz with $f(0, 0) = 0$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous with $h(0, 0) = 0$. Suppose that this system is zero state observable and lossless with a continuously differentiable positive definite storage function. Let $Q \in \mathbb{R}^{m\times m}$ be a symmetric positive definite matrix. Determine whether the below claim is true or false. *If you think that the claim is true, provide a proof; if you think that it is false, find a counterexample.*

Claim. The origin of the closed-loop system under the feedback $u = -Qy$ is asymptotically stable.

Let V be the storage function.

lossless $\Rightarrow$ $\dot{V} = y^T u$

$u = -Qy$ $\Rightarrow$ $\dot{V} = -y^T Q y \leq -\lambda_{min}(Q)\|y\|^2$

Hence, solutions converge to the largest invariant set within $\{\dot{V} = 0\}$. (LaSalle)

$$\left.\begin{aligned} \dot{V} \equiv 0 \Rightarrow\ & y \equiv 0 \\ \Rightarrow\ & u \equiv 0 \end{aligned}\right\} \Rightarrow x \equiv 0 \text{ (by ZSO)}$$

Therefore the origin is GAS. ∎

Q2. 30%

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -4x_2 - \mathrm{sat}(x_1) + u\\ y &= x_2\end{aligned}$$

where $\mathrm{sat}(\cdot)$ is the saturation function.[1]

a) Is this system input to state stable? Explain. *Hint: Consider* $u(t) = 4x_2(0) + \mathrm{sat}(x_1(t))$.

b) Is this system $\mathcal{L}_\infty$ stable? Explain.

c) Is this system $\mathcal{L}_2$ stable? Explain. *Hint: Consider* $V(x) = \int_0^{x_1} \mathrm{sat}(\alpha)d\alpha + \frac{1}{2}x_2^2$.

a) Let $u(t) = 4x_2(0) + \mathrm{sat}(x_1(t))$ & $x_2(0) \neq 0$

Then $\dot{x}_2 = 0 \Rightarrow \dot{x}_1 = x_2(0)$

$\Rightarrow x_1(t) = x_1(0) + x_2(0)t$

$\Rightarrow \|x(t)\| \to \infty$ as $t \to \infty$

However, $|u(t)| \leq |4x_2(0)| + 1$ for all t.

We've obtained:

$\to$ bounded input
$\to$ unbounded state
$\Bigg\} \Rightarrow \boxed{\text{NOT ISS}}$

b) $\dot{y} = -4y - \mathrm{sat}(x_1) + u$

$\Rightarrow y(t) = e^{-4t}y(0) + \int_0^t e^{-4(t-\tau)}[-\mathrm{sat}(x_1(\tau)) + u(\tau)]d\tau$

$\Rightarrow |y(t)| \leq |y(0)| + \underbrace{\left(\int_0^t e^{-4(t-\tau)}d\tau\right)}_{\leq 1/4}\left(1 + \sup_{\tau\in[0,t]}|u(\tau)|\right)$

$\Rightarrow \sup_{\tau\in[0,t]}|y(t)| \leq \left[|y(0)| + \frac{1}{4}\right] + \frac{1}{4}\sup_{\tau\in[0,t]}|u(\tau)|$

$\Rightarrow \boxed{\mathcal{L}_\infty \text{ stable}}$

c) $\dot{V} = \dot{x}_1 \mathrm{sat}(x_1) + \dot{x}_2 x_2$

$= x_2 \mathrm{sat}(x_1) + (-4x_2 - \mathrm{sat}(x_1) + u)x_2$

$= -4x_2^2 + ux_2 = -4y^2 + uy$

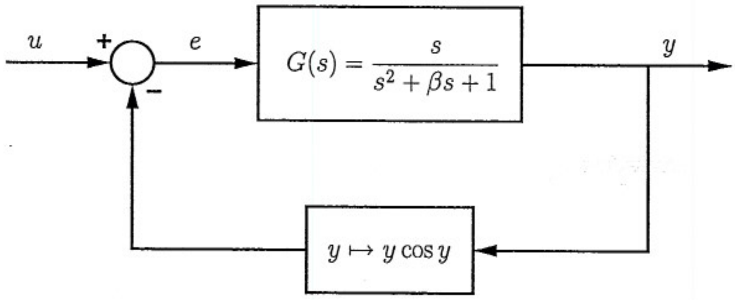
$\Rightarrow$ system output strictly passive with linear $\rho(\cdot)$ function.

$\Rightarrow \boxed{\mathcal{L}_2 \text{ stable}}$

[1] $\mathrm{sat}(\alpha) = \begin{cases} \min\{\alpha, 1\} & \text{for } \alpha \geq 0\\ \max\{\alpha, -1\} & \text{for } \alpha < 0\end{cases}$

Q3. 20%

Consider the block diagram below.



a) Find the range of β for which the open-loop system (i.e. the map from e to y) is $\mathcal{L}_2$ stable.

b) Find the range of β for which the closed-loop system (i.e. the map from u to y) is $\mathcal{L}_2$ stable. *Hint: Use the small gain theorem and note that we can write* $G(j\omega) = \frac{1}{\beta + j(\omega - \omega^{-1})}$.

a) For $\mathcal{L}_2$ stability we need BIBO stability for $G(s)$. Hence we need all the poles λ_i of $G(s)$ satisfy $Re\{\lambda_i\} < 0$

$\Rightarrow \boxed{\beta > 0}$

b) Let $\gamma_1 = \mathcal{L}_2$ gain of $G(s)$

$\gamma_2 = \mathcal{L}_2$ gain of "$y \mapsto y\cos y$"

$\gamma_2 = 1$ since $|y \cos y| \leq 1 \cdot |y|$

$$\gamma_1 = \max_{\omega \in \mathbb{R}} |G(j\omega)| = \max_{\omega} \left(\frac{1}{\beta^2 + (\omega - \omega^{-1})^2} \right)^{1/2}$$

$$= \frac{1}{\beta}$$

$\gamma_1 \gamma_2 < 1 \Rightarrow \frac{1}{\beta} < 1$

$\Rightarrow \boxed{\beta > 1}$

Q4. 30%

Part I. Using $V(x) = 0.25x_1^4 + 0.5x_2^2$ show that the below system is passive.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + u \\ y &= x_2 \end{aligned}$$

Part II. Consider now the identical pair of systems below.

$$\text{System (1)}: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 + u \\ y = x_2 \end{cases} \qquad \text{System (2)}: \begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 \\ \dot{\hat{x}}_2 = -\hat{x}_1^3 + \hat{u} \\ \hat{y} = \hat{x}_2 \end{cases}$$

Suppose that the systems are coupled via $u = \hat{y} - y$ and $\hat{u} = y - \hat{y}$.

a) Show that the origin $(x, \hat{x}) = 0$ is stable.

b) Show that the systems asymptotically synchronize. That is, for all initial conditions $x(0)$ and $\hat{x}(0)$, the solutions satisfy $\|x(t) - \hat{x}(t)\| \to 0$ as $t \to \infty$.

Part I

$\dot{V} = x_1^3 \dot{x}_1 + x_2 \dot{x}_2 = \cancel{x_1^3 x_2} - \cancel{x_1^3 x_2} + x_2 u$

$\Rightarrow \dot{V} = uy \Rightarrow$ system passive (lossless).

Part II

a) Let $W(x, \hat{x}) = V(x) + V(\hat{x})$

$\Rightarrow W$ is pos. def. & radially unbounded.

$\dot{W} = \dot{V}(x) + \dot{V}(\hat{x}) = uy + \hat{u}\hat{y}$

$= (\hat{y} - y)y + (y - \hat{y})\hat{y}$

$= -(y - \hat{y})^2$

$\Rightarrow \dot{W} \le 0 \Rightarrow$ the origin is stable. ▨

b) LaSalle $\Rightarrow$ solutions converge to the largest invariant set within $\{\dot{W} = 0\}$

$\dot{W} \equiv 0 \Rightarrow y - \hat{y} \equiv 0 \Rightarrow u \equiv 0$ & $\hat{u} \equiv 0$

also, $x_2 \equiv \hat{x}_2 \Rightarrow \dot{x}_2 \equiv \dot{\hat{x}}_2$

$\Rightarrow -x_1^3 \equiv -\hat{x}_1^3 \Rightarrow x_1 \equiv \hat{x}_1$

$\Rightarrow x \equiv \hat{x}$

▨