First name:____________________________

Last name:____________________________

Student ID:____________________________

Signature:____________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + x_2^2 \,. \end{aligned}$$

(a) Obtain its linearization $\dot{\eta} = A\eta$ at the origin.

(b) Find a quadratic Lyapunov function $V : \mathbb{R}^2 \to \mathbb{R}$ that satisfies (for the linear system) $\dot{V}(\eta) = -\|\eta\|^2$.

(c) Using this V show that the origin of the nonlinear system is exponentially stable.

(d) Find a positive number c for which the set $\{x \in \mathbb{R}^2 : V(x) \leq c\}$ is forward invariant with respect to the nonlinear system.

Q2.

Suppose the system $\dot{x} = f(x)$ satisfies the Lipschitz condition $\|f(x) - f(y)\| \leq L\|x - y\|$ for some (fixed) $L > 0$ and all x, $y \in \mathbb{R}^n$. Furthermore, suppose the origin is an equilibrium point. Using the function $V(x) = 0.5\|x\|^2$ and the comparison lemma, show that this system does not exhibit finite escape times.

Q3.

Consider the second-order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix}.$$

Investigate the stability properties of the origin with respect to the parameter $\theta \in [0,\ 2\pi)$. That is, find the ranges of θ for which the origin is stable, asymptotically stable, and unstable. Justify your answer. *Hint: You may want to use* $V(x) = 0.25x_1^4 + 0.25x_2^4$.

Q4.

(a) $\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\dfrac{x_1}{1+x_1^2} \end{cases}$ **(b)** $\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\dfrac{x_1}{1+x_1^2} - x_2^3 \end{cases}$

For each of the above systems answer the following questions.

(i) Does the system have periodic orbits? Does it have limit cycles? Explain.

(ii) Is the origin stable? Is it asy. stable? Is it globally asy. stable? Explain.

Hint: $\displaystyle\int_0^{x_1} \frac{z}{1+z^2}\, dz = \frac{1}{2}\ln(1+x_1^2).$