EE555 First Midterm Exam 30 Nov 2021

First name:____________________________

Last name: KEY

Student ID:____________________________

Signature:____________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - x_2 + x_2^2.\end{aligned}$$

(a) Obtain its linearization $\dot{\eta} = A\eta$ at the origin.

(b) Find a quadratic Lyapunov function $V : \mathbb{R}^2 \to \mathbb{R}$ that satisfies (for the linear system) $\dot{V}(\eta) = -\|\eta\|^2$.

(c) Using this V show that the origin of the nonlinear system is exponentially stable.

(d) Find a positive number c for which the set $\{x \in \mathbb{R}^2 : V(x) \leq c\}$ is forward invariant with respect to the nonlinear system.

a) $\boxed{\dot{\eta} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \eta}$

b) Solve the Lyap eqn. $A^T P + PA = -I$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2p_2 & p_1 - (p_2 + p_3) \\ p_1 - (p_2 + p_3) & 2(p_2 - p_3) \end{bmatrix}$$

$\Rightarrow p_1 = \frac{3}{2},\ p_2 = \frac{1}{2},\ p_3 = 1$

$\Rightarrow P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{pmatrix}$

$\Rightarrow V(\eta) = \eta^T P \eta = \boxed{\frac{3}{2}\eta_1^2 + \eta_1\eta_2 + \eta_2^2}$

c) $V(x) = \frac{3}{2}x_1^2 + x_1 x_2 + x_2^2 = x^T P x$

$\Rightarrow \lambda_{min}(P)\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\|x\|^2 \quad (1)$

$\dot{V} = \langle \nabla V(x), f(x) \rangle$

$= \left\langle \begin{bmatrix} 3x_1 + x_2 \\ 2x_2 + x_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ -x_1 - x_2 + x_2^2 \end{bmatrix} \right\rangle$

$= -x_1^2 - x_2^2 + 2x_2^3 + x_1 x_2^2$

Hence,

$\dot{V} = -\|x\|^2 + 2x_2^3 + x_1 x_2^2$

$\leq -\|x\|^2 + 3\|x\|^3$

$= -\frac{1}{2}\|x\|^2 - \frac{1}{2}\|x\|^2\underbrace{(1 - 6\|x\|)}_{\text{positive for } \|x\| < \frac{1}{6}}$

$\Rightarrow \dot{V} \leq -\frac{1}{2}\|x\|^2$ for $\|x\| < \frac{1}{6} \quad (2)$

(1) & (2) & Thm. 4.10 $\Rightarrow$ exp. stability. ∎

d) A conservative (but valid) choice would be $c = \lambda_{min}(P) \cdot \frac{1}{36}$. Because:

$V(x) \leq \frac{\lambda_{min}(P)}{36} \overset{(1)}{\Rightarrow} \|x\|^2 \leq \frac{1}{36}$

$\Rightarrow \|x\| \leq \frac{1}{6}$

$\overset{(2)}{\Rightarrow} \dot{V} < 0$ ∎

Q2.

Suppose the system $\dot{x} = f(x)$ satisfies the Lipschitz condition $\|f(x) - f(y)\| \leq L\|x - y\|$ for some (fixed) $L > 0$ and all x, $y \in \mathbb{R}^n$. Furthermore, suppose the origin is an equilibrium point. Using the function $V(x) = 0.5\|x\|^2$ and the comparison lemma, show that this system does not exhibit finite escape times.

Since $f(0) = 0$ we can write

$$\|f(x)\| = \|f(x) - f(0)\|$$
$$\leq L\|x - 0\|$$
$$= L\|x\|.$$

Then we have

$$\dot{V} = x^T \dot{x} = x^T f(x)$$
$$\leq L\|x\|^2$$
$$= 2LV.$$

By comparison lemma

$$\frac{1}{2}\|x(t)\|^2 = V(x(t))$$
$$\leq e^{2Lt} V(x(0))$$
$$= e^{2Lt} \left\{ \frac{1}{2} \|x(0)\|^2 \right\}$$

Hence,

$$\|x(t)\| \leq e^{Lt} \|x(0)\| \quad \text{for all } t \geq 0.$$

That is $x(t)$ remains finite during any finite interval. ∎

Q3.

Consider the second-order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix}.$$

Investigate the stability properties of the origin with respect to the parameter $\theta \in [0, 2\pi)$. That is, find the ranges of θ for which the origin is stable, asymptotically stable, and unstable. Justify your answer. *Hint: You may want to use* $V(x) = 0.25x_1^4 + 0.25x_2^4$.

$\dot{V} = \langle \nabla V(x), f(x) \rangle$

$= [x_1^3 \;\; x_2^3] \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix}$

$= \cos\theta\,(x_1^6 + x_2^6)$ (1)

There are three cases :

$\cos\theta > 0$

Note that $V(0) = 0$ and the set $U = \{V(x) > 0\}$ contains points that are arbitrarily close to the origin. Moreover, by (1) we see that $\dot{V} > 0$ in U. Hence the origin is unstable by Chetaev's Thm.

$\cos\theta = 0$

Note that V is pos. def. and $\dot{V} = 0$.

Since $\dot{V} \leq 0$ by Lyapunov's Thm the origin is stable. However, $\dot{V} = 0$ means $V(x(t)) =$ constant. That is, $V(x(t)) \not\to 0$.

Hence the origin is not asy. stable.

$\cos\theta < 0$

V pos. def. & rad. unbounded
$\dot{V}$ neg. def
$\Bigg\} \Rightarrow$ GAS.

To summarize :

$\theta \in$	the origin
$[0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$	unstable
$\{\frac{\pi}{2}, \frac{3\pi}{2}\}$	stable but not AS
$(\frac{\pi}{2}, \frac{3\pi}{2})$	GAS

Q4.

(a) $\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\dfrac{x_1}{1+x_1^2} \end{cases}$ (b) $\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\dfrac{x_1}{1+x_1^2} - x_2^3 \end{cases}$

For each of the above systems answer the following questions.

(i) Does the system have periodic orbits? Does it have limit cycles? Explain.

(ii) Is the origin stable? Is it asy. stable? Is it globally asy. stable? Explain.

Hint: $\displaystyle\int_0^{x_1} \frac{z}{1+z^2}\,dz = \frac{1}{2}\ln(1+x_1^2)$.

a) Let $V(x) = \frac{1}{2}x_2^2 + \frac{1}{2}\ln(1+x_1^2)$

Note that V is pos. def. and rad. unbounded.

$\dot{V} = \langle \nabla V, f(x) \rangle = \dfrac{x_1}{1+x_1^2} \cdot \dot{x}_1 + x_2 \dot{x}_2 = 0$

$\dot{V} = 0 \Rightarrow$ each solution $x(t)$ remains in the set $\{V(x) = c\} \supset \mathbb{R}^2$ where $c = V(x(0))$.

Since the sets $\{V(x) = \text{const.}\}$ are closed curves, every (nontrivial) solution generates a periodic orbit. We cannot have isolated periodic orbits because every solution is periodic. Hence there are no limit cycles.

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$\left.\begin{array}{l} V \text{ pos. def} \\ \& \ \dot{V} \le 0 \end{array}\right\} \Rightarrow$ the origin is $\boxed{\text{stable}}$

Since the solutions are periodic they cannot converge to the origin. Hence the origin is $\boxed{\text{not AS.}}$

b) Using the same V this time we have

$\dot{V} = -x_2^4$. Since $\dot{V} \le 0$ the origin is stable. Consider now the set

$S = \{\dot{V} = 0\}$

$x(t) \in S \ \forall t \Rightarrow -x_2^4(t) \equiv 0$

$\Rightarrow x_2(t) \equiv 0$

$\Rightarrow \dot{x}_2(t) \equiv 0$

$\Rightarrow -\dfrac{x_1(t)}{1+x_1(t)^2} - x_2(t)^3 \equiv 0$

$\Rightarrow x_1(t) \equiv 0$

Hence, no solution can stay in S other than $x(t) \equiv 0$. Then (because V is pos. def. & rad. unbounded) the origin is $\boxed{\text{GAS.}}$ (Corollary 4.2)

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GAS $\Rightarrow$ every solution satisfies $x(t) \to 0$

$\Rightarrow$ we cannot have periodic orbits

$\Rightarrow$ we cannot have limit cycles.