First name:____________________________

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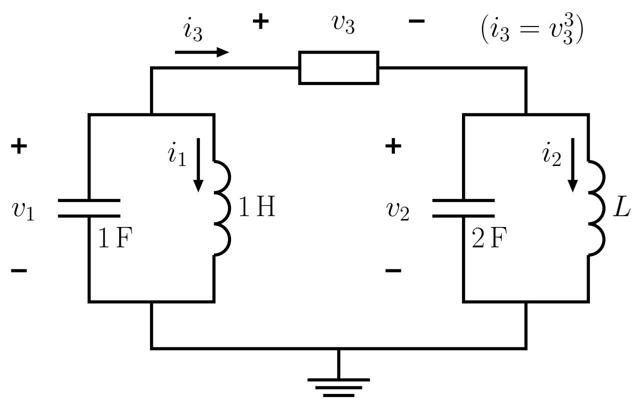
Read before you start:

- There are five questions.
- The examination is open-book.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Q5	Total

Q1.

Consider the following circuit where two LC oscillators are coupled via a nonlinear resistor whose i-v characteristics are described by the relation $i_3 = v_3^3$. Study the stability properties of the origin $x = [i_1\ v_1\ i_2\ v_2]^T = 0$ of the system in terms of the inductance $L > 0$ of the second oscillator. That is, determine for what value(s) of the inductance the system is unstable / stable / asymptotically stable.



Sol'n. Let $V(x) = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}C_1 v_1^2 + \frac{1}{2}L_2 i_2^2 + \frac{1}{2}C_2 v_2^2 = \frac{1}{2}i_1^2 + \frac{1}{2}v_1^2 + \frac{1}{2}L i_2^2 + v_2^2$, i.e., the total stored energy in the circuit. Note that V is positive definite. One can compute $\dot{V} = -i_3 v_3 = -(v_1 - v_2)^4$. Hence $\dot{V}$ is negative semidefinite and the origin is stable for all inductance values $L > 0$. To study asymptotic stability let us employ the invariance principle. Suppose $\dot{V} \equiv 0$. This implies $v_1(t) \equiv v_2(t)$ as well as $i_3(t) \equiv 0$. Under the condition $i_3(t) \equiv 0$ the oscillators become decoupled and we have to have $v_1(t) = A_1 \cos(\omega_1 t + \phi_1)$ and $v_2(t) = A_2 \cos(\omega_2 t + \phi_2)$ for some A_1, ϕ_1, A_2, ϕ_2 where $\omega_1 = 1/\sqrt{L_1 C_1} = 1$ and $\omega_2 = 1/\sqrt{L_2 C_2} = 1/\sqrt{2L}$ are the natural frequencies (in rad/sec) of the uncoupled oscillators. Now, if $\omega_1 \neq \omega_2$ (i.e., if $L \neq 1/2$) then $v_1(t) \equiv v_2(t)$ implies $v_1(t) \equiv v_2(t) \equiv 0$. The capacitor currents then must be zero, which by KCL yields $i_1(t) \equiv i_2(t) \equiv 0$ because $i_3(t) \equiv 0$. Consequently, the origin is asymptotically stable provided that $L \neq 1/2$. If $L = 1/2\,\text{H}$ on the other hand the decoupled oscillators have the same natural frequency ($\omega_1 = \omega_2 = 1$ rad/sec) and nonzero steady state oscillations $v_1(t) = v_2(t) = A\cos(t + \phi)$ are possible. In other words, when $L = 1/2\,\text{H}$ the origin is not asymptotically stable.

Q2.

Consider the LTV system $\dot{x} = A(t)x$ where $x \in \mathbb{R}^n$ and the matrix $A(t) \in \mathbb{R}^{n\times n}$ is a continuous function of time.

(a) Show that this system cannot exhibit finite escape times.

(b) Show that no solution $x(t)$ simultaneously satisfies $x(0) \neq 0$ and $x(T) = 0$ for some finite time $T > 0$.

Sol'n. (a) A short answer to this question is this: We can write $x(t) = \Phi(t,\, t_0)x(t_0)$, where Φ is the state transition matrix. Recall that the state transition matrix of a linear system is well defined for all times. Therefore the solution $x(t)$ is well defined for all times.

An alternative answer is as follows. Given an (arbitrary) interval of time $[t_0,\, t_1]$ define

$$L = \max_{t\in[t_0,\, t_1]} \|A(t)\| \,.$$

Since $A(t)$ is continuous we have $L < \infty$. Then we can write for all $t \in [t_0,\, t_1]$ and all $x,\, y \in \mathbb{R}^n$

$$\|A(t)x - A(t)y\| = \|A(t)(x - y)\| \leq \|A(t)\| \cdot \|x - y\| \leq L\|x - y\| \,.$$

That is, the system satisfies Lipschitz property. Therefore for each initial condition $x(t_0)$ a unique solution exists on $[t_0,\, t_1]$ which rules out the possibility of finite escape times.

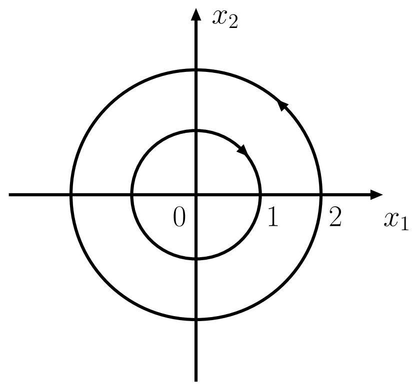
(b) A short answer to this question is this: We can write $x(T) = \Phi(T,\, 0)x(0)$. Hence $x(T) = 0$ implies $x(0) \in \operatorname{null}\Phi(T,\, 0)$. Recall that the state transition matrix of a linear system is always nonsingular, i.e., $\operatorname{null}\Phi(T,\, 0) = \{0\}$. Therefore $x(0) = 0$.

A more insightful answer is as follows. Suppose that there exists a solution $x(t)$ satisfying $x(0) \neq 0$ and $x(T) = 0$. Using this solution define on the the interval $[0,\, T]$ the function $\psi(t) = x(T - t)$. Note that $\psi(0) = 0$ and $\psi(T) \neq 0$. Also define the matrix $B(t) = -A(T - t)$. Since $A(t)$ is continuous, so is $B(t)$. Note that we can write

$$\dot{\psi} = \frac{d}{dt}\{x(T - t)\} = -A(T - t)x(T - t) = B(t)\psi \,.$$

Therefore $\psi(t)$ is a solution of the system $\dot{\psi} = B(t)\psi$ for the initial condition $\psi(0) = 0$. We know by the previous part that this solution is unique. But this contradicts the fact that the constant function $\hat{\psi}(t) \equiv 0$ is another possible solution starting from the same initial condition.

Q3.



Let $\dot{x} = f(x)$ be a second-order autonomous system, where $x = [x_1 \ x_2]^T \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable. Suppose that this system has periodic orbits, two of which are shown in the phase plane above. The outer periodic orbit satisfies $x_1(t)^2 + x_2(t)^2 = 4$ and rotates in ccw direction, whereas the inner one satisfies $x_1(t)^2 + x_2(t)^2 = 1$ and rotates in cw direction. Either prove or find a counterexample for the below claim.

Claim. The ring $\mathcal{R} = \{x \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 4\}$ must contain at least one equilibrium point.

Sol'n. FALSE. Below is a counterexample in polar coordinates (r, θ).

$$\begin{aligned} \dot{r} &= \sin((r-1)\pi) \\ \dot{\theta} &= 2r - 3\,. \end{aligned}$$

Q4.

For $x = [x_1\ x_2\ \cdots\ x_n]^T \in \mathbb{R}^n$ let us introduce the notation $x^3 := [x_1^3\ x_2^3\ \cdots\ x_n^3]^T$. Consider the system

$$\dot{x} = -Px^3$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Either prove or find a counterexample for the below claim.

Claim. The origin of this system must be asymptotically stable.

Sol'n. TRUE. Let $V(x) = \frac{1}{4}\sum_{i=1}^n x_i^4$. Note that V is positive definite ($V > 0$) and radially unbounded. We can write

$$\dot{V} = \sum_{i=1}^n x_i^3 \dot{x}_i = x^{3T}\dot{x} = -x^{3T}Px^3 .$$

That is, $\dot{V}$ is negative definite ($\dot{V} < 0$). Hence by Lyapunov's theorem the origin is GAS.

Alternatively, let $V(x) = \frac{1}{2}x^T P^{-1} x$. Note that $V > 0$ because P^{-1} is symmetric positive definite. We can write

$$\dot{V} = x^T P^{-1}\dot{x} = -x^T P^{-1} P x^3 = -x^T x^3 = -\sum_{i=1}^n x_i^4 < 0$$

whence GAS follows.

Q5.

Let $h : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satisfies:

- $h(0) = 0$ and $h(z) \neq 0$ for $z \neq 0$.
- $\left[\frac{\partial h}{\partial z}\right]_{z=0} \neq 0.$

(a) Show that the origin of the below second-order system cannot be asymptotically stable.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1)\,. \end{aligned}$$

(b) Show that the origin of the below third-order system must be unstable.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -h(x_1)\,. \end{aligned}$$

Sol'n. **(a)** Let us construct $V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} h(s)ds$, which is not necessarily positive definite. However, it is continuous and satisfies $V(0) = 0$. Moreover, we have $\dot{V} = 0$ along the solutions of the system. Suppose now the origin is asymptotically stable. This means we can find some $\varepsilon > 0$ such that the solution $x(t)$ starting from the initial state $(x_1(0),\, x_2(0)) = (0,\, \sqrt{2\varepsilon})$ satisfies $x(t) \to 0$. Since V is zero at zero and continuous, this implies $V(x(t)) \to 0$. But this cannot happen because $\dot{V} = 0$ implies $V(x(t)) = V(x(0)) = \varepsilon$ for all $t \geq 0$. Hence the origin cannot be asymptotically stable.

(b) Let $\alpha = [\partial h/\partial z]_{z=0}$. The Jacobian of the righthand side at $x = 0$ reads

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha & 0 & 0 \end{bmatrix}$$

which admits the characteristic polynomial $d(s) = s^3 + \alpha$. Since $\alpha \neq 0$ this means that the linearization of the system has at least one eigenvalue on the open right half complex plane. Therefore the origin is necessarily unstable.