First name:________________________

Last name: Solution

Student ID:________________________

Signature:________________________

Read before you start:

- There are four questions.
- The examination is open-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

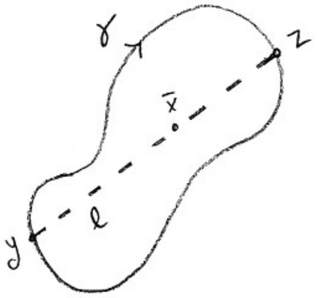
Q1	Q2	Q3	Q4	Total

Q1. 20%

Consider a planar system $\dot{x} = f(x)$ and a compact set $M \subset \mathbb{R}^2$, where M contains no equilibrium of the system and is positively invariant with respect to its solutions. Either prove or find a counterexample for the following claim.

Claim: M cannot be convex.[1]

By Poincare-Bendixson criterion M contains a periodic orbit γ. By index theorem γ should encircle an equilibrium point $\bar{x}$.



So, we can find a line segment ℓ satisfying:

→ its endpoints y, z belongs to γ

→ $\bar{x}$ belongs to ℓ

Now,

$$\left.\begin{array}{l} \left.\begin{array}{l} y, z \in \gamma \\ \gamma \subset M \end{array}\right\} \Rightarrow y, z \in M \\ \\ \left.\begin{array}{l} \bar{x} \in \ell \\ \bar{x} \notin M \end{array}\right\} \Rightarrow \ell \not\subset M \end{array}\right\} \Rightarrow \begin{array}{l} M \text{ cannot} \\ \text{be convex.} \end{array}$$

Hence, the claim is correct. ∎

[1] A set $C \subset \mathbb{R}^n$ is convex if for any two points $y, z \in C$ the straight line segment with endpoints y and z lies entirely in C. For instance, full moon (○) is convex whereas crescent (☾) is not.

Q2. 25%

Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= g(x_2)\\ \dot{x}_2 &= -x_1 - g(x_2)\end{aligned}$$

where g is continuous and satisfies $x_2^2 \leq x_2 g(x_2) \leq K x_2^2$ for some $K > 1$.

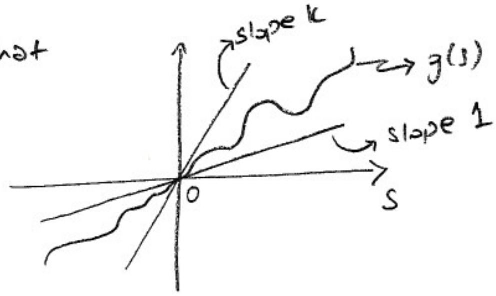
a) Show that the below function is positive definite.

$$V(x) = \frac{1}{2}x_1^2 + \int_0^{x_2} g(s)ds$$

b) Using this V as an energy function show that the origin is stable.

c) Employing the invariance principle establish asymptotic stability of the origin.

a) Note that



Hence, $\frac{1}{2}x_2^2 \leq \int_0^{x_2} g(s)ds \leq \frac{K}{2}x_2^2$

$\Rightarrow \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \leq V(x) \leq \frac{1}{2}x_1^2 + \frac{K}{2}x_2^2$

$\Rightarrow \frac{1}{2}\|x\|^2 \leq V(x) \leq \frac{K}{2}\|x\|^2$

$\Rightarrow V$ pos. def.

b) $\dot{V}(x) = \langle \nabla V(x), f(x) \rangle$

$= \begin{bmatrix} x_1 & g(x_2) \end{bmatrix} \begin{bmatrix} g(x_2) \\ -x_1 - g(x_2) \end{bmatrix}$

$= x_1 g(x_2) - g(x_2)x_1 - g^2(x_2)$

$= -g^2(x_2) \leq 0$

$\Rightarrow \dot{V}$ neg. semi def.

$\Rightarrow$ the origin is stable.

c) Let $S = \{x \in \mathbb{R}^2 : \dot{V}(x) = 0\}$

Note that $\dot{V}(x) = 0 \Rightarrow g(x_2) = 0 \Rightarrow x_2 = 0$

Hence, $S = \{x \in \mathbb{R}^2 : x_2 = 0\}$. Let $x(t)$ be a solution that belongs identically to S. Then

$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$.

$\Rightarrow x(t) \equiv 0$ is the only solution that can stay identically in S. By invariance principle, therefore, the origin is asy. stable.

Q3. 25%

Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= g(x_2)\\ \dot{x}_2 &= -x_1 - g(x_2)\end{aligned}$$

where g is continuous and satisfies $x_2^2 \le x_2 g(x_2) \le K x_2^2$ for some $K > 1$. Show that the origin is globally asymptotically stable by using the below function

$$V(x) = \frac{1}{2}x_1^2 + \int_0^{x_2} g(s)ds + \varepsilon x_1 x_2$$

where $\varepsilon > 0$ is your design parameter. *Hint:* $|x_1 x_2| \le \frac{\alpha x_1^2}{2} + \frac{x_2^2}{2\alpha}$ *for all* $\alpha > 0$.

Note that $\varepsilon |x_1 x_2| \le \frac{\varepsilon}{2}(x_1^2 + x_2^2)$

$\Rightarrow \frac{1-\varepsilon}{2}\|x\|^2 \le V(x) \le \frac{K+\varepsilon}{2}\|x\|^2 \qquad (1)$

For V pos. def. we need $\varepsilon < 1$.

$\dot{V}(x) = \langle \nabla V(x), f(x) \rangle$

$= \begin{bmatrix} x_1 + \varepsilon x_2 & g(x_2) + \varepsilon x_1 \end{bmatrix} \begin{bmatrix} g(x_2) \\ -x_1 - g(x_2) \end{bmatrix}$

$= -g^2(x_2) + \varepsilon x_2 g(x_2) - \varepsilon x_1^2 - \varepsilon x_1 g(x_2)$

$\le -x_2^2 + K\varepsilon x_2^2 - \varepsilon x_1^2 + K\varepsilon |x_1 x_2|$

$\le -(1-K\varepsilon)x_2^2 - \varepsilon x_1^2 + \frac{K\varepsilon}{2}\left(\alpha x_1^2 + \frac{x_2^2}{\alpha}\right)$

$= -\left(1 - K\varepsilon - \frac{K\varepsilon}{2\alpha}\right)x_2^2 - \left(\varepsilon - \frac{\alpha K \varepsilon}{2}\right)x_1^2$

$\downarrow \alpha = \frac{1}{K}$

$= -\left(1 - K\varepsilon - \frac{K^2\varepsilon}{2}\right)x_2^2 - \frac{\varepsilon}{2}x_1^2$

$\downarrow \varepsilon \le \frac{1}{2}\left(K + \frac{K^2}{2}\right)^{-1}$

$\le -\frac{1}{2}x_2^2 - \frac{\varepsilon}{2}x_1^2$

Hence, choosing $\varepsilon \le \frac{1}{2}\left[K + \frac{K^2}{2}\right]^{-1}$ (which automatically implies $\varepsilon < 1$, since $K > 1$) we can write

$\dot{V}(x) \le -\frac{\varepsilon}{2}\|x\|^2 \qquad (2)$

Now,

$(1) \Rightarrow V$ pos. def. & radially unbounded

$(2) \Rightarrow \dot{V}$ neg. def.

$\Rightarrow$ the origin is GAS $\square$

Q4. 30%

Consider the system $\dot{x} = f(x)$, where f is continuous and $f(\lambda x) = \lambda^2 f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$.

a) Show that if $\phi(t)$ is a solution of this system then so is $\psi(t) := \lambda\phi(\lambda t)$ for each $\lambda > 0$.

b) Can this system display finite escape times?

c) Can this system have no equilibrium? Can it have exactly two equilibria?

d) Can this system have a limit cycle?

e) Show for the origin of this system that asymptotic stability implies global asymptotic stability.

a) $\dot{\psi}(t) = \frac{d}{dt}\left\{\lambda \phi(\lambda t)\right\} = \lambda\left\{\frac{d}{dt}\phi(\lambda t)\right\}$

$= \lambda\left\{\lambda\dot{\phi}(\lambda t)\right\} = \lambda^2 f(\phi(\lambda t))$

$= f(\lambda\phi(\lambda t)) = f(\psi(t))$

b) YES, take, for instance, $\dot{x} = x^2 =: f(x)$.

c) NO, because

$f(0) = f(\lambda \cdot 0) = \lambda^2 f(0)$ for all $\lambda > 0$.

$\Rightarrow f(0) = 0 \Rightarrow x = 0$ is an equilibrium.

NO, because

let $\bar{x} \neq 0$ be an equilibrium. That is, $f(\bar{x}) = 0$. Then $0 = 4f(\bar{x}) = f(2\bar{x})$.

Hence, the system has "at least" three equilibria : $\{0, \bar{x}, 2\bar{x}\}$

d) NO, because

let $\phi(t)$ be a solution defining a periodic orbit γ. Since $\psi(t) = \lambda\phi(\lambda t)$ is also a solution, $\lambda\gamma$ is also a periodic orbit for all $\lambda > 0$. This means γ cannot be isolated. Hence, the system cannot have a limit cycle.

e) Let the origin be asy. stable. Then there must exist some $\delta > 0$ such that $\|x(0)\| \leq \delta \Rightarrow \|x(t)\| \to 0$ as $t \to \infty$. For GAS we need to show all solutions satisfy $\|x(t)\| \to 0$ as $t \to \infty$.

Now, given any solution $\phi(t)$ with $\phi(0) \neq 0$ let $\lambda := \frac{\delta}{\|\phi(0)\|}$ and define

$\psi(t) := \lambda\phi(\lambda t)$. Since

$\|\psi(0)\| = \frac{\delta}{\|\phi(0)\|}\|\phi(0)\| = \delta$ we can

write :

$\lim_{t\to\infty} \psi(t) = 0$

$\Rightarrow \lim_{t\to\infty} \lambda\phi(\lambda t) = 0$

$\Rightarrow \lim_{t\to\infty} \phi(t) = 0$