First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is open-book.
- No computer/calculator is allowed.
- The duration of the examination is 100 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1. 20%

Consider a planar system $\dot{x} = f(x)$ and a compact set $M \subset \mathbb{R}^2$, where M contains no equilibrium of the system and is positively invariant with respect to its solutions. Either prove or find a counterexample for the following claim.

Claim: M cannot be convex.[1]

[1] A set $C \subset \mathbb{R}^n$ is convex if for any two points y, $z \in C$ the straight line segment with endpoints y and z lies entirely in C. For instance, full moon (○) is convex whereas crescent (☾) is not.

Q2. 25%

Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= g(x_2)\\ \dot{x}_2 &= -x_1 - g(x_2)\end{aligned}$$

where g is continuous and satisfies $x_2^2 \leq x_2 g(x_2) \leq K x_2^2$ for some $K > 1$.

a) Show that the below function is positive definite.

$$V(x) = \frac{1}{2}x_1^2 + \int_0^{x_2} g(s)ds$$

b) Using this V as an energy function show that the origin is stable.

c) Employing the invariance principle establish asymptotic stability of the origin.

Q3. 25%

Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= g(x_2) \\ \dot{x}_2 &= -x_1 - g(x_2) \end{aligned}$$

where g is continuous and satisfies $x_2^2 \leq x_2 g(x_2) \leq K x_2^2$ for some $K > 1$. Show that the origin is globally asymptotically stable by using the below function

$$V(x) = \frac{1}{2}x_1^2 + \int_0^{x_2} g(s)ds + \varepsilon x_1 x_2$$

where $\varepsilon > 0$ is your design parameter. *Hint:* $|x_1 x_2| \leq \dfrac{\alpha x_1^2}{2} + \dfrac{x_2^2}{2\alpha}$ *for all* $\alpha > 0$.

Q4. 30%

Consider the system $\dot{x} = f(x)$, where f is continuous and $f(\lambda x) = \lambda^2 f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$.

a) Show that if $\phi(t)$ is a solution of this system then so is $\psi(t) := \lambda\phi(\lambda t)$ for each $\lambda > 0$.

b) Can this system display finite escape times?

c) Can this system have no equilibrium? Can it have exactly two equilibria?

d) Can this system have a limit cycle?

e) Show for the origin of this system that asymptotic stability implies global asymptotic stability.