First name:____________________________

Last name:____________________________

Student ID:____________________________

Signature:____________________________

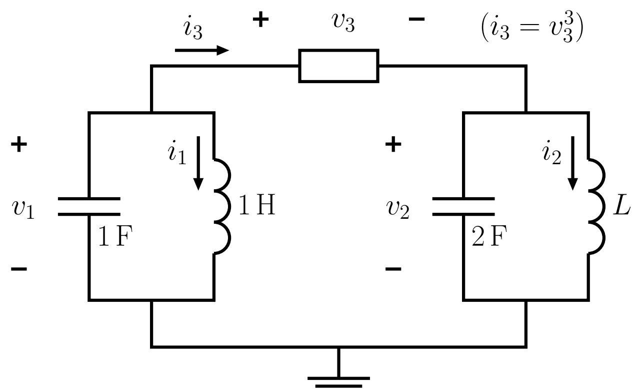
Read before you start:

- There are five questions.
- The examination is open-book.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Total**

Q1.

Consider the following circuit where two LC oscillators are coupled via a nonlinear resistor whose i-v characteristics are described by the relation $i_3 = v_3^3$. Study the stability properties of the origin $x = [i_1\ v_1\ i_2\ v_2]^T = 0$ of the system in terms of the inductance $L > 0$ of the second oscillator. That is, determine for what value(s) of the inductance the system is unstable / stable / asymptotically stable.



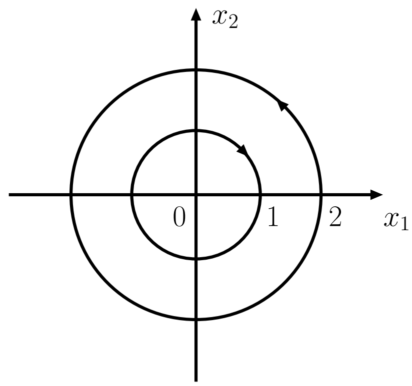
Q2.

Consider the LTV system $\dot{x} = A(t)x$ where $x \in \mathbb{R}^n$ and the matrix $A(t) \in \mathbb{R}^{n \times n}$ is a continuous function of time.

(a) Show that this system cannot exhibit finite escape times.

(b) Show that no solution $x(t)$ simultaneously satisfies $x(0) \neq 0$ and $x(T) = 0$ for some finite time $T > 0$.

Q3.



Let $\dot{x} = f(x)$ be a second-order autonomous system, where $x = [x_1 \; x_2]^T \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable. Suppose that this system has periodic orbits, two of which are shown in the phase plane above. The outer periodic orbit satisfies $x_1(t)^2 + x_2(t)^2 = 4$ and rotates in ccw direction, whereas the inner one satisfies $x_1(t)^2 + x_2(t)^2 = 1$ and rotates in cw direction. Either prove or find a counterexample for the below claim.

Claim. The ring $\mathcal{R} = \{x \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 4\}$ must contain at least one equilibrium point.

Q4.

For $x = [x_1 \; x_2 \; \cdots \; x_n]^T \in \mathbb{R}^n$ let us introduce the notation $x^3 := [x_1^3 \; x_2^3 \; \cdots \; x_n^3]^T$. Consider the system

$$\dot{x} = -Px^3$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Either prove or find a counterexample for the below claim.

Claim. The origin of this system must be asymptotically stable.

Q5.

Let $h : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satisfies:

- $h(0) = 0$ and $h(z) \neq 0$ for $z \neq 0$.
- $\left[\dfrac{\partial h}{\partial z}\right]_{z=0} \neq 0$.

(a) Show that the origin of the below second-order system cannot be asymptotically stable.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1)\,. \end{aligned}$$

(b) Show that the origin of the below third-order system must be unstable.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -h(x_1)\,. \end{aligned}$$