EE555 First Midterm Exam 16 Nov 2022

First name:____________________

Last name: KEY

Student ID:____________________

Signature:____________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.

For each of the below dynamics determine whether the system displays finite escape times.

$$\text{(a)} \begin{cases} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_1 x_2^2 \end{cases} \qquad \text{(b)} \begin{cases} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_1^2 x_2 \end{cases}$$

(a) Finite escape times occur.

(b) Finite escape times do not occur.

[See HW2 Problem 2 Sol'n.]

Q2.

Consider the second-order system

$$\dot{x} = [\,A - \|x\|^2 I\,]\,x$$

where $A \in \mathbb{R}^{2\times 2}$ is a nonsingular matrix.

(a) Find a general condition on A so that the system has a periodic orbit.

(b) Find a general condition on A so that the system has no periodic orbit.

Let λ_1, λ_2 denote the eigenvalues of A.

(a) Let $V(x) = x^T x = \|x\|^2$. We can write

$$\dot{V} = 2x^T\dot{x} = 2x^T[A - \|x\|^2 I]x$$
$$= 2\,[x^T A x - \|x\|^4]$$
$$\leq 2\,(\|A\|\cdot\|x\|^2 - \|x\|^4)$$
$$= 2\,(\|A\| - V)\,V$$

Hence the set $M = \{x : \|x\|^2 \leq \|A\|\}$ is forward invariant.

Note that $x=0$ is the unique equilibrium of the system if A has no (purely) real eigenval. Also note that $\dot{y} = Ay$ is the linearization of the system at the origin. Hence if the condition

$$\boxed{Re\{\lambda_i\} > 0 \quad \& \quad Im\{\lambda_i\} \neq 0 \qquad i=1,2}$$

holds then by Poincare-Bendixson Thm the set M contains a limit cycle.

(b) Suppose the condition

$$\boxed{Re\{\lambda_i\} < 0 \qquad i=1,2}$$

holds. That is, A is Hurwitz. Then we can find a symmetric pos. def. $P \in \mathbb{R}^{2\times 2}$ satisfying $A^TP + PA = -I$. Let $V(x) = x^T P x$. We can write

$$\dot{V} = x^T P\dot{x} + \dot{x}^T P x$$
$$= x^T P[A - \|x\|^2 I]x + x^T[A - \|x\|^2 I]^T P x$$
$$= x^T[A^TP + PA]x - 2\|x\|^2 x^T P x$$
$$= -\|x\|^2 - 2\|x\|^2\cdot V$$
$$< 0.$$

Hence the origin is GAS. Consequently, the system cannot have any periodic orbits.

Q3.

Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= -ax_1 - 2x_2^3 \\ \dot{x}_2 &= x_1 - bx_2\,. \end{aligned}$$

For each of the below cases determine if the origin is unstable, stable, asymptotically stable, globally asymptotically stable. (For some cases you may want to use $V(x) = x_1^2 + x_2^4$.)

(a) $a = 0$ and $b = 0$.

(b) $a = 1$ and $b = 1$.

(c) $a = 0$ and $b = -1$.

(d) $a = 1$ and $b = 0$.

We can write

$$\begin{aligned} \dot{V} &= 2x_1\dot{x}_1 + 4x_2^3\dot{x}_2 \\ &= 2x_1[-ax_1 - 2x_2^3] + 4x_2^3[x_1 - bx_2] \\ &= -2ax_1^2 - 4bx_2^4 \qquad (1) \end{aligned}$$

(a) When $a, b = 0$ (1) yields $\dot{V} = 0$. Since $\dot{V} \le 0$, by Lyapunov Thm. the origin is stable. Now, $\dot{V} = 0$ means $V(x(t)) = $ const. Hence $V(x(t)) \not\to 0$. That is, $x(t) \not\to 0$. The origin therefore is NOT asy. stable.

(b) When $a, b = 1$ we have $\dot{V} = -2x_1^2 - 4x_2^4$. Therefore,

$$\begin{cases} V \text{ is pos. def. \& rad. unbounded} \\ \dot{V} \text{ is neg. def.} \end{cases}$$

and the origin is GAS.

(c) The linearization at the origin reads

$$\dot{y} = \underbrace{\begin{bmatrix} -a & 0 \\ 1 & -b \end{bmatrix}}_{A} y$$

When $a = 0$ & $b = -1$, the matrix A has an eigenvalue ($\lambda = 1$) on the open right half plane. The linearization test then tells us that the origin $x = 0$ is unstable.

(d) When $a = 1$ & $b = 0$ we have $\dot{V} = -2x_1^2 \le 0$

Now, $\dot{V} \equiv 0 \Rightarrow x_1 \equiv 0 \Rightarrow \dot{x}_1 \equiv 0 \Rightarrow x_2 \equiv 0$.

Hence the largest invariant set in $\{\dot{V} = 0\}$ is the origin. Then by invariance principle (and V is rad. unb.) the origin is GAS.

Q4.

Consider the third-order system

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3\\ \dot{x}_2 &= x_1 x_3 - x_2\\ \dot{x}_3 &= 1 - x_1 x_2\,.\end{aligned}$$

(a) Find the (unique) equilibrium point of this system.

(b) Is the equilibrium point asymptotically stable? Explain.

(c) Is it globally asymptotically stable? Explain.

(a) Solving $f(x)=0$ yields

$$\boxed{x_{eq} = \begin{bmatrix}1\\1\\1\end{bmatrix}}$$

(b) Linearize at $x = \begin{bmatrix}1\\1\\1\end{bmatrix}$:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2x_1 - 3x_1^2 & 0 & 0\\ x_3 & -1 & x_1\\ -x_2 & -x_1 & 0\end{bmatrix}_{x=\begin{bmatrix}1\\1\\1\end{bmatrix}}$$

$$= \begin{bmatrix} -1 & 0 & 0\\ 1 & -1 & 1\\ -1 & -1 & 0\end{bmatrix} = A$$

$$|sI - A| = (s+1)(s^2+s+1)$$

$$\Rightarrow \lambda_1 = -1\,,\ \lambda_{2,3} = -\frac{1}{2} \mp j\frac{\sqrt{3}}{2}$$

$$\Rightarrow Re\{\lambda_i\} < 0 \quad \forall i$$

$\Rightarrow$ the equilibrium is AS.

(c) NO. Observe that

$x_1(0) = 0 \Rightarrow x_1(t) = 0$ for all $t \geq 0$.

Hence $x_1(0) = 0 \Rightarrow x(t) \not\to \begin{bmatrix}1\\1\\1\end{bmatrix}$.