First name:____________________________

Last name:____________________________

Student ID:____________________________

Signature:____________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ given as

$$V(x) = x_1^2 - x_1 x_2^2 + x_2^4 \,.$$

(a) Show that V is positive definite.

(b) Show that $\nabla V(x) = 0$ only when $x = 0$.

(c) For each of the below systems determine whether the origin is stable, asy. stable, unstable.

$$\text{(i)} \begin{cases} \dot{x}_1 &= x_2^2 - 2x_1 \\ \dot{x}_2 &= 2x_1x_2 - 4x_2^3 \end{cases} \qquad \text{(ii)} \begin{cases} \dot{x}_1 &= 2x_1x_2 - 4x_2^3 \\ \dot{x}_2 &= 2x_1 - x_2^2 \end{cases}$$

(d) Find a sufficient condition on α so that the origin of the below system is GAS.

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -7 & \alpha \end{bmatrix} \nabla V(x)$$

Q2.

For each of the below systems study the stability of the origin.

(a) $$\begin{cases} \dot{x}_1 &= 2x_1^3 - x_2 \\ \dot{x}_2 &= -x_1^3 - x_2 \end{cases}$$

Hint: Consider $V(x) = \frac{1}{4}x_1^4 - \frac{1}{2}x_2^2$.

(b) $$\begin{cases} \dot{x}_1 &= -2x_1^3 - x_2 \\ \dot{x}_2 &= x_1^3 \end{cases}$$

Hint: Consider $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

(c) $$\begin{cases} \dot{x}_1 &= -2x_1^3 - x_2 \\ \dot{x}_2 &= x_1^3 + 3(x_4 - x_2) \\ \dot{x}_3 &= -x_4 \\ \dot{x}_4 &= x_3^3 + 3(x_2 - x_4) \end{cases}$$

Hint: Consider $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \frac{1}{4}x_3^4 + \frac{1}{2}x_4^2$.

Q3.

Consider the following system

$$\begin{aligned} \dot{x}_1 &= 3x_2 - x_1 \\ \dot{x}_2 &= x_1 - x_2^3 + x_2 \,. \end{aligned}$$

(a) Find all the equilibrium points of the system.

(b) Study the stability of each equilibrium using linearization.

(c) Show that this system cannot have finite escape times.

Q4.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable and $f(x) = 0$ only for $x = 0$. Suppose the origins of both $\dot{x} = f(x)$ and $\dot{y} = -f(y)$ are stable. Show that the system $\dot{x} = f(x)$ has a periodic orbit.