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Read before you start:

- There are five questions.
- The examination is open-book.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Total**

Q1.

Consider the following system of order $2n$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -Px_1 - Qx_2\end{aligned}$$

where $x_1, x_2 \in \mathbb{R}^n$ and $P, Q \in \mathbb{R}^{n\times n}$. The matrix P is symmetric positive definite and Q is symmetric positive semidefinite.

(a) By choosing an appropriate Lyapunov function find conditions[1] on the pair (P, Q) under which the origin is stable.

(b) Using the same Lyapunov function find conditions on the pair (P, Q) under which the origin is asymptotically stable.

Sol'n. **(a)** Let $V(x_1, x_2) = \frac{1}{2}x_1^T Px_1 + \frac{1}{2}x_2^T x_2$. Note that V is positive definite and yields $\dot{V} = x_1^T P\dot{x}_1 + x_2^T \dot{x}_2 = x_1^T Px_2 + x_2^T(-Px_1 - Qx_2) = -x_2^T Qx_2$. Since $Q \geq 0$ we have $\dot{V} \leq 0$. Hence the origin is stable for all (P, Q).

(b) To establish asymptotic stability we can use the invariance principle. For this we need the implication $\dot{V} \equiv 0 \implies x \equiv 0$ where $x = [x_1^T\ x_2^T]^T$ is the overall state of the system. Now, we can proceed as follows. $\dot{V} \equiv 0 \implies x_2^T Qx_2 \equiv 0$ and since Q is symmetric positive semidefinite we can write $x_2^T Qx_2 \equiv 0 \implies Qx_2 \equiv 0$. Under $Qx_2 \equiv 0$ the state evolves via

$$\dot{x} = \begin{bmatrix} 0 & I \\ -P & 0 \end{bmatrix} x\,.$$

Therefore if the above system is observable from the output $y = [0\ Q]x$ then we can claim $\dot{V} \equiv 0 \implies x \equiv 0$. By the eigenvector test (sometimes called the PBH test) for observability it can be shown that the observability of the pair $\left([0\ Q], \begin{bmatrix} 0 & I \\ -P & 0 \end{bmatrix}\right)$ is equivalent to the observability of the pair (Q, P). Hence the origin is asymptotically stable if the pair (Q, P) is observable.

[1] Conditions you find here (Q1) and in the following question (Q2) should be as unrestrictive as possible.

Q2.

Consider the following system of order $2n$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -Px_1 - Qx_2 + Ru \\ y &= Rx_2 \end{aligned}$$

where $x_1,\, x_2,\, u,\, y \in \mathbb{R}^n$ and $P,\, Q,\, R \in \mathbb{R}^{n\times n}$. The matrix P is symmetric positive definite and $Q,\, R$ are symmetric positive semidefinite.

(a) Find conditions on the triple $(P,\, Q,\, R)$ under which the system is passive.

(b) Find conditions on the triple $(P,\, Q,\, R)$ under which the system is output strictly passive.

(c) Find conditions on the triple $(P,\, Q,\, R)$ under which the system is input-to-state stable.

(d) Suppose both Q and R are positive definite. Find an upper bound on the $\mathcal{L}_2$ gain of the system.

Sol'n. **(a)** Let $V(x_1,\, x_2) = \frac{1}{2}x_1^T P x_1 + \frac{1}{2}x_2^T x_2$ be our storage function. We have $\dot{V} = -x_2^T Q x_2 + x_2^T R u = -x_2^T Q x_2 + y^T u$. Since Q is positive semidefinite we can therefore write $\dot{V} \le y^T u$. Hence the system is passive for all $(P,\, Q,\, R)$.

(b) Output strict passivity requires the existence of a function $\rho : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\dot{V} = -x_2^T Q x_2 + y^T u \le -y^T \rho(y) + y^T u$ and $y^T \rho(y) > 0$ for $y \ne 0$. Such a function exists if the implication $x_2^T Q x_2 \ne 0 \implies y \ne 0$ holds. Since Q is symmetric positive semidefinite we have $x_2^T Q x_2 \ne 0 \iff Qx_2 \ne 0$. Hence the system is OSP if the implication $Qx_2 \ne 0 \implies Rx_2 \ne 0$ holds or, equivalently, if $\operatorname{null} Q \subset \operatorname{null} R$.

(c) The system is LTI. Therefore it is ISS if and only if the origin of the unforced $(u = 0)$ system is GAS. This situation is already studied in Q1(b). Hence the system is ISS if the pair $(Q,\, P)$ is observable.

(d) We can write

$$\begin{aligned} y^T u &= x_2^T Q x_2 + \dot{V} \\ &\ge \lambda_{\min}(Q)\|x_2\|^2 + \dot{V} \\ &\ge \frac{\lambda_{\min}(Q)}{\lambda_{\max}(R^2)} x_2^T R^2 x_2 + \dot{V} \\ &= \frac{\lambda_{\min}(Q)}{\lambda_{\max}(R^2)} \|y\|^2 + \dot{V}\,. \end{aligned}$$

Hence the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is no greater than $\dfrac{\lambda_{\max}(R^2)}{\lambda_{\min}(Q)}$ by Lemma 6.5.

Q3.

Consider the following system

$$\dot{x} = Ax - B\rho(Cx)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and the function $\rho : \mathbb{R}^m \to \mathbb{R}^m$ is continuous. Suppose there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying $PB = C^T$. Further assume that ρ satisfies $y^T\rho(y) > 0$ for all nonzero $y \in \mathbb{R}^m$. Show that this system cannot display finite escape times.

Sol'n. Let us construct the positive definite function $V(x) = x^T Px$, which yields $\dot{V} = 2x^T P\dot{x} = 2x^T PAx - 2x^T PB\rho(Cx) = 2x^T PAx - 2(Cx)^T\rho(Cx) \leq 2x^T PAx$. We can then proceed as

$$\begin{aligned} \dot{V} &\leq 2x^T PAx \\ &\leq 2\|P\|\cdot\|A\|\cdot\|x\|^2 \\ &\leq \frac{2\|P\|\cdot\|A\|}{\lambda_{\min}(P)}V\,. \end{aligned}$$

Defining $\gamma := \|P\|\cdot\|A\|/\lambda_{\min}(P)$ allows us to write $\dot{V} \leq 2\gamma V$ which implies $V(x(t)) \leq e^{2\gamma t}V(x(0))$ for all $t \geq 0$. Since $\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2$ we can then write

$$\|x(t)\| \leq e^{\gamma t}\sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}\|x(0)\|\,.$$

In other words, the solution $x(t)$ is defined for all $t \geq 0$.

Q4.

Consider the following linear system

$$\begin{aligned} x^+ &= Fx \\ y &= Hx \end{aligned}$$

where $F \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{1\times n}$. Suppose F is nonsingular and the pair $(H,\, F)$ is observable. Show that applying Glad's observer construction to this system leads to the observer dynamics $\hat{x}^+ = F\hat{x} + L(y - H\hat{x})$ for some constant matrix $L \in \mathbb{R}^{n\times 1}$. Find a closed-form expression for this L (in terms of F and H).

Sol'n. We begin by writing

$$\begin{aligned} H\eta &= y \\ HF^{-1}\eta &= HF^{-1}\hat{x} \\ HF^{-2}\eta &= HF^{-2}\hat{x} \\ &\vdots \\ HF^{-(n-1)}\eta &= HF^{-(n-1)}\hat{x} \end{aligned}$$

which can be compactly written as

$$W\eta = W\hat{x} + e_1(y - H\hat{x})\,.$$

where the matrix $W \in \mathbb{R}^{n\times n}$ and the unit vector $e_1 \in \mathbb{R}^n$ are defined as

$$W = \begin{bmatrix} H \\ HF^{-1} \\ \vdots \\ HF^{-(n-1)} \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then we have $\eta = \hat{x} + W^{-1}e_1(y - H\hat{x})$. The inverse W^{-1} exists by the observability assumption. Finally, $\hat{x}^+ = F\eta = F\hat{x} + FW^{-1}e_1(y - H\hat{x})$ which tells us that we have to have $L = FW^{-1}e_1$.

Q5.

Consider the following system

$$x^+ = f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(0) = 0$. Suppose the origin is globally exponentially stable. Let the function $V : \mathbb{R}^n \to \mathbb{R}$ be constructed as

$$V(x) = \|x\|^2 + \|f(x)\|^2 + \|f(f(x))\|^2 + \|f(f(f(x)))\|^2 + \cdots$$

Show that there exist positive constants c_1, c_2, c_3 such that V satisfies

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2 \tag{1}$$

$$V(f(x)) - V(x) \leq -c_3\|x\|^2 \tag{2}$$

for all $x \in \mathbb{R}^n$.

Sol'n. Since the origin is globally exponentially stable there exist constants $0 < \lambda < 1$ and $M > 0$ such that every solution satisfies $\|x(k)\| \leq M\lambda^k\|x(0)\|$. Note that we can write

$$\begin{aligned} V(x) &= \|x\|^2 + \|f(x)\|^2 + \|f(f(x))\|^2 + \|f(f(f(x)))\|^2 + \cdots \\ &\leq \sum_{k=0}^{\infty}(M\lambda^k\|x\|)^2 \\ &= M^2\left(\sum_{k=0}^{\infty}\lambda^{2k}\right)\|x\|^2 \\ &= \frac{M^2}{1-\lambda^2}\|x\|^2 . \end{aligned}$$

Hence we can let $c_2 = \dfrac{M^2}{1-\lambda^2}$. Likewise,

$$\begin{aligned} V(x) &= \|x\|^2 + \|f(x)\|^2 + \|f(f(x))\|^2 + \|f(f(f(x)))\|^2 + \cdots \\ &\geq \|x\|^2 . \end{aligned}$$

That is, we can let $c_1 = 1$. Finally,

$$\begin{aligned} V(x) &= \|x\|^2 + \|f(x)\|^2 + \|f(f(x))\|^2 + \|f(f(f(x)))\|^2 + \cdots \\ &= \|x\|^2 + V(f(x)) \end{aligned}$$

which allows us to pick $c_3 = 1$.