First name:________________

Last name: KEY

Student ID:________________

Signature:________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1. 25%

Consider the system $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$. Assume that we can write $f(x) = Ax + g(x)$ for some $A \in \mathbb{R}^{n\times n}$ and $g : \mathbb{R}^n \to \mathbb{R}^n$. Further assume that the following hold.

- $\|g(x)\| \le \gamma \|x\|^2$ for some $\gamma > 0$.
- $A^T P + PA = -I$ for some symmetric positive definite $P \in \mathbb{R}^{n\times n}$.

a) Show that the origin is an equilibrium of the system.

b) Using the Lyapunov function $V(x) = x^T P x$ establish asymptotic stability of the origin.

c) Obtain an estimate of the region of attraction in the form $\{x \in \mathbb{R}^n : V(x) < c\}$.

a) $\|g(0)\| \le \gamma \|0\|^2 \Rightarrow g(0) = 0$

$\Rightarrow f(0) = A \cdot 0 + g(0) = 0$

b) $\lambda_{min}(P)\|x\|^2 \le V(x) \le \lambda_{max}(P)\|x\|^2 \quad (1)$

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$
$$= (x^T A^T + g(x)^T) P x + x^T P (Ax + g(x))$$
$$= x^T (A^T P + PA) x + 2x^T P g(x)$$
$$= -\|x\|^2 + 2x^T P g(x)$$
$$\le -\|x\|^2 + 2\|x\| \cdot \|P\| \cdot \|g(x)\|$$
$$\le -\|x\|^2 + 2\gamma \|P\| \cdot \|x\|^3$$
$$= -\|x\|^2 \{1 - 2\gamma \|P\| \cdot \|x\|\}$$

Let $D := \{x \in \mathbb{R}^n : \|x\| < r\}$

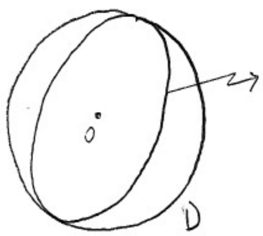
with $r := \dfrac{1}{2\gamma \|P\|}$

Note that D is a neighborhood of the origin.

Hence $\dot{V}(x) < 0$ for $x \in D - \{0\}$ (2)

(1) & (2) $\Rightarrow$ asy. stability ▨

c)



largest level surface of $V(x)$ that fits in D is the region of attraction.

$\Rightarrow$ region of attraction

$= \{x \in \mathbb{R}^n : V(x) < \lambda_{min}(P) \cdot r^2\}$

Q2. 20%

Consider the discrete-time system $x(k+1) = f(x(k))$ where $x(k)$ for $k \in \{0, 1, \ldots\}$ denotes the solution of the system starting from the initial condition $x(0) \in \mathbb{R}^n$. Assume that there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the conditions below.

- $\alpha_1\|x\|^2 \leq V(x) \leq \alpha_2\|x\|^2$ for some $\alpha_1,\ \alpha_2 > 0$.
- $V(f(x)) - V(x) \leq -\alpha_3\|x\|^2$ for some $\alpha_3 > 0$.

Show that the origin is globally exponentially stable. That is, there exist some $c > 0$ and $0 < \gamma < 1$ such that any solution of the system satisfies $\|x(k)\| \leq c\gamma^k\|x(0)\|$ for all $k \in \{0, 1, \ldots\}$.

$$V(x_{k+1}) - V(x_k) = V(f(x_k)) - V(x_k)$$

$$\leq -\alpha_3 \|x_k\|^2$$

$$\leq -\frac{\alpha_3}{\alpha_2} V(x_k)$$

$$\Rightarrow V(x_{k+1}) \leq \left(1 - \frac{\alpha_3}{\alpha_2}\right) V(x_k)$$

$$\Rightarrow V(x_k) \leq \left(1 - \frac{\alpha_3}{\alpha_2}\right)^k V(x_0)$$

$$\Rightarrow \alpha_1 \|x_k\|^2 \leq \left(1 - \frac{\alpha_3}{\alpha_2}\right)^k \alpha_2 \|x_0\|^2$$

$$\Rightarrow \|x_k\| \leq \sqrt{\alpha_2/\alpha_1} \left(\sqrt{1 - \frac{\alpha_3}{\alpha_2}}\right)^k \|x_0\|$$

$\boxtimes$

Q3. 30%

For each of the below systems determine whether the origin is stable or unstable. (If the origin turns out to be stable determine whether it is asymptotically stable or not.)

a) $\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 \end{cases}$ *Hint: Consider* $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

b) $\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 - x_2^3 \end{cases}$ *Hint: Consider* $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

c) $\begin{cases} \dot{x}_1 &= x_1^3 - x_2^3 \\ \dot{x}_2 &= -x_1^3 - x_2^3 \end{cases}$ *Hint: Consider* $V(x) = \frac{1}{4}x_1^4 - \frac{1}{4}x_2^4$.

a) $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ pos. def.

$$\dot{V}(x) = x_1^3\dot{x}_1 + x_2\dot{x}_2$$
$$= x_1^3x_2 + x_2(-x_1^3) = 0$$

$\dot{V}$ neg. semi def. $\Rightarrow$ the origin is $\boxed{\text{stable}}$

$\dot{V} = 0 \Rightarrow V(x(t)) = V(x(0)) \quad \forall t$

$\Rightarrow x(t) \not\to 0$

$\Rightarrow$ the origin is $\boxed{\text{not asy. stable}}$

b) $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ pos. def. & radially unbounded

$$\dot{V}(x) = x_1^3x_2 + x_2(-x_1^3 - x_2^3) = -x_2^4$$

$\dot{V}$ neg. semi def. $\Rightarrow$ the origin is stable

Apply LaSalle's principle:

$\dot{V} \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$

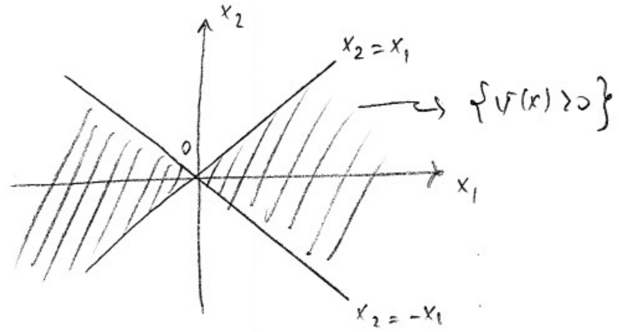
$\Rightarrow$ the origin is the largest invariant set within $\{\dot{V} = 0\}$

Hence the origin is $\boxed{\text{GAS}}$

c) $V(x) = \frac{1}{4}x_1^4 - \frac{1}{4}x_2^4$, $V(0) = 0$

$$\dot{V}(x) = x_1^3\dot{x}_1 - x_2^3\dot{x}_2$$
$$= x_1^3(x_1^3 - x_2^3) - x_2^3(-x_1^3 - x_2^3)$$
$$= x_1^6 + x_2^6$$

$\Rightarrow \dot{V} > 0$ everywhere (except at the origin) & the set $\{V(x) > 0\}$ contains points that are arbitrarily close to the origin



The origin is $\boxed{\text{unstable}}$ by Chetaev's Thm.

Q4. 25%

For each of the below situations determine whether the claim is true (T) or false (F). (No explanation is required.)

a) Consider the first order ($x \in \mathbb{R}^1$) system $\dot{x} = f(x)$ where f is continuous. For this system the number of asymptotically stable equilibria is two. *Claim: The number of all equilibria must be at least three.*

b) Consider the first order system $\dot{x} = f(x)$ where f is continuous. The origin is the only equilibrium. Moreover it is asymptotically stable. *Claim: The origin must be globally asymptotically stable.*

c) Consider the first order system $\dot{x} = f(x)$. *Claim: This system may have a (nontrivial) periodic solution.*

d) Consider the system $\dot{x} = f(x)$ with an equilibrium at the origin. The linearization of this system at the origin is unstable. *Claim: The origin cannot be asymptotically stable.*

e) Consider the LTV system $\dot{x} = A(t)x$ where $A(t)$ is a continuous function of time. *Claim: The system cannot exhibit finite escape times.*

f) Consider the LTV system $\dot{x} = A(t)x$ where $A(t)$ is a continuous function of time. All the eigenvalues of $A(t)$ are constant (i.e. $\lambda_i(t) = \lambda_i$) and satisfy $\mathrm{Re}(\lambda_i) < 0$. *Claim: The origin must be asymptotically stable.*

g) Consider the second order LTI system $\dot{x} = Ax$. This system has a periodic orbit. *Claim: The origin must be stable.*

h) Consider the second order system $\dot{x} = f(x)$ where f is continuously differentiable and $f(0) = 0$. This system has a periodic orbit that encircles the origin. *Claim: The origin must be stable.*

Your answer:

a	b	c	d	e	f	g	h
T	T	F	F	T	F	T	F