EE555 First Midterm Exam 12 Nov 2025

First name:________________________

Last name: KEY________________________

Student ID:________________________

Signature:________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ given as

$$V(x) = x_1^2 - x_1 x_2^2 + x_2^4.$$

(5) **(a)** Show that V is positive definite.

(b) Show that $\nabla V(x) = 0$ only when $x = 0$.

(10) **(c)** For each of the below systems determine whether the origin is stable, asy. stable, unstable.

$$\text{(i)} \begin{cases} \dot{x}_1 = x_2^2 - 2x_1 \\ \dot{x}_2 = 2x_1 x_2 - 4x_2^3 \end{cases} \qquad \text{(ii)} \begin{cases} \dot{x}_1 = 2x_1 x_2 - 4x_2^3 \\ \dot{x}_2 = 2x_1 - x_2^2 \end{cases}$$

(10) **(d)** Find a sufficient condition on α so that the origin of the below system is GAS.

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -7 & \alpha \end{bmatrix} \nabla V(x)$$

a) $0 < \frac{1}{2}x_1^2 + \frac{1}{2}x_2^4 + \frac{1}{2}(x_1 - x_2^2)^2$

$= x_1^2 - x_1 x_2^2 + x_2^4 = V(x)$

b) $\nabla V(x) = \begin{bmatrix} 2x_1 - x_2^2 \\ -2x_1 x_2 + 4x_2^3 \end{bmatrix}$

$\nabla V(x) = 0 \Rightarrow 2x_1 - x_2^2 = 0 \Rightarrow 2x_1 = x_2^2 \quad (1)$

$\nabla V(x) = 0 \Rightarrow 0 = -2x_1 x_2 + 4x_2^3$ (1)

$= -x_2^3 + 4x_2^3$

$= 3x_2^3 \Rightarrow x_2 = 0 \overset{(1)}{\Rightarrow} x_1 = 0$

c) i) $\dot{x} = -\nabla V(x) \Rightarrow \dot{V} = -\|\nabla V(x)\|^2 < 0$

$V > 0$ & $\dot{V} < 0 \Rightarrow$ the origin is $\boxed{\text{AS.}}$

ii) $\dot{x} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S} \nabla V(x) \Rightarrow \dot{V} = \nabla V(x)^T S \, \nabla V(x) = 0$

$V > 0$ & $\dot{V} \leq 0 \Rightarrow$ the origin is $\boxed{\text{stable.}}$

$\dot{V} = 0 \Rightarrow V(x(t)) = \text{constant}$

$\Rightarrow V(x(t)) \not\to 0$

$\Rightarrow x(t) \not\to 0 \Rightarrow$ the origin is $\boxed{\text{not AS.}}$

d) $\dot{V} = -\nabla V(x)^T \overbrace{\begin{bmatrix} 1 & -1 \\ 7 & -\alpha \end{bmatrix}}^{P} \nabla V(x)$

$= -\nabla V(x)^T \hat{P} \, \nabla V(x)$

where $\hat{P} = \frac{1}{2}[P + P^T] = \begin{bmatrix} 1 & 3 \\ 3 & -\alpha \end{bmatrix}$

LPM: 1 & $-\alpha - 9$

$-\alpha - 9 > 0 \Rightarrow \boxed{\alpha < -9}$

Hence, $\alpha < -9 \Rightarrow \hat{P} > 0 \Rightarrow \dot{V} < 0$

Since $V > 0$ is rad. unbounded, GAS follows.

Q2.

For each of the below systems study the stability of the origin.

(8) (a) $\begin{cases} \dot{x}_1 = 2x_1^3 - x_2 \\ \dot{x}_2 = -x_1^3 - x_2 \end{cases}$ *Hint: Consider* $V(x) = \frac{1}{4}x_1^4 - \frac{1}{2}x_2^2$.

(8) (b) $\begin{cases} \dot{x}_1 = -2x_1^3 - x_2 \\ \dot{x}_2 = x_1^3 \end{cases}$ *Hint: Consider* $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

(9) (c) $\begin{cases} \dot{x}_1 = -2x_1^3 - x_2 \\ \dot{x}_2 = x_1^3 + 3(x_4 - x_2) \\ \dot{x}_3 = -x_4 \\ \dot{x}_4 = x_3^3 + 3(x_2 - x_4) \end{cases}$ *Hint: Consider* $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \frac{1}{4}x_3^4 + \frac{1}{2}x_4^2$.

a) $\dot{V} = x_1^3 \dot{x}_1 - x_2 \dot{x}_2 = x_1^3\{2x_1^3 - \cancel{x_2}\} - x_2\{-\cancel{x_1^3} - x_2\}$

$= 2x_1^6 + x_2^2 > 0$ on $\mathbb{R}^2$

The set $\{V(x) > 0\}$ has points arbitrarily close to the origin. Hence $\dot{V} > 0$ implies the origin is $\boxed{\text{unstable}}$ by Chetaev's Thm.

b) $\dot{V} = x_1^3 \dot{x}_1 + x_2 \dot{x}_2 = x_1^3\{-2x_1^3 - \cancel{x_2}\} + \cancel{x_2\{x_1^3\}}$

$= -2x_1^6 \leq 0$

Now, $\dot{V} \equiv 0 \Rightarrow -2x_1^6 \equiv 0 \Rightarrow x_1 \equiv 0 \Rightarrow \dot{x}_1 \equiv 0$

Then $x_2 = -2x_1^3 - \dot{x}_1 \equiv 0$

Hence $\dot{V} \equiv 0 \Rightarrow x \equiv 0$ (1)

Combining:

$V > 0$ and rad. unb. & $\dot{V} \leq 0$ & (1)

yields $\boxed{\text{GAS}}$ by LaSalle's invariance thm.

c) $\dot{V} = x_1^3 \dot{x}_1 + x_2 \dot{x}_2 + x_3^3 \dot{x}_3 + x_4 \dot{x}_4$

$= x_1^3\{-2x_1^3 - \cancel{x_2}\} + x_2\{\cancel{x_1^3} + 3(x_4 - x_2)\}$

$+ \cancel{x_3^3\{-x_4\}} + x_4\{\cancel{x_3^3} + 3(x_2 - x_4)\}$

$= -2x_1^6 + 3x_2(x_4 - x_2) - 3x_4(x_4 - x_2)$

$= -2x_1^6 - 3(x_4 - x_2)^2 \leq 0$

Now, $\dot{V} \equiv 0 \Rightarrow x_1 \equiv 0$ & $x_2 \equiv x_4$

$x_1 \equiv 0 \Rightarrow \dot{x}_1 \equiv 0 \Rightarrow x_2 = -2x_1^3 - \dot{x}_1 \equiv 0$

$x_2 \equiv 0 \Rightarrow x_4 \equiv 0 \Rightarrow \dot{x}_4 \equiv 0$

$x_3^3 = \dot{x}_4 + 3(x_4 - x_2) \equiv 0 \Rightarrow x_3 \equiv 0$

Hence $\dot{V} \equiv 0 \Rightarrow x \equiv 0$ (2)

Combining:

$V > 0$ and rad. unb. & $\dot{V} \leq 0$ & (2)

yields $\boxed{\text{GAS.}}$

Q3.

Consider the following system

$$\left.\begin{aligned} \dot{x}_1 &= 3x_2 - x_1 \\ \dot{x}_2 &= x_1 - x_2^3 + x_2 . \end{aligned}\right\} =: f(x)$$

(5) (a) Find all the equilibrium points of the system.

(10) (b) Study the stability of each equilibrium using linearization.

(10) (c) Show that this system cannot have finite escape times.

a) $f(x) = 0 \Rightarrow 3x_2 - x_1 = 0 \Rightarrow x_1 = 3x_2 \quad (1)$

$f(x) = 0 \Rightarrow 0 = x_1 - x_2^3 + x_2$

$= 3x_2 - x_2^3 + x_2 \quad (1)$

$= x_2(4 - x_2^2)$

$\Rightarrow x_2 \in \{0, \mp 2\} \overset{(1)}{\Rightarrow} \boxed{x_{eq} \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ -2 \end{bmatrix} \right\}}$

b) $A = \frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 3 \\ 1 & -3x_2^2 + 1 \end{bmatrix}$

$x_{eq} = [0 \ 0]^T$

$A = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \Rightarrow |sI - A| = s^2 - 4 = (s+2)(s-2)$

$\Rightarrow \lambda_1 = -2 \ \& \ \lambda_2 = 2$

$\Rightarrow Re\{\lambda_2\} > 0 \Rightarrow$ the origin is $\boxed{\text{unstable}}$

$x_{eq} = \mp [6 \ 2]^T$

$A = \begin{bmatrix} -1 & 3 \\ 1 & -11 \end{bmatrix} \Rightarrow |sI - A| = s^2 + 12s + 8 = (s+6)^2 - 28$

$\lambda_{1,2} = -6 \mp \sqrt{28} \Rightarrow Re\{\lambda_i\} < 0$ for $i = 1, 2$

Both equilibria $x = \mp \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ are $\boxed{\text{asy. stable}}$

c) We can write

$\dot{x} = \underbrace{\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}}_{B} x + \begin{bmatrix} 0 \\ -x_2^3 \end{bmatrix}$

Let $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 = \frac{1}{2}x^T x = \frac{1}{2}\|x\|^2$

Then $\dot{V} = x^T \dot{x} = x^T B x + x^T \begin{bmatrix} 0 \\ -x_2^3 \end{bmatrix}$

$= x^T B x - x_2^4$

$\leq x^T B x \leq \|B\| \cdot \|x\|^2 = 2\|B\| \cdot V$

$\Rightarrow \dot{V} \leq 2\|B\| \cdot V$ — comparison lemma

$\Rightarrow V(x(t)) \leq e^{2\|B\| t} V(x(0))$

$\Rightarrow \|x(t)\| \leq e^{\|B\| t} \|x(0)\|$

$\Rightarrow$ Solutions exist for all $t \geq 0$

$\Rightarrow \boxed{\text{NO finite esc. time}}$ ☒

Q4.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable and $f(x) = 0$ only for $x = 0$. Suppose the origins of both $\dot{x} = f(x)$ and $\dot{y} = -f(y)$ are stable. Show that the system $\dot{x} = f(x)$ has a periodic orbit.

(Notation: $B_r := \{x \in \mathbb{R}^2 : \|x\| < r\}$).

Proof By stability of $x=0$ $\exists \delta > 0$ s.t.

$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq 1 \quad \forall t \geq 0$

Let R be the closure of the following set

$\{y \in \mathbb{R}^2 : \|x(0)\| = \delta$ and $x(t) = y$ for some $t \geq 0\}$

Note that R is forward invariant (by construction) and also bounded since $R \subset B_2$.

Now, by stability of $y=0$ $\exists \alpha > 0$ s.t.

$\|y(0)\| \leq \alpha \Rightarrow \|y(t)\| \leq \frac{\delta}{2} \quad \forall t \geq 0 \qquad (1)$

Observe that $R \cap B_\alpha = \emptyset$. Otherwise we can find a solution $x(t)$ and some time $t_1 > 0$ s.t. $\|x(0)\| = \delta$ and $\|x(t_1)\| < \alpha$. Then using this sol^n we can construct the solution $y(t) := x(t_1 - t)$ on $t \in [0, t_1]$ which solves $\dot{y} = -f(y)$ and satisfies $\|y(0)\| < \alpha$ and $\|y(t_1)\| = \delta$ which contradicts (1).

Since $R \cap B_\alpha = \emptyset$, the set R contains no equil. point. The result follows from Poin.-Bend. Thm. $\blacksquare$