First name:____________________

Last name: Sol'n

Student ID:____________________

Signature:____________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.

For each of the below dynamics determine whether the system has a periodic orbit.

(a) $\begin{cases} \dot{x}_1 = -x_2 + x_1(1 - 2x_1^2 - 3x_2^2) \\ \dot{x}_2 = x_1 \end{cases}$ (b) $\begin{cases} \dot{x}_1 = x_2 - x_1(1 - 2x_1^2 - 3x_2^2) \\ \dot{x}_2 = -x_1 \end{cases}$

a) Let $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$

$$= \cancel{-x_1x_2} + x_1^2(1-2x_1^2-3x_2^2) + \cancel{x_2x_1}$$

$$= x_1^2(1-2(x_1^2+x_2^2)) - x_1^2x_2^2$$

$$\le x_1^2(1-4V)$$

$\Rightarrow$ The bounded set $M := \{x \in \mathbb{R}^2 : V(x) \le \frac{1}{4}\}$ is forward invariant. M contains a single equilibrium point, $x=0$. The linearization at the origin yields

$$\dot{\eta} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \eta$$

The eigenvalues $\lambda_{1,2}$ of A?

$$d(s) = |sI - A| = \begin{vmatrix} s-1 & 1 \\ -1 & s \end{vmatrix} = s^2 - s + 1$$

$$\Rightarrow d(s) = (s-\tfrac{1}{2})^2 + \tfrac{3}{4} \Rightarrow \lambda_{1,2} = \tfrac{1}{2} \mp j\tfrac{\sqrt{3}}{2}$$

$\Rightarrow$ $Re\{\lambda_i\} > 0$ for both $i=1,2$.

By Poincare - Bendixson Thm therefore this system has at least one periodic orbit.

b) Let $\dot{x} = f(x)$ denote the system in part (a) which, we have shown, admits a periodic solution $\phi(t)$. Now, the dynamics of the second system satisfies $\dot{x} = -f(x)$, for which $\psi(t) = \phi(-t)$ is a solution. Note that $\psi(t)$ must be periodic because $\phi(t)$ is. Hence the second system also has a periodic orbit.

Q2.

Consider the system

$$\begin{aligned}\dot{x}_1 &= (x_1x_2 - 1)x_1^3 + (x_1x_2 - 1 + x_2^2)x_1 \\ \dot{x}_2 &= -x_2\,.\end{aligned}$$

(a) Show that the set $\{x \in \mathbb{R}^2 : x_1x_2 \geq 2\}$ is forward invariant.

(b) Find all the equilibrium points of this system.

(c) Study the stability of each equilibrium point.

(d) Does this system have a GAS equilibrium point?

a) Let $V(x) = x_1x_2$. Let's analyze $\dot{V}$ at the border $x_1x_2 = 2$.

$$\begin{aligned}\dot{V} &= x_1\dot{x}_2 + \dot{x}_1x_2 \\ &= -x_1x_2 + (x_1x_2 - 1)x_1^3x_2 + (x_1x_2 - 1 + x_2^2)x_1x_2 \\ &= x_1x_2\left[-1 + (x_1x_2 - 1)x_1^2 + (x_1x_2 - 1 + x_2^2)\right] \quad \swarrow x_1x_2 = 2 \\ &= 2\left[-1 + x_1^2 + 1 + x_2^2\right] \\ &= 2(x_1^2 + x_2^2) > 0\end{aligned}$$

Hence $\dot{V} > 0$ when $V(x) = 2$. This means the set $\{V(x) \geq 2\}$ is forward invariant.

b) The origin $x=0$ is the unique equilibrium.

c) Linearize at $x=0$.

$$\dot{\mu} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{A} \mu \quad \Rightarrow \quad \lambda_{1,2} = -1$$

All the eigenvalues of A are on the open left half plane. Therefore by linearization test, the origin is AS.

d) NO. Because by part (a) we know that if $x(0)$ satisfies $x_1(0)x_2(0) \geq 2$ then $x(t)$ cannot be converging to the origin.

Q3.

Consider the second order system

$$\dot{x} = A\nabla V(x) =: f(x)$$

where $A \in \mathbb{R}^{2\times 2}$ and $V : \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable, positive definite function whose gradient satisfies $\nabla V(x) \neq 0$ for all $x \neq 0$. For each of the following cases study the stability of the origin of this system.

(a) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

Note that $\dot{V} = \langle \nabla V(x), f(x) \rangle$

$= [\nabla V(x)]^T A \nabla V(x)$ (*)

a) A is symmetric. Also,

$d(s) = |sI - A| = (s+2)^2 - 1 = (s+1)(s+3)$

$\Rightarrow \lambda_{1,2} = -1, -3$. Therefore A is neg. def.

Hence $\dot{V} < 0$ by (*). By Lyapunov's Thm the origin is AS.

b) $\dot{V} = [\nabla V(x)]^T A \nabla V(x)$

$= [\nabla V(x)]^T \left[\frac{A+A^T}{2} \right] \nabla V(x)$

$= [\nabla V(x)]^T \nabla V(x) = \|\nabla V(x)\|^2 > 0$

We have $\dot{V} > 0$ everywhere (except the origin)

The origin is unstable by Chetaev's Thm.

) $\dot{V} = [\nabla V(x)]^T \left[\frac{A+A^T}{2} \right] \nabla V(x) = 0$.

We have $\dot{V} \leq 0$. Hence the origin is stable.

But since $\dot{V} = 0$, $V(x(t))$ is constant

and $x(t) \not\to 0$. That is, the origin is

not AS.

Q4.

For each of the below systems study the stability of the origin.

(a) $\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = x_1^3 - x_2^3 \end{cases}$ *Hint: Consider* $V(x) = \frac{1}{4}x_1^4 - \frac{1}{2}x_2^2$.

(b) $\begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = x_1^3 \end{cases}$ *Hint: Consider* $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$. $\Rightarrow$ V is pos. def.

(c) $\begin{cases} \dot{x}_1 = -x_1^3 - x_2 \\ \dot{x}_2 = x_1^3 - x_2|x_2| \end{cases}$ *Hint: Consider* $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$. $\Rightarrow$ V is pos. def.

a) $\dot{V} = x_1^3 \dot{x}_1 - x_2 \dot{x}_2$

$= x_1^3 (x_1 + x_2) - x_2 (x_1^3 - x_2^3)$

$= x_1^4 + x_2^4 > 0$

$V(0) = 0$ and the set $\{V(x) > 0\}$ contains points arbitrarily close to the origin. Since $\dot{V} > 0$ everywhere (except the origin) the origin is unstable by Chetaev's Thm.

b) $\dot{V} = x_1^3 \dot{x}_1 + x_2 \dot{x}_2$

$= x_1^3 (-x_1 - x_2) + x_2 x_1^3$

$= -x_1^4 \leq 0$

Now, V pos. def & $\dot{V}$ neg semidef $\Rightarrow$ stability. Also,

$\dot{V} \equiv 0 \Rightarrow -x_1^4 \equiv 0 \Rightarrow x_1 \equiv 0$

$\Rightarrow \dot{x}_1 \equiv 0 \Rightarrow x_2 \equiv 0 \Rightarrow x \equiv 0$

By La Salle's invariance principle the origin is AS. Moreover, V being radially unbounded, we have GAS.

c) $\dot{V} = x_1^3 \dot{x}_1 + x_2 \dot{x}_2$

$= x_1^3 (-x_1^3 - x_2) + x_2 (x_1^3 - x_2 |x_2|)$

$= -x_1^6 - x_2^2 |x_2|$

$= -x_1^6 - |x_2|^3 < 0$

Now, V pos. def. & $\dot{V}$ neg. def. & V rad. unb. yield that the origin is GAS.