EE555 First Midterm Exam 24 November 2015

First name:____________________

Last name: KEY

Student ID:____________________

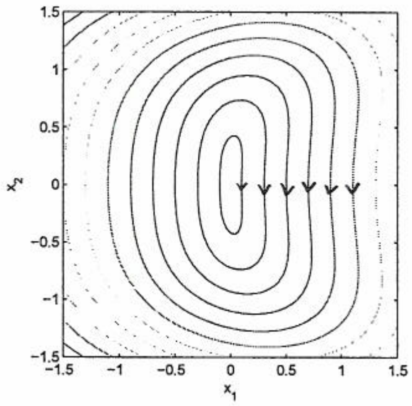
Signature:____________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1.



Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= \partial V/\partial x_2 \\ \dot{x}_2 &= -\partial V/\partial x_1\end{aligned}$$

where $V(x) = 3x_1^2 - x_1x_2^2 + x_2^4$. The level surfaces of V are shown in the figure above.

(5) a) Find all the equilibrium points of this system.

(10) b) Is the origin stable? Is it asymptotically stable?

(5) c) Does the system have periodic orbits? Does it have limit cycles?

(5) d) Sketch the phase portrait.

Note that we can write $V(x) = 2.5x_1^2 + 0.5x_2^4 + 0.5(x_1 - x_2^2)^2$. $\Rightarrow$ V is pos. def.

a) $\frac{\partial V}{\partial x_1} = 6x_1 - x_2^2 = 0 \Rightarrow x_1 = \frac{1}{6}x_2^2$

$\frac{\partial V}{\partial x_2} = 4x_2^3 - 2x_1x_2 \Rightarrow 4x_2^3 - 2\left(\frac{1}{6}x_2^2\right)x_2 = 0$

$\Rightarrow \frac{11}{3}x_2^3 = 0 \Rightarrow x_2 = 0 \Rightarrow x_1 = 0$

Therefore the origin $x = \begin{bmatrix}0\\0\end{bmatrix}$ is the only equilibrium.

b) $\langle \nabla V(x), f(x) \rangle = \begin{bmatrix}\partial V/\partial x_1 & \partial V/\partial x_2\end{bmatrix}\begin{bmatrix}\partial V/\partial x_2 \\ -\partial V/\partial x_1\end{bmatrix} = 0$

$\Rightarrow \dot{V}$ neg. semidefinite $\Rightarrow$ $\boxed{\text{the origin stable}}$

$\dot{V} = 0 \Rightarrow V(x(t)) = \text{constant} \Rightarrow V(x(t)) \not\to 0$

$\Rightarrow x(t) \not\to 0 \Rightarrow$ $\boxed{\text{the origin NOT AS}}$

c) YES. Since $V(x(t)) = \text{constant}$ and the only equilibrium is the origin, each level surface describes a periodic orbit.

NO. Because every solution is periodic.

d) See figure.

Q2.

Consider the second-order system

$$\dot{x}_1 = \partial V/\partial x_2 - \partial V/\partial x_1$$
$$\dot{x}_2 = -\partial V/\partial x_1$$

where $V(x) = 3x_1^2 - x_1 x_2^2 + x_2^4$.

(8) a) Is the origin stable?

(12) b) Is the origin asymptotically stable? Is it globally asymptotically stable?

(5) c) Is the origin exponentially stable? Is it globally exponentially stable?

a) $\langle \nabla V(x), f(x) \rangle = -\left(\frac{\partial V}{\partial x_1}\right)^2 \leq 0$

$\Rightarrow \dot{V}$ neg. semidefinite $\Rightarrow$ $\boxed{\text{the origin stable.}}$

b) Apply LaSalle's invariance principle.

Let $x(t)$ belong identically to the set $\{\dot{V}=0\}$

In this set we have $\frac{\partial V}{\partial x_1} = 6x_1 - x_2^2 = 0$ (1)

Eq. (1) implies:

$\dot{x}_1 = \partial V/\partial x_2 = 4x_2^3 - 2x_1 x_2$ (2)

& $\dot{x}_2 = 0$ (3)

(3) $\Rightarrow$ $x_2 =$ constant. Then (1) $\Rightarrow$ $x_1 =$ constant.

$\Rightarrow \dot{x}_1 = 0$ (4)

Now, (1), (2), (4) $\Rightarrow$ $4x_2^3 - 2\left(\frac{1}{6}x_2^2\right)x_2 = 0 \Rightarrow x_2 = 0$

Then (1) $\Rightarrow$ $x_1 = 0$.

Therefore, the only solution that stays identically in $\{\dot{V}=0\}$ is $x(t) \equiv 0$. $\boxed{\text{The origin is AS.}}$

V is radially unbounded $\Rightarrow$ $\boxed{\text{The origin is GAS}}$

c) $\left.\begin{aligned} \dot{x}_1 &= 4x_2^3 - 2x_1 x_2 - 6x_1 + x_2^2 \\ \dot{x}_2 &= -6x_1 + x_2^2 \end{aligned}\right\} = f(x)$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} -6 & 0 \\ -6 & 0 \end{bmatrix} = A$$

The eigenvalues of A are $\lambda_1 = -6$, $\lambda_2 = 0$.

Not all eigenvalues have negative real parts.

Therefore the origin of the linearization not ES.

$\Rightarrow$ $\boxed{\text{the origin of the actual system NOT ES.}}$

NOT ES $\Rightarrow$ $\boxed{\text{NOT GES.}}$

Q3.

Consider the LTV system

$$\dot{x} = A(t)x$$

the origin of which is uniformly asymptotically stable. That is, there exist a constant $c > 0$ and a class-$\mathcal{KL}$ function β such that the solutions of the system satisfy

$$\|x(t_0)\| \leq c \implies \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad \text{for all} \quad t \geq t_0 .$$

(15) a) Show that the origin is globally uniformly asymptotically stable.

(10) b) Show that the origin is globally exponentially stable.

Note that if $x(t)$ is a solution then $\hat{x}(t) = \alpha x(t)$ is also a solution for any scalar α.

a) Given $x(t) \not\equiv 0$ let $\alpha = \dfrac{c}{\|x(t_0)\|}$

and define $\hat{x}(t) := \alpha x(t)$.

$\|\hat{x}(t_0)\| = \alpha \|x(t_0)\| = c$. Therefore:

$$\|\hat{x}(t)\| \leq \beta(\|\hat{x}(t_0)\|, t - t_0) = \beta(c, t - t_0)$$

Define $\hat{\beta}(s,t) := \frac{s}{c}\beta(c,t)$. Note that $\hat{\beta} \in \mathcal{KL}$.

Now,

$$\|x(t)\| = \frac{1}{\alpha}\|\hat{x}(t)\| \leq \frac{1}{\alpha}\beta(c, t - t_0) = \frac{\|x(t_0)\|}{c}\beta(c, t - t_0)$$

Therefore $\|x(t)\| \leq \hat{\beta}(\|x(t_0)\|, t - t_0)$ ∎

b) Let $M := 2\hat{\beta}(1,0)$ & $T > 0$ be such that $\hat{\beta}(1,T) \leq \frac{1}{2}$. Given any $x(t) \not\equiv 0$ define $\hat{x}(t) = \dfrac{x(t)}{\|x(t_0)\|}$. Note that $\|\hat{x}(t_0)\| = 1$. Then

$$\|\hat{x}(t)\| \leq \hat{\beta}(1, t - t_0) \leq \hat{\beta}(1,0) \quad \text{for } t \geq t_0$$

Hence we can write

$$\|\hat{x}(t)\| \leq \frac{M}{2} \quad \text{for } t \in [t_0, t_0 + T] \quad \&$$

$$\|\hat{x}(t_0 + T)\| \leq \hat{\beta}(1,T) \leq \frac{1}{2}$$

This implies

$$\|x(t)\| \leq \frac{M}{2}\|x(t_0)\| \quad \text{for } t \in [t_0, t_0 + T] \quad (1)$$

$$\& \quad \|x(t_0 + T)\| \leq \frac{1}{2}\|x(t_0)\| \quad (2)$$

Since M is independent of t_0, (1)&(2) imply

$$\|x(t)\| \leq \frac{M}{2}\|x(t_0 + T)\|$$

$$\leq \frac{M}{4}\|x(t_0)\| \quad \text{for } t \in [t_0 + T, t_0 + 2T]$$

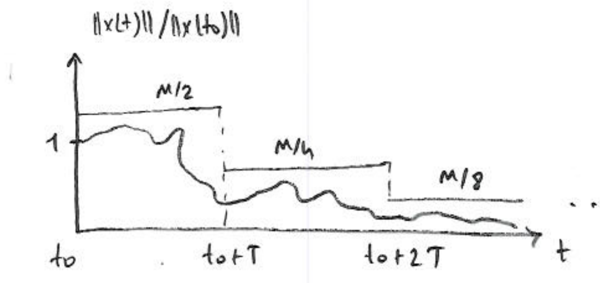
Also,

$$\|x(t_0 + 2T)\| \leq \frac{1}{2}\|x(t_0 + T)\| \leq \frac{1}{4}\|x(t_0)\| .$$

By induction we can write

$$\|x(t)\| \leq \frac{M}{2^k}\|x(t_0)\| \quad \text{for } t \in [t_0 + (k-1)T, t_0 + kT] \quad (3)$$

$$\& \quad \|x(t_0 + kT)\| \leq \frac{1}{2^k}\|x(t_0)\| \quad (4)$$



Let $\lambda = \dfrac{\ln 2}{T}$. Then (3)&(4) imply

$$\|x(t)\| \leq M e^{-\lambda(t - t_0)}\|x(t_0)\| .$$ ∎

Q4.

For each of the below situations determine whether the claim is true (T) or false (F). (No explanation is required.)

a) Consider the system $\dot{x} = f(x)$ where f is locally Lipschitz and the origin is an equilibrium point. Each solution of this system satisfies $\lim_{t\to\infty} \|x(t)\| \to 0$. *Claim: The origin must be asymptotically stable.*

b) Consider the LTV system $\dot{x} = A(t)x$ where $A(t)$ is continuous. Each solution of this system satisfies $\lim_{t\to\infty} \|x(t)\| \to 0$. *Claim: The origin must be asymptotically stable.*

c) Consider the LTV system $\dot{x} = A(t)x$ where $A(t)$ is continuous. Each solution of this system satisfies $\lim_{t\to\infty} \|x(t)\| \to 0$. *Claim: The origin must be uniformly stable.*

d) Let α_1, α_2 be class-$\mathcal{K}_\infty$ functions. Define $\alpha_3(s) := \min\{\alpha_1(s), \alpha_2(s)\}$. *Claim: α_3 must be a class-$\mathcal{K}_\infty$ function.*

e) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite, continuous function. *Claim: There must exist $\alpha \in \mathcal{K}$ such that $V(x) \leq \alpha(\|x\|)$ for all $x \in \mathbb{R}^n$.*

f) Let positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ and $\alpha \in \mathcal{K}$ satisfy $\alpha(\|x\|) \leq V(x)$ for all $x \in \mathbb{R}^n$. *Claim: V must be radially unbounded.*

g) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite, continuously differentiable function. Suppose $\nabla V(x) \neq 0$ for all $x \neq 0$. *Claim: The origin of the system $\dot{x} = -\nabla V(x)$ must be asymptotically stable.*

h) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable and $f(x) = 0$ only for $x = 0$. The origins of both $\dot{x} = f(x)$ and $\dot{x} = -f(x)$ are stable. *Claim: The system $\dot{x} = f(x)$ must have a periodic orbit.*

Your answer:

a	b	c	d	e	f	g	h
F	T	F	T	T	F	T	T