Problem 1 Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. A solution $x(t)$ is known to satisfy $x(0) \neq 0$ and $x(T) = 0$ for some finite $T > 0$. Show that f does not satisfy Lipschitz condition at the origin, that is, no $L > 0$ and $\varepsilon > 0$ exist such that $\|f(x) - f(y)\| \leq L\|x - y\|$ for all $x, y \in \{\eta \in \mathbb{R}^n : \|\eta\| \leq \varepsilon\}$.

Sol'n. Let $\phi : [0, T] \to \mathbb{R}^n$ be a solution of $\dot{x} = f(x)$ with $\phi(0) \neq 0$ and $\phi(T) = 0$ for some $T > 0$. Without loss of generality let $\phi(t) \neq 0$ for all $t \in [0, T)$. Define $\psi_1 : [0, T] \to \mathbb{R}^n$ as $\psi_1(t) := \phi(-t + T)$. Note that we can write

$$\begin{aligned}\dot{\psi}_1(t) &= -\dot{\phi}(-t + T)\\ &= -f(\phi(-t + T))\\ &= -f(\psi_1(t)).\end{aligned}$$

Hence ψ_1 is a solution of the system $\dot{x} = -f(x)$. Moreover, $\psi_1(0) = \phi(T) = 0$ and $\psi_1(t) = \phi(-t + T) \neq 0$ for $t \in (0, T]$. Since $-f(0) = 0$, the constant function $\psi_2 : [0, T] \to \mathbb{R}^n$ with $\psi_2(t) \equiv 0$ is also a solution of the system $\dot{x} = -f(x)$. Therefore the system $\dot{x} = -f(x)$ admits two different solutions ψ_1 and ψ_2 starting from the origin $\psi_1(0) = \psi_2(0) = 0$. This nonuniquess of solutions implies that the righthand side $-f$ cannot be Lipschitz at the origin (Theorem 3.1). Therefore f cannot be Lipshitz at the origin either.

Problem 2 For each of the below systems determine whether or not finite escape times occur.

(a) $\begin{cases}\dot{x}_1 = -x_1 + x_1^2 x_2\\ \dot{x}_2 = -x_2\end{cases}$ **(b)** $\begin{cases}\dot{x}_1 = x_2^2 x_1\\ \dot{x}_2 = x_2 + 10\end{cases}$

Sol'n. **(a)** First recall that the scalar system $\dot{y} = y^2$ is prey to finite escape times. In particular, for $y(0) = 1$ the solution $y : [0, 1) \to \mathbb{R}$ reads $y(t) = (1 - t)^{-1}$. That is, we have $y(t) \to \infty$ as $t \to 1$. Now returning to our system we can write

$$\dot{x}_1 = -x_1 + x_2(0)e^{-t}x_1^2.$$

Choose $x_1(0) = 1$ and $x_2(0) = 2e$. Then we can write for $t \in [0, 1]$

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2e^{1-t}x_1^2\\ &\geq -x_1 + 2x_1^2\\ &\geq x_1^2.\end{aligned}$$

Hence, for some $\bar{t} \in (0, 1]$ we should have $x_1(t) \to \infty$ as $t \to \bar{t}$. That is, finite escape times do occur for the system.

(b) Note that $x_2(t) = -10 + (10 + x_2(0))e^t$. Therefore for any finite time interval $x_2(t)$ remains bounded. Now, given an arbitrary initial condition $x_2(0)$ let us define $a(t) := (-10 + (10 + x_2(0))e^t)^2$. Then we can write

$$\dot{x}_1 = a(t)x_1 =: f(t, x_1).$$

Given an arbitrary finite time $T > 0$ let us define $L := \max_{t \in [0, T]} |a(t)|$. Note that $L < \infty$ because a is continuous. Then we can write

$$\begin{aligned}|f(t, y) - f(t, z)| &= |a(t)(y - z)|\\ &\leq L|y - z|\end{aligned}$$

for all $y, z \in \mathbb{R}$ and $t \in [0, T]$. Then by Theorem 3.2 for any initial condition $x_1(0)$ the solution $x_1(t)$ is defined for all $t \in [0, T]$. But T was arbitrary. Therefore the solution $x_1(t)$ must exist for all $t \in [0, \infty)$. Hence the finite escape times are ruled out for this system.

Problem 3 Consider the system $\dot{x} = f(x)$ where f is continuous and $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$.

(a) Show that if $\phi(t)$ is a solution to this system so is $\lambda\phi(t)$ for each $\lambda > 0$.

(b) Show that this system cannot exhibit finite escape times.

(c) Show that no solution $x(t)$ exists satisfying $x(0) \neq 0$ and $x(T) = 0$ for some finite $T > 0$.

Sol'n. (a) Given some solution $\phi(\cdot)$ and a positive number λ let us define $\psi(t) := \lambda\phi(t)$. We can write

$$\dot{\psi}(t) = \lambda\dot{\phi}(t) = \lambda f(\phi(t)) = f(\lambda\phi(t)) = f(\psi(t))\,.$$

(b)&(c) Let us define the set $\mathcal{R} := \{x \in \mathbb{R}^n : \frac{1}{2} \leq \|x\| \leq 2\}$ and $L := \max_{x\in\mathcal{R}} \|f(x)\|$. Since f is continuous and $\mathcal{R}$ is compact we have $L < \infty$. Now consider any solution $x(\cdot)$ with $\|x(0)\| = 1$. Let $T > 0$ be such that $x(t) \in \mathcal{R}$ for all $t \in [0, T]$. We can write for $t \in [0, T]$

$$\begin{aligned} \|x(t)\| &= \left\| x(0) + \int_0^t f(x(\tau))d\tau \right\| \\ &\leq \|x(0)\| + \int_0^t \|f(x(\tau))\|d\tau \\ &\leq 1 + Lt\,. \end{aligned} \tag{1}$$

Likewise, again for $t \in [0, T]$,

$$\begin{aligned} \|x(t)\| &\geq \|x(0)\| - \int_0^t \|f(x(\tau))\|d\tau \\ &\geq 1 - Lt\,. \end{aligned} \tag{2}$$

Eq. (1) allows us to write

$$\|x(0)\| = 1 \implies \|x(t)\| \leq 2 \quad \text{for all} \quad t \in \left[0,\ \frac{1}{L}\right]$$

Likewise, by eq. (2) we can write

$$\|x(0)\| = 1 \implies \|x(t)\| \geq \frac{1}{2} \quad \text{for all} \quad t \in \left[0,\ \frac{1}{2L}\right]$$

Combining these we can write

$$\|x(0)\| = 1 \implies \frac{1}{2} \leq \|x(t)\| \leq 2 \quad \text{for all} \quad t \in \left[0,\ \frac{1}{2L}\right] \tag{3}$$

Now given an arbitrary solution $\phi(t)$ with $\phi(0) \neq 0$, let $\lambda := \|\phi(0)\|$. Define $\psi(t) := \lambda^{-1}\phi(t)$. By part (a) ψ is also a solution. Note that $\|\psi(0)\| = 1$. Hence we can write by (3)

$$\begin{aligned} \|\psi(t)\| \in [2^{-1}, 2] \quad \forall t \in [0, (2L)^{-1}] &\implies \lambda\|\psi(t)\| \in [2^{-1}\lambda, 2\lambda] \quad \forall t \in [0, (2L)^{-1}] \\ &\implies \|\phi(t)\| \in [2^{-1}\|\phi(0)\|, 2\|\phi(0)\|] \quad \forall t \in [0, (2L)^{-1}] \end{aligned}$$

By time invariance we can therefore write

$$2^{-k}\|\phi(0)\| \leq \|\phi(t)\| \leq 2^{k}\|\phi(0)\| \quad \forall t \in [0,\, k(2L)^{-1}]$$

for all $k = 1,\, 2,\, \ldots$

Problem 4 Exercise 4.3.

Sol'n. See end of document.

Problem 5 Exercise 4.7.

Sol'n. See end of document.

Problem 6 Exercise 4.21.

Sol'n. See end of document.

• **4.3** **(1)** Let $V(x) = (1/2)(x_1^2 + x_2^2)$.

$$\dot{V} = x_1(-x_1 + x_1 x_2) - x_2^2$$

In the set $\{\|x\|_2 \le r^2\}$, we have $|x_1| \le r$. Hence,

$$\dot{V} \le -x_1^2 - x_2^2 + r|x_1|\,|x_2| = -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -r/2 \\ -r/2 & 1 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}$$

$\dot{V}$ is negative definite for $r < 2$. Thus, the origin is asymptotically stable. To investigate global asymptotic stability, note that the solution of the second equation is $x_2(t) = \exp(-t)x_2(0)$, which when substituted in the first equation yields

$$\dot{x}_1 = [-1 + \exp(-t)x_2(0)]x_1$$

This is a linear time-varying system whose solution does not have a finite escape time. After some finite time the coefficient of x_1 on the right-hand side will be less than a negative number. Hence, $\lim_{t\to\infty} x_1(t) = 0$. Thus, the origin is globally asymptotically stable.
(2) Let $V(x) = (1/2)(x_1^2 + x_2^2)$.

$$\dot{V} = -(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) = -2V(1 - 2V)$$

In the region $V(x) < 1/2$, $\dot{V}$ is negative definite. Hence, the origin is asymptotically stable. For $V > 1/2$, $\dot{V}$ is positive. Hence, trajectories starting in the region $V(x) > 1/2$ cannot approach the origin. In fact, they grow unbounded. Thus, the origin is not globally asymptotically stable.
(3) Let $V(x) = x^T P x = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$, where P is a positive definite symmetric matrix.

$$\dot{V} = -2p_{12}x_1^2 + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2(p_{22} - p_{12})x_2^2 + \text{Higher order terms}$$

Near the origin, the quadratic term dominates the higher-order terms. Thus, $\dot{V}$ will be negative definite in the neighborhood of the origin if the quadratic term is negative definite. Choosing $p_{12} = 1$, $p_{22} = 2$, and $p_{11} = 3$ makes $V(x)$ positive definite and $\dot{V}(x)$ negative definite. Hence, the origin is asymptotically stable. It is not globally asymptotically stable since the origin is not the unique equilibrium point. The set $\{x_1^2 = 1\}$ is an equilibrium set.
(4) Let $V(x) = x_1^2 + (1/2)x_2^2$.

$$\dot{V} = -2x_1^2 - 2x_1x_2 + 2x_1x_2 - x_2^4 = -x_1^2 - x_2^4$$

Hence, the origin is globally asymptotically stable.

• **4.7** **(a)** Let $\nabla V(x) = g(x)$. Then, $\dot{V} = -g^T(x)Q\phi(x)$. Choose $g(x) = Px$ so that $V(x) = (1/2)x^T Px$. We need to choose $P = P^T > 0$ such that $\dot{V} = -x^T PQ\phi(x)$ is negative definite. Choosing $P = Q^{-1}$ yields

$$\dot{V} = -x^T\phi(x) = -\sum_{i=1}^{n} x_i\phi_i(x_i)$$

$\dot{V}$ is negative definite in the neighborhood of the origin because $y\phi(y) > 0$ for $y \ne 0$. Hence, the origin is asymptotically stable.
(b) The function $V(x)$ is radially unbounded. The origin will be globally asymptotically stable if $\dot{V}$ is negative definite for all x. This will be the case if $y\phi_i(y) > 0$ for all $y \ne 0$.
(c) The function ϕ_2 satisfies the condition $y\phi_i(y) > 0$ for all $y \ne 0$. The function ϕ_1 satisfies the condition only near $y = 0$ because $\phi_1(y)$ vanishes at $y = 1$. Thus, we can only show asymptotic stability of the origin using the Lyapunov function $V(x) = x^T Q^{-1} x = x_1^2 + 2x_1x_2 + 2x_2^2$.

• **4.21**
(a)

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = -(\nabla V)^T(\nabla V) \le 0$$

$$\dot{V}(x) = 0 \Leftrightarrow \nabla V(x) = 0 \Leftrightarrow \dot{x} = 0$$

Hence, $\dot{V}(x) = 0$ if and only if x is an equilibrium point.
(b) Every solution starts in a set Ω_c with $c \ge V(x_0)$. Since $\dot{V} \le 0$ in Ω_c, the solution remains in Ω_c for all $t \ge 0$. Since Ω_c is compact, we conclude by Theorem 3.3 that the solution is defined for all $t \ge 0$.
(c) By LaSalle's theorem, $x(t) \to M = \{p_1, \ldots, p_r\}$ as $t \to \infty$. Since the points $p_1, \ldots, p_r$ are isolated, we conclude from Exercise 4.20 that $x(t) \to p_i$ as $t \to \infty$ for some $p_i \in M$.