Problem 1 Exercise 6.11.

Problem 2 Exercise 6.16.

Problem 3 Exercise 6.19.

Problem 4 Consider the following linear system

$$\begin{aligned} x^+ &= Ax \\ y &= Cx \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{1\times n}$. Suppose A is nonsingular and the pair (C, A) is observable. Show that applying Glad's observer construction to this system leads to the observer dynamics $\hat{x}^+ = A\hat{x} + L(y - C\hat{x})$. What is the observer gain L? What can be said about the eigenvalues of the matrix $A - LC$?

Problem 5 Let the origin of the continuous-time linear system $\dot{x} = Ax$ be asymptotically stable. Show that the origin of the discrete-time system obtained by Euler approximation $x^+ = x + \varepsilon Ax$ is asymptotically stable for small enough $\varepsilon > 0$ (a) by eigenvalue analysis and (b) by Lyapunov equation.

Problem 6 Consider the system $x^+ = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Suppose $f(\lambda x) = \lambda f(x)$ for all $\lambda \geq 0$ and $x \in \mathbb{R}^n$. Show that the asymptotic stability of the origin implies global exponential stability.

Problem 7 Consider the system $x^+ = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Suppose the origin is globally exponentially stable. Show that there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\begin{aligned} c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2 \\ V(f(x)) - V(x) \leq -c_3\|x\|^2 \end{aligned}$$

for some positive numbers c_1, c_2, c_3. *Hint: Try* $V(x) := \sum_{k=0}^{\infty} \|f^k(x)\|^2$ *where* $f^0(x) = x$.