First name:____________________________

Last name:____________________________

Student ID:____________________________

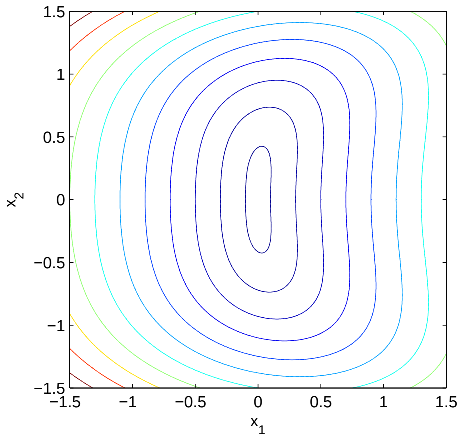
Signature:____________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.



Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= \partial V/\partial x_2 \\ \dot{x}_2 &= -\partial V/\partial x_1 \end{aligned}$$

where $V(x) = 3x_1^2 - x_1 x_2^2 + x_2^4$. The level surfaces of V are shown in the figure above.

a) Find all the equilibrium points of this system.

b) Is the origin stable? Is it asymptotically stable?

c) Does the system have periodic orbits? Does it have limit cycles?

d) Sketch the phase portrait.

Note that we can write $V(x) = 2.5x_1^2 + 0.5x_2^4 + 0.5(x_1 - x_2^2)^2$.

Q2.

Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= \partial V/\partial x_2 - \partial V/\partial x_1 \\ \dot{x}_2 &= -\partial V/\partial x_1\end{aligned}$$

where $V(x) = 3x_1^2 - x_1x_2^2 + x_2^4$.

a) Is the origin stable?

b) Is the origin asymptotically stable? Is it globally asymptotically stable?

c) Is the origin exponentially stable? Is it globally exponentially stable?

Q3.

Consider the LTV system

$$\dot{x} = A(t)x$$

the origin of which is uniformly asymptotically stable. That is, there exist a constant $c > 0$ and a class-$\mathcal{KL}$ function β such that the solutions of the system satisfy

$$\|x(t_0)\| \leq c \implies \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad \text{for all} \quad t \geq t_0 \,.$$

a) Show that the origin is globally uniformly asymptotically stable.

b) Show that the origin is globally exponentially stable.

Note that if $x(t)$ is a solution then $\hat{x}(t) = \alpha x(t)$ is also a solution for any scalar α.

Q4.

For each of the below situations determine whether the claim is true (T) or false (F). (No explanation is required.)

a) Consider the system $\dot{x} = f(x)$ where f is locally Lipschitz and the origin is an equilibrium point. Each solution of this system satisfies $\lim_{t\to\infty} \|x(t)\| \to 0$. *Claim: The origin must be asymptotically stable.*

b) Consider the LTV system $\dot{x} = A(t)x$ where $A(t)$ is continuous. Each solution of this system satisfies $\lim_{t\to\infty} \|x(t)\| \to 0$. *Claim: The origin must be asymptotically stable.*

c) Consider the LTV system $\dot{x} = A(t)x$ where $A(t)$ is continuous. Each solution of this system satisfies $\lim_{t\to\infty} \|x(t)\| \to 0$. *Claim: The origin must be uniformly stable.*

d) Let α_1, α_2 be class-$\mathcal{K}_\infty$ functions. Define $\alpha_3(s) := \min\{\alpha_1(s), \alpha_2(s)\}$. *Claim: α_3 must be a class-$\mathcal{K}_\infty$ function.*

e) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite, continuous function. *Claim: There must exist $\alpha \in \mathcal{K}$ such that $V(x) \leq \alpha(\|x\|)$ for all $x \in \mathbb{R}^n$.*

f) Let positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ and $\alpha \in \mathcal{K}$ satisfy $\alpha(\|x\|) \leq V(x)$ for all $x \in \mathbb{R}^n$. *Claim: V must be radially unbounded.*

g) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite, continuously differentiable function. Suppose $\nabla V(x) \neq 0$ for all $x \neq 0$. *Claim: The origin of the system $\dot{x} = -\nabla V(x)$ must be asymptotically stable.*

h) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable and $f(x) = 0$ only for $x = 0$. The origins of both $\dot{x} = f(x)$ and $\dot{x} = -f(x)$ are stable. *Claim: The system $\dot{x} = f(x)$ must have a periodic orbit.*

Your answer:

a	b	c	d	e	f	g	h