Problem 1. Exercises 1.19, 1.20(a), 1.21(a) from the textbook.[1]

Sol'n. See the end of file.

Problem 2. 2.1(1), 2.1(5), 2.1(6).

Sol'n. See the end of file.

Problem 3. 2.15.

Sol'n. See the end of file.

Problem 4. 2.17(2).

Sol'n. See the end of file.

Problem 5. 2.20(3), 2.20(4), 2.20(5).

Sol'n. See the end of file.

Problem 6. Consider the system

$$\begin{aligned} \dot{x}_1 &= ax_1 - x_1 x_2 \\ \dot{x}_2 &= bx_1^2 - cx_2 \end{aligned}$$

where $c > a > 0$. Let $D = \{x \in \mathbb{R}^2 : x_2 \geq 0\}$.

(a) Let $b > 0$. Show that every trajectory starting in D stays in D for all future time.

(b) Let $b > 0$. Show that there is no periodic orbit through any $x \in D$. *Hint: Use Lemma 2.2.*

(c) Still letting $b > 0$, show that there can be no periodic orbits (in $\mathbb{R}^2$).

(d) This time let $b \leq 0$. Show that there can be no periodic orbits for this case, either.

Sol'n. (a) On the border of D ($x_2 = 0$) we have $\dot{x} = [ax_1 \; bx_1^2]^T$, that is, the velocity vector never points outside the set D. Therefore, given that it starts in D, no solution can leave D.

(b) For $x \in D$ we can write

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} &= a - x_2 - c \\ &\leq a - c \\ &< 0\,. \end{aligned}$$

Since $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is strictly negative in D, by Lemma 2.2 there can be no periodic orbit that entirely lies in D. By part (a) it is not possible that there is a periodic orbit that lies partly in D and partly out of it. Hence there is no periodic orbit through any $x \in D$.

(c) Suppose there is a periodic orbit γ. We know by part (b) that γ must lie entirely in $\mathbb{R}^2 \backslash D$. Corollary 2.1 tells us that inside γ there should be an equilibrium point, which implies the

[1]H.K. Khalil. *Nonlinear Systems (Third Edition)*. Prentice Hall, 2002.

existence of an equilibrium point in $\mathbb{R}^2 \backslash D$. The system has three equilibria: (0, 0), ($\sqrt{ac/b}$, a), and ($-\sqrt{ac/b}$, a); all of which are in D. Hence γ cannot exist.

(d) For $b \leq 0$ the origin is the only equilibrium point. The linearization at the origin yields the following A matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

whose eigenvalues are at $\lambda_1 = a > 0$ and $\lambda_2 = -c < 0$. Therefore the origin is a saddle. As a result, by Corollary 2.1, the existence of a periodic orbit is ruled out.

Problem 7. 2.24.

Sol'n. See the end of file.

• **1.19** (a)

$$\frac{d}{dt}\left(\int_0^h A(\lambda)\,d\lambda\right) = w_i - k\sqrt{\rho g h}$$
$$A(h)\dot{h} = u - k\sqrt{\rho g h}$$

Let $x = h$.

$$\dot{x} = \frac{1}{A(x)}[u - k\sqrt{\rho g x}], \quad y = x$$

(b) $x = p - p_a = \rho g h$.

$$\dot{x} = \frac{\rho g}{A(x/\rho g)}(u - k\sqrt{x}), \quad y = x/(\rho g)$$

(c) At equilibrium,

$$0 = u_{ss} - k\sqrt{\rho g x_{ss}}, \quad y_{ss} = x_{ss} = r$$

Hence, $u_{ss} = k\sqrt{\rho g r}$

• **1.20** (a) From the equations $\dot{v} = w_i - w_o$ and $p = p_a + (\rho g/A)v$, we have

$$\dot{p} = \frac{\rho g}{A}\dot{v} = \frac{\rho g}{A}(w_i - w_o) = \frac{\rho g}{A}[\phi^{-1}(\Delta p) - k\sqrt{\Delta p}]$$

Using $x = \Delta p$ as the state variable, we obtain

$$\dot{x} = \frac{\rho g}{A}[\phi^{-1}(x) - k\sqrt{x}]$$

• **1.21** (a) We have

$$\dot{v}_1 = w_p - w_1, \quad \dot{v}_2 = w_1 - w_2$$
$$\dot{p}_1 = \frac{\rho g}{A_1}\dot{v}_1, \quad \dot{p}_2 = \frac{\rho g}{A_2}\dot{v}_2$$
$$w_1 = k_1\sqrt{p_1 - p_2}, \quad w_2 = k_2\sqrt{p_2 - p_a}, \quad p_1 - p_a = \phi(w_p)$$

Let $x_1 = p_1 - p_a$ and $x_2 = p_2 - p_a$.

$$\dot{x}_1 = \dot{p}_1 = \frac{\rho g}{A_1}(w_p - w_1) = \frac{\rho g}{A_1}\left[\phi^{-1}(x_1) - k_1\sqrt{x_1 - x_2}\right]$$
$$\dot{x}_2 = \dot{p}_2 = \frac{\rho g}{A_2}(w_1 - w_2) = \frac{\rho g}{A_2}\left[k_1\sqrt{x_1 - x_2} - k_2\sqrt{x_2}\right]$$

• **2.1** (1)

$$0 = -x_1 + 2x_1^3 + x_2, \quad 0 = -x_1 - x_2$$
$$x_2 = -x_1 \Rightarrow 0 = 2x_1(x_1^2 - 1) \Rightarrow x_1 = 0,\ 1,\ \text{or}\ -1$$

There are three equilibrium points at $(0,0)$, $(1,-1)$, and $(-1,1)$. Determine the type of each point using linearization.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + 6x_1^2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = -1 \pm j \Rightarrow (0,0) \text{ is a stable focus}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)} = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2 \pm \sqrt{8} \Rightarrow (1,-1) \text{ is a saddle}$$

Similarly, $(-1,1)$ is a saddle.

(5)

$$0 = (x_1 - x_2)(1 - x_1^2 - x_2^2), \quad 0 = (x_1 + x_2)(1 - x_1^2 - x_2^2)$$

$\{x_1^2 + x_2^2 = 1\}$ is an equilibrium set and $(0,0)$ is an isolated equilibrium point.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 1 - 3x_1^2 - x_2^2 + 2x_1x_2 & -2x_1x_2 - 1 + x_1^2 + 3x_2^2 \\ 1 - 3x_1^2 - x_2^2 - 2x_1x_2 & -2x_1x_2 + 1 - x_1^2 - 3x_2^2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Eigenvalues are $1 \pm j$; hence, $(0,0)$ is unstable focus.

(6)

$$0 = -x_1^3 + x_2, \quad 0 = x_1 - x_2^3$$
$$x_2 = x_1^3 \Rightarrow x_1(1 - x_1^8) = 0 \Rightarrow x_1 = 0 \text{ or } x_1^8 = 1$$

The equation $x_1^8 = 1$ has two real roots at $x_1 = \pm 1$. Thus, there are three equilibrium points at $(0,0)$, $(1,1)$, $(-1,-1)$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \text{Eigenvalues} : 1, -1 \ \Rightarrow \ (0,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(1,1)} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}; \quad \text{Eigenvalues} : -2, -4 \ \Rightarrow \ (1,1) \text{ is a stable node}$$

Similarly, $(-1,-1)$ is a stable node.

• **2.15** The solution of the state equation

$$\begin{aligned} \dot{x}_1 &= x_2, \quad x_1(0) = x_{10} \\ \dot{x}_2 &= k, \quad x_2(0) = x_{20} \end{aligned}$$

where $k = \pm 1$, is given by

$$\begin{aligned} x_2(t) &= kt + x_{20} \\ x_1(t) &= \tfrac{1}{2}kt^2 + x_{20}t + x_{10} \end{aligned}$$

Eliminating t between the two equations, we obtain

$$x_1 = \frac{1}{2k}x_2^2 + c$$

where $c = x_{10} - x_{20}^2/(2k)$. This is the equation of the trajectories in the x_1–x_2 plane. Different trajectories correspond to different values of c. Figures 2.24 and 2.25 show the phase portraits for $u = 1$ and $u = -1$, respectively. The two portraits are superimposed in Figure 2.26. From Figure 2.26 we see that trajectories can reach the origin through only two curves, which are highlighted. The curve in the lower half corresponds to $u = 1$ and the curve in the upper half corresponds to $u = -1$. We will refer to these curves as the switching curves. To move any point in the plane to the origin, we can switch between ± 1. For example, to move the point A to the origin, we apply $u = -1$ until the trajectory hits the switching curve, then we switch to $u = 1$. Similarly, to move the point B to the origin, we apply $u = 1$ until the trajectory hits the switching curve, then we switch to $u = -1$. When the trajectory reaches the origin we can keep it there by switching to $u = 0$ which makes the origin an equilibrium point.

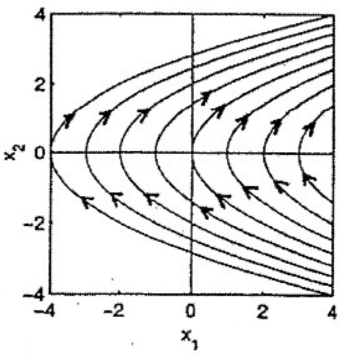

Figure 2.24: Exercise 2.15.

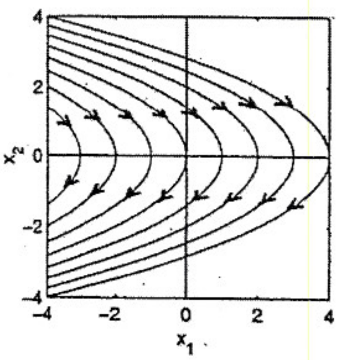

Figure 2.25: Exercise 2.15.

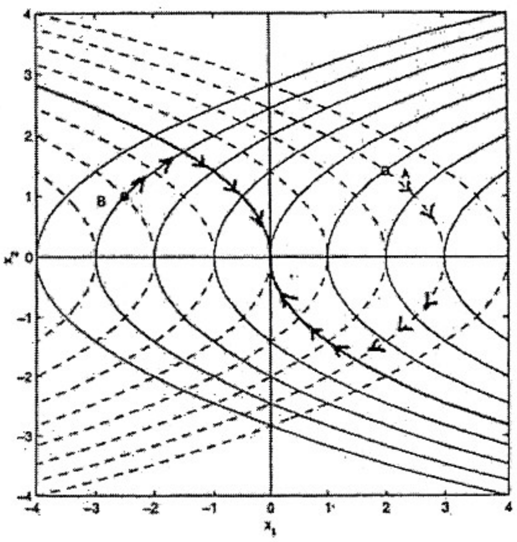

Figure 2.26: Exercise 2.15.

• 2.17 (2) Let $V(x) = x_1^2 + x_2^2$.

$$f(x) \cdot \nabla V(x) = 2x_2^2(2 - 3x_1^2 - 2x_2^2) = 4x_2^2(1 - x_1^2 - x_2^2) - 2x_1^2x_2^2 \leq 4x_2^2(1 - x_1^2 - x_2^2)$$

Hence, $f(x) \cdot \nabla V(x) \leq 0$ for $x_1^2 + x_2^2 \geq 1$. In particular, all trajectories starting in $M = \{V(x) \leq 1\}$ stay in M for all future time. M contains only one equilibrium point at the origin. Linearization at the origin yields the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$, whose eigenvalues are 1 and 1. Since $f(x)$ is an analytic function of x, we conclude that the origin is unstable node. By the Poincaré-Bendixson criterion, there is a periodic orbit in M.

• 2.20 (3) The equilibrium points of the system are the roots of

$$0 = 1 - x_1x_2^2, \quad 0 = x_1$$

These equations have no real roots. Thus, there are no equilibrium points. Since, by Corollary 2.1, a closed orbit must enclose an equilibrium point, we conclude that there are no closed orbits.

(4) The x_1-axis is an equilibrium set. Therefore, a periodic orbit cannot cross the x_1-axis; it must lie entirely in the upper or lower halves of the plane. However, there are no equilibrium points other than the x_1-axis. Since, by Corollary 2.1, a periodic orbit must enclose an equilibrium point, we conclude that there are no periodic orbits.

(5) The equilibrium points are the roots of

$$0 = x_2 \cos x_1, \quad 0 = \sin x_1$$

The equilibrium point are $(\pm n\pi, 0)$ for $n = 0, 1, 2, \cdots$. Linearization at the equilibrium points yields the matrix $\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ where $a = \pm 1$. Hence, all equilibrium points are saddles. Since, by Corollary 2.1, a periodic orbit must enclose equilibrium points such that $N - S = 1$, we conclude that there are no periodic orbits.

• **2.24** Suppose M does not contain an equilibrium point. Then, by the Poincaré-Bendixson criterion, there is a periodic orbit in M. But, by Corollary 2.1, the periodic orbit must contain an equilibrium point: A contradiction. Thus, M contains an equilibrium point.