First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is open-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1. 25%

Consider the second-order linear time-varying system

$$\begin{aligned} \dot{x}_1 &= -2x_1 - g(t)x_2 \\ \dot{x}_2 &= x_1 - g(t)x_2 \end{aligned}$$

where $g(t) = 2 + \cos(\omega t)$ with $\omega > 0$.

a) Using the function $V(t, x) = x_1^2 + g(t)x_2^2$ find a condition on ω for which the origin is exponentially stable.

b) For $\omega = 1$ find a pair of positive constants (c, λ) such that all solutions of the system satisfy $\|x(t)\| \leq ce^{-\lambda t}\|x(0)\|$.

Q2. 25%

$$\text{System (1)} : \begin{cases} \dot{x}_1 &= f_1(x_1, u_1) \\ y_1 &= x_1 \end{cases} \qquad \text{System (2)} : \begin{cases} \dot{x}_2 &= f_2(x_2, u_2) \\ y_2 &= x_2 \end{cases}$$

It is known that the above systems (1) and (2), with continuous right-hand sides, are finite-gain $\mathcal{L}_2$ stable with gains γ_1 and γ_2, respectively. For the below interconnection

$$\text{System (3)} : \begin{cases} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_2, x_1) \end{cases}$$

prove the following claims.

a) System (3) cannot have a nonzero equilibrium point if $\gamma_1 \gamma_2 < 1$.

b) System (3) cannot have a (nontrivial) periodic solution if $\gamma_1 \gamma_2 < 1$.

Q3. 25%

Let $V_1 : \mathbb{R}^n \to \mathbb{R}$ and $V_2 : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, positive definite functions. Consider the following system.

$$\begin{aligned} \dot{x}_1 &= \nabla V_2(x_2) \\ \dot{x}_2 &= -\nabla V_1(x_1) - \nabla V_2(x_2) + u \\ y &= h(x) \end{aligned}$$

a) Find a nontrivial output function h (i.e., $h(x) \equiv 0$ is not allowed) for which the system is finite-gain $\mathcal{L}_2$ stable. What is the gain? *Hint: Try* $V(x) = V_1(x_1) + V_2(x_2)$.

b) Show that if the system is input-to-state stable then ∇V_1 cannot be bounded, that is, no $K > 0$ exists such that $\|\nabla V_1(x_1)\| \leq K$ for all $x_1 \in \mathbb{R}^n$.

Q4. 25%

Consider the system

$$\dot{x} = Fxx^T x + Guu^T u$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $F \in \mathbb{R}^{n \times n}$, and $G \in \mathbb{R}^{n \times m}$.

a) Suppose there exists a symmetric positive definite matrix P satisfying the Lyapunov equation $F^T P + PF + I = 0$. Show that the system is input-to-state stable. *Hint: Try* $V(x) = x^T Px$.

b) Show that if F is not full rank then the system cannot be input-to-state stable.