Acknowledgment. Problems 3-4 are due to Professor Andrew Teel. More problems and their solutions are available on the web site: `http://www.ece.ucsb.edu/~teel/ECE236/`

Problem 1 Exercise 4.54.

Sol'n. See end of document.

Problem 2 Consider the system below

$$\begin{aligned}\dot{x}_1 &= -ax_1 + x_2\\ \dot{x}_2 &= bx_1 - g(x_1) - x_2 + u\end{aligned}$$

where $a > 0$, $g(0) = 0$, and $sg(s) \geq 0$ for all s. Using the below function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \int_0^{x_1} g(s)ds$$

find a condition on the parameter b so that the system is input-to-state stable.

Sol'n. Note that

$$\begin{aligned}\langle \nabla V(x),\, f(x)\rangle &= -ax_1^2 + x_1x_2 - ax_1g(x_1) + x_2g(x_1) + bx_1x_2 - x_2g(x_1) - x_2^2 + x_2u\\ &\leq -ax_1^2 + (b+1)x_1x_2 - x_2^2 + x_2u\,.\end{aligned}$$

Hence we can write

$$\dot{V}(x) \leq -x^TQx + x_2u$$

where

$$Q = \begin{bmatrix} a & -\frac{b+1}{2}\\ -\frac{b+1}{2} & 1\end{bmatrix}$$

Suppose now $Q > 0$. Then we have $\lambda_{\min}(Q) > 0$ and we can write

$$\begin{aligned}\dot{V}(x) &\leq -\lambda_{\min}(Q)\|x\|^2 + \|x\||u|\\ &= -\frac{\lambda_{\min}(Q)}{2}\|x\|^2 - \|x\|\left(\frac{\lambda_{\min}(Q)}{2}\|x\| - |u|\right)\,.\end{aligned}$$

Therefore

$$\dot{V}(x) \leq -\frac{\lambda_{\min}(Q)}{2}\|x\|^2 \quad \text{when} \quad \|x\| \geq \frac{2}{\lambda_{\min}(Q)}|u|$$

which implies ISS by Theorem 4.19. Finally we ask when is $Q > 0$? We can show easily that if $|b+1| < 2\sqrt{a}$ then $Q > 0$.

Problem 3 Consider the below system

$$\dot{y} = f(y) + g(y)u$$

where f is continuously differentiable and has a bounded Jacobian matrix, g is continuous and bounded, and the origin of $\dot{y} = f(y)$ is globally exponentially stable. Using Theorem 4.14 show that the system is input-to-state stable with a linear gain function. [By gain function we mean the class-$\mathcal{K}$ function γ of the inequality (4.47) in the text.]

Sol'n. By Theorem 4.14 there exists a Lyapunov function V_1 satisfying $c_1\|y\|^2 \leq V_1(y) \leq c_2\|y\|^2$ and $\langle \nabla V_1(y),\, f(y)\rangle \leq -c_3\|y\|^2$ and $\|\nabla V_1(y)\| \leq c_4\|y\|$. Since g is bounded we can find some $c_5 > 0$ such that $\|g(y)\| \leq c_5$ for all y. Now we can write

$$\begin{aligned}
\langle \nabla V_1(y),\, f(y) + g(y)u\rangle &= \langle \nabla V_1(y),\, f(y)\rangle + \langle \nabla V_1(y),\, g(y)u\rangle \\
&\leq -c_3\|y\|^2 + \|\nabla V_1(y)\| \cdot \|g(y)\| \cdot \|u\| \\
&\leq -c_3\|y\|^2 + c_4c_5\|y\|\|u\| \\
&= -\frac{c_3}{2}\|y\|^2 - \|y\|\left(\frac{c_3}{2}\|y\| - c_4c_5\|u\|\right).
\end{aligned}$$

Hence $\dot{V}_1 < 0$ when $\|y\| \geq 2c_4c_5c_3^{-1}\|u\|$. By Theorem 4.19 therefore the system is finite-gain ISS with $\gamma = 2c_1^{-1/2}c_2^{1/2}c_3^{-1}c_4c_5$.

Problem 4 Consider the second-order system

$$\begin{aligned}
\dot{x}_1 &= f(x_1) + g(x_2)x_2^4 \\
\dot{x}_2 &= -x_2|x_2|^3 + \varepsilon x_1
\end{aligned}$$

where f and g satisfy the conditions of the previous problem. Show that the origin is globally asymptotically stable when $\varepsilon = 0$. Show also that if $\varepsilon \neq 0$ is sufficiently small we can still establish GAS. *Hint: You may want to use the Lyapunov function* $V(x) = V_1(x_1) + \frac{K}{5}|x_2|^5$ *where* $K > 0$ *is your design parameter and you already know what* V_1 *is from the previous problem.*

Sol'n. First case, $\varepsilon = 0$. Similar to the solution of the previous problem the system $\dot{x}_1 = f(x_1) + g(u)u^4$ can be shown to be ISS. Also, note that the origin of the system $\dot{x}_2 = -x_2|x_2|^3$ is GAS, which can be shown with the Lyapunov function $V_2(x_2) = \frac{1}{2}x_2^2$. Then the origin of the cascade connection is GAS by Lemma 4.7.

Second case, $\varepsilon \neq 0$. Using the suggested Lyapunov function V we can write

$$\begin{aligned}
\dot{V} &= \nabla V_1(x_1)f(x_1) + \nabla V_1(x_1)g(x_2)x_2^4 - Kx_2^8 + K\varepsilon x_1x_2^4 \\
&\leq -c_3x_1^2 + c_4c_5|x_1|x_2^4 - Kx_2^8 + K|\varepsilon||x_1|x_2^4 \\
&= -[|x_1| \;\; x_2^4]Q[|x_1| \;\; x_2^4]^T
\end{aligned}$$

where

$$Q = \begin{bmatrix} c_3 & -\frac{c_4c_5+K|\varepsilon|}{2} \\ -\frac{c_4c_5+K|\varepsilon|}{2} & K \end{bmatrix}$$

Note that $Q > 0$ guarantees GAS. Hence if we let $K > \dfrac{(c_4c_5+1)^2}{4c_3}$ then for $|\varepsilon| < \frac{1}{K}$ we achieve what we desire.

Problem 5 Exercise 5.10.

Sol'n. See end of document.

Problem 6 Exercise 5.13.

Sol'n. See end of document.

Problem 7 Exercise 5.15(4).

Sol'n. See end of document.

Problem 8 Exercise 5.16. *Hint: See the Lyapunov function construction of Problem 2.*

Sol'n. See end of document.

• **4.54** (1) The system is not input-to-state stable since with $u(t) \equiv c > 1$ and $x(0) > 0$, $x(t) \to \infty$ as $t \to \infty$.

(2) Let $V(x) = \frac{1}{2}x^2$.

$$\dot{V} = -x^4 + ux^4 - x^6 \le -x^4, \quad \forall\ |x| \ge \sqrt{u}$$

By Theorem 4.19, the system is input-to-state stable.

(3) The system is not input-to-state stable since with $u(t) \equiv 1$ and $x(0) > 0$, $x(t) \to \infty$ as $t \to \infty$.

(4) With $u = 0$, the origin of $\dot{x} = x - x^3$ is unstable. Hence, the system is not input-to-state stable.

• **5.10** (1) Let $V(x) = \frac{1}{2}x^2$.

$$\dot{V} = -(1+u)x^4 \le -(1-r_u)x^4, \quad \forall\ |u| \le r_u < 1$$

By Theorem 5.2, we conclude that the system is small-signal $\mathcal{L}_\infty$ stable for sufficiently small $|x(0)|$. Taking $u(t) \equiv -2$, it can be seen that the system is not $\mathcal{L}_\infty$ stable. The origin of the unforced system in not exponentially stable. However, the origin of the forced system is asymptotically stable for $|u| < 1$, which implies that $|y(t)| \le \beta(|x_0|, t)$. Therefore, the system is small-signal finite-gain $\mathcal{L}_\infty$ stable.
(2) We saw in Exercise 4.54 that the system is input-to-state stable. Using Theorem 5.3, we conclude that the system is $\mathcal{L}_\infty$ stable. The origin of the unforced system in not exponentially stable. However, the origin of the forced system is asymptotically stable for $|u| < 1$, which implies that $|y(t)| \le \beta(|x_0|, t) + |u(t)|$ Therefore, the system is small-signal finite-gain $\mathcal{L}_\infty$ stable.
(3) Since $|y| \le \frac{1}{2}$, the system is finite-gain $\mathcal{L}_\infty$ stable.
(4) With $V = \frac{1}{2}x^2$, we have

$$\dot{V} = -x^2 - x^4 + x^3 u \le -x^2, \quad \forall\ |x| \ge |u|$$

By Theorem 4.19 we conclude that the system is input-to-state stable. Using $|y| = |x \sin(u)| \le |x|$, we conclude by Theorem 5.3 that the system is $\mathcal{L}_\infty$ stable.

• **5.13** Since $W(x)$ is positive definite and radially unbounded, it follows from Lemma 4.3 that there is a class $\mathcal{K}_\infty$ function α such that $W(x) \ge \alpha(\|x\|)$ for all $x \in R^n$. Since $|\psi(u)|$ is positive definite, it follows from Lemma 4.3 that there is a class $\mathcal{K}$ function ρ_0 such that $|\psi(u)| \le \rho_0(\|u\|)$ for all $u \in R^m$. Hence,

$$\frac{\partial V}{\partial x} f(x,u) \le -\alpha(\|x\|) + \rho_0(\|u\|) \le -\frac{1}{2}\alpha(\|x\|), \quad \forall\ \|x\| \ge \alpha^{-1}(2\rho_0(\|u\|))$$

We conclude from Theorem 4.19 that the system is input-to-state stable. Furthermore, $\|h(x,u)\|$ is a positive definite function of $\begin{bmatrix} x \\ u \end{bmatrix}$. It follows from Lemma 4.3 that there is a class $\mathcal{K}_\infty$ function ρ_1 such that

$$\|h(x,u)\| \le \rho_1\left(\left\| \begin{matrix} x \\ u \end{matrix} \right\|\right) \le \rho_1(2\|x\|) + \rho_1(2\|u\|), \quad \forall\ (x,u)$$

Thus, (5.23) is satisfied globally with $\eta = 0$. We conclude from Theorem 5.3 that the system is $\mathcal{L}_\infty$ stable.

• **5.15** (4)

$$f(x) = \begin{bmatrix} x_2 \\ -(1+x_1^2)x_2 - x_1^3 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}, \quad h(x) = x_1 x_2$$

Let $W(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

$$\frac{\partial W}{\partial x} f(x) = [\ x_1^3 \quad x_2\ ] \begin{bmatrix} x_2 \\ -(1+x_1^2)x_2 - x_1^3 \end{bmatrix} = -(1+x_1^2)x_2^2 \le -x_1^2 x_2^2 = -h^2(x)$$

$$\frac{\partial W}{\partial x} G = [\ x_1^3 \quad x_2\ ] \begin{bmatrix} 0 \\ x_1 \end{bmatrix} = x_1 x_2 = h(x)$$

Thus, $W(x)$ satisfies (5.32)–(5.33) globally with $k = 1$. It follows from Example 5.9 that the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to 1.

• **5.16** (a) Let $V(x) = \int_0^{x_1} \sigma(y)\, dy + \frac{1}{2}(x_1^2 + x_2^2)$.

$$\begin{aligned} \dot{V} &= -x_1^2 - x_1\sigma(x_1) - x_2^2 + x_2 u \ \le\ -x_1^2 - x_2^2 + x_2 u \\ &\le -(1-\theta)\|x\|_2^2 - \theta\|x\|_2^2 + \|x\|_2|u| \ \le\ -(1-\theta)\|x\|_2^2; \quad \forall\ \|x\|_2 \ge |u|/\theta \end{aligned}$$

where $0 < \theta < 1$. It follows from Theorem 4.19 that the system is input-to-state stable. Since $|y| = |x_2| \le \|x\|_2$, we conclude from Theorem 5.3 that the the system is $\mathcal{L}_\infty$ stable.
(b) Let $V(x) = \alpha[\int_0^{x_1} \sigma(y)\, dy + \frac{1}{2}(x_1^2 + x_2^2)]$.

$$\begin{aligned} \mathcal{H} &= \frac{\partial V}{\partial x} f + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x} G G^T \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}h^T h \\ &= \alpha[-x_1^2 - x_1\sigma(x_1) - x_2^2] + \frac{\alpha^2}{2\gamma^2}x_2^2 + \frac{1}{2}x_2^2 \ \le\ \left(-\alpha + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2}\right)x_2^2 \end{aligned}$$

Choosing $\alpha = \gamma = 1$ yields $\mathcal{H} \le 0$. Hence, the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less than or equal to one.