First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1. 20%

Consider the system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz with $f(0, 0) = 0$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous with $h(0, 0) = 0$. Suppose that this system is zero state observable and lossless with a continuously differentiable positive definite storage function. Let $Q \in \mathbb{R}^{m\times m}$ be a symmetric positive definite matrix. Determine whether the below claim is true or false. *If you think that the claim is true, provide a proof; if you think that it is false, find a counterexample.*

Claim. The origin of the closed-loop system under the feedback $u = -Qy$ is asymptotically stable.

Q2. 30%

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4x_2 - \mathrm{sat}(x_1) + u \\ y &= x_2\end{aligned}$$

where sat(·) is the saturation function.[1]

a) Is this system input to state stable? Explain. *Hint: Consider* $u(t) = 4x_2(0) + \mathrm{sat}(x_1(t))$.

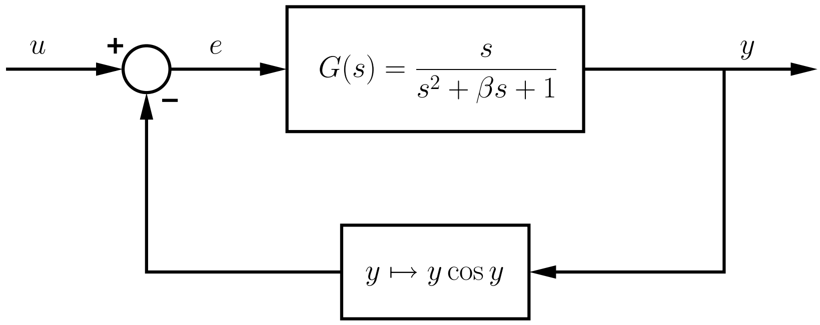
b) Is this system $\mathcal{L}_\infty$ stable? Explain.

c) Is this system $\mathcal{L}_2$ stable? Explain. *Hint: Consider* $V(x) = \int_0^{x_1} \mathrm{sat}(\alpha) d\alpha + \frac{1}{2}x_2^2$.

[1] $\mathrm{sat}(\alpha) = \begin{cases} \min\{\alpha, 1\} & \text{for} \quad \alpha \geq 0 \\ \max\{\alpha, -1\} & \text{for} \quad \alpha < 0 \end{cases}$

Q3. 20%

Consider the block diagram below.



a) Find the range of β for which the open-loop system (i.e. the map from e to y) is $\mathcal{L}_2$ stable.

b) Find the range of β for which the closed-loop system (i.e. the map from u to y) is $\mathcal{L}_2$ stable.
Hint: Use the small gain theorem and note that we can write $G(j\omega) = \dfrac{1}{\beta + j(\omega - \omega^{-1})}$.

Q4. 30%

Part I. Using $V(x) = 0.25x_1^4 + 0.5x_2^2$ show that the below system is passive.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + u \\ y &= x_2 \end{aligned}$$

Part II. Consider now the identical pair of systems below.

$$\text{System (1)} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 + u \\ y = x_2 \end{cases} \qquad \text{System (2)} : \begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 \\ \dot{\hat{x}}_2 = -\hat{x}_1^3 + \hat{u} \\ \hat{y} = \hat{x}_2 \end{cases}$$

Suppose that the systems are coupled via $u = \hat{y} - y$ and $\hat{u} = y - \hat{y}$.

a) Show that the origin $(x, \hat{x}) = 0$ is stable.

b) Show that the systems asymptotically synchronize. That is, for all initial conditions $x(0)$ and $\hat{x}(0)$, the solutions satisfy $\|x(t) - \hat{x}(t)\| \to 0$ as $t \to \infty$.