First name:________________________

Last name: Solution

Student ID:________________________

Signature:________________________

Read before you start:

- There are four questions.
- The examination is open-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1. 25%

Consider the second-order linear time-varying system

$$\begin{aligned}\dot{x}_1 &= -2x_1 - g(t)x_2\\ \dot{x}_2 &= x_1 - g(t)x_2\end{aligned}$$

where $g(t) = 2 + \cos(\omega t)$ with $\omega > 0$.

a) Using the function $V(t, x) = x_1^2 + g(t)x_2^2$ find a condition on ω for which the origin is exponentially stable.

b) For $\omega = 1$ find a pair of positive constants (c, λ) such that all solutions of the system satisfy $\|x(t)\| \le ce^{-\lambda t}\|x(0)\|$.

a) Note that $1 \le g(t) \le 3$

$\Rightarrow \quad \|x\|^2 \le V(t,x) \le 3\|x\|^2 \quad (1)$

$\dot{V} = \frac{\partial V}{\partial t} + \langle \nabla V, f(x) \rangle$

$= -\omega \sin(\omega t) x_2^2 + \begin{bmatrix} 2x_1 & 2g(t)x_2 \end{bmatrix} \begin{bmatrix} -2x_1 - g(t)x_2 \\ x_1 - g(t)x_2 \end{bmatrix}$

$= -\omega \sin(\omega t) x_2^2 - 4x_1^2 - \cancel{2g(t)x_1x_2} + \cancel{2g(t)x_1x_2} - 2g(t)^2 x_2^2$

$\le \omega x_2^2 - 4x_1^2 - 2x_2^2 = -4x_1^2 - (2-\omega)x_2^2 \quad (2)$

Let $\boxed{\omega < 2}$ then $\varepsilon := 2 - \omega > 0$ and

$(2) \Rightarrow \quad \dot{V} \le -\varepsilon\|x\|^2 \quad (3)$

Then (1) & (3) & Thm. 4.10 $\Rightarrow$ GES.

b) For $\omega = 1$ $\quad (3) \Rightarrow \quad \dot{V} \le -\|x\|^2$

Then $\quad \dot{V} \le -\|x\|^2$
$\quad\quad \le -\frac{1}{3} V \quad$ by (1)

$\Rightarrow \quad V(t, x(t)) \le e^{-\frac{t}{3}} V(0, x(0)) \quad (4)$

Now, by (1) $\quad \|x(t)\|^2 \le V(t, x(t))$
$\quad\quad \le e^{-\frac{1}{3}t} V(0, x(0)) \quad$ by (4)
$\quad\quad \le e^{-\frac{1}{3}t} 3\|x(0)\|^2 \quad$ by (1)

$\Rightarrow \quad \|x(t)\| \le \sqrt{3}\, e^{-\frac{t}{6}} \|x(0)\|$

That is, $\boxed{(c, \lambda) = (\sqrt{3}, \frac{1}{6})}$ will do.

Q2. 25%

$$\text{System (1)}: \begin{cases} \dot{x}_1 &= f_1(x_1, u_1) \\ y_1 &= x_1 \end{cases} \qquad \text{System (2)}: \begin{cases} \dot{x}_2 &= f_2(x_2, u_2) \\ y_2 &= x_2 \end{cases}$$

It is known that the above systems (1) and (2), with continuous right-hand sides, are finite-gain $\mathcal{L}_2$ stable with gains γ_1 and γ_2, respectively. For the below interconnection

$$\text{System (3)}: \begin{cases} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_2, x_1) \end{cases}$$

prove the following claims.

a) System (3) cannot have a nonzero equilibrium point if $\gamma_1\gamma_2 < 1$.

b) System (3) cannot have a (nontrivial) periodic solution if $\gamma_1\gamma_2 < 1$.

System (3) has no external inputs. Hence, by small gain theorem $\|x\|_{\mathcal{L}_2} \leq \beta_0$ where constant β_0 is determined by initial condition $x(0)$.

b) Suppose not. Then there exists a solution $x(t) \not\equiv 0$ such that $x(t+T) = x(t)$ for some $T>0$ and all t. Let $c := \int_0^T x(t)^T x(t)\,dt$. Note that $c>0$ since $x(t) \not\equiv 0$.

Now, $\|x\|_{\mathcal{L}_2} = \left[\int_0^\infty x(t)^T x(t)\,dt \right]^{1/2}$

$$= \left[\sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} x(t)^T x(t)\,dt \right]^{1/2}$$

$$= \left[\sum_{n=0}^{\infty} \int_0^T x(t)^T x(t)\,dt \right]^{1/2}$$

$$= \left[\sum_{n=0}^{\infty} c \right]^{1/2} = \infty$$

This contradicts $\|x\|_{\mathcal{L}_2} \leq \beta_0 < \infty$.

a) This is a special case of part (b).

Q3. 25%

Let $V_1 : \mathbb{R}^n \to \mathbb{R}$ and $V_2 : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, positive definite functions. Consider the following system.

$$\begin{aligned}\dot{x}_1 &= \nabla V_2(x_2)\\ \dot{x}_2 &= -\nabla V_1(x_1) - \nabla V_2(x_2) + u\\ y &= h(x)\end{aligned}$$

a) Find a nontrivial output function h (i.e., $h(x) \equiv 0$ is not allowed) for which the system is finite-gain $\mathcal{L}_2$ stable. What is the gain? *Hint: Try* $V(x) = V_1(x_1) + V_2(x_2)$.

b) Show that if the system is input-to-state stable then ∇V_1 cannot be bounded, that is, no $K > 0$ exists such that $\|\nabla V_1(x_1)\| \le K$ for all $x_1 \in \mathbb{R}^n$.

a) $\langle \nabla V(x), f(x,u)\rangle = \left\langle \begin{bmatrix}\nabla V_1(x_1)\\ \nabla V_2(x_2)\end{bmatrix}, f(x,u)\right\rangle$

$= \cancel{\langle \nabla V_1(x_1), \nabla V_2(x_2)\rangle - \langle \nabla V_2(x_2), \nabla V_1(x_1)\rangle}$

$- \langle \nabla V_2(x_2), \nabla V_2(x_2)\rangle + \langle \nabla V_2(x_2), u\rangle$

$\Rightarrow \langle \nabla V(x), f(x,u)\rangle = -\|\nabla V_2(x_2)\|^2 + u^T \nabla V_2(x_2)$

Now, if we let $y = \boxed{\nabla V_2(x_2) =: h(x)}$

we can write $\dot{V} + y^T y = u^T y$. The result follows from Lemma 6.5. The gain is $\boxed{\gamma = 1}$

b) Given the initial condition $x(0) = \begin{bmatrix} x_1(0)\\ x_2(0)\end{bmatrix}$

choose $u(t) = \nabla V_1(x_1(t)) + \nabla V_2(x_2(0))$. Note that if ∇V_1 is bounded then $u(t)$ will also be bounded. Under $u(t)$ the dynamics read

$$\dot{x}_1 = \nabla V_2(x_2(t))$$
$$\dot{x}_2 = 0$$

which implies $x_1(t) = x_1(0) + \nabla V_2(x_2(0))\, t$ &

$$x_2(t) = x_2(0)$$

Hence, $\|x_1(t)\| \to \infty$ for bounded input. That is, the system cannot be ISS if ∇V_1 is bounded. ∎

Q4. 25%

Consider the system

$$\dot{x} = Fxx^T x + Guu^T u$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $F \in \mathbb{R}^{n\times n}$, and $G \in \mathbb{R}^{n\times m}$.

a) Suppose there exists a symmetric positive definite matrix P satisfying the Lyapunov equation $F^T P + PF + I = 0$. Show that the system is input-to-state stable. *Hint: Try* $V(x) = x^T Px$.

b) Show that if F is not full rank then the system cannot be input-to-state stable.

a) Let $V(x) = x^T P x$.

$$\langle \nabla V(x), f(x,u) \rangle = x^T P F x x^T x + x^T P G u u^T u + x^T x x^T F^T P x + u^T u u^T G^T P x$$

$$= \|x\|^2 \{ x^T \underbrace{(PF + F^T P)}_{=-I} x \} + \|u\|^2 \{ 2x^T P G u \}$$

$$\Rightarrow \dot{V} = -\|x\|^4 + 2\|u\|^2 x^T P G u$$

$$\leq -\|x\|^4 + 2\|P\|\cdot\|G\|\cdot\|x\|\cdot\|u\|^3$$

$$= -\frac{1}{2}\|x\|^4 - \frac{1}{2}\|x\| \left\{ \|x\|^3 - 4\|P\|\cdot\|G\|\cdot\|u\|^3 \right\}$$

$$\Rightarrow \dot{V} \leq -\frac{1}{2}\|x\|^4 \quad \text{when} \quad \|x\| \geq \left\{ 4\|P\|\cdot\|G\| \right\}^{1/3} \|u\|$$

The result follows by Thm. 4.19.

b) If F is not full rank then there exists $0 \neq \eta \in \mathbb{R}^n$ such that $F\eta = 0$. Therefore, η is an equilibrium point of $\dot{x} = f(x,u)\big|_{u=0}$.

Hence the origin is not the unique equilibrium point. This implies that the origin cannot be GAS for $\dot{x} = f(x,0)$. Finally,

no GAS $\Rightarrow$ no ISS ▨