First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are five questions.
- The examination is open-book.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Total**

Q1.

Consider the following system of order $2n$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -Px_1 - Qx_2\end{aligned}$$

where $x_1,\, x_2 \in \mathbb{R}^n$ and $P,\, Q \in \mathbb{R}^{n\times n}$. The matrix P is symmetric positive definite and Q is symmetric positive semidefinite.

(a) By choosing an appropriate Lyapunov function find conditions[1] on the pair $(P,\, Q)$ under which the origin is stable.

(b) Using the same Lyapunov function find conditions on the pair $(P,\, Q)$ under which the origin is asymptotically stable.

[1] Conditions you find here (Q1) and in the following question (Q2) should be as unrestrictive as possible.

Q2.

Consider the following system of order $2n$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -Px_1 - Qx_2 + Ru \\ y &= Rx_2 \end{aligned}$$

where $x_1,\ x_2,\ u,\ y \in \mathbb{R}^n$ and $P,\ Q,\ R \in \mathbb{R}^{n\times n}$. The matrix P is symmetric positive definite and $Q,\ R$ are symmetric positive semidefinite.

(a) Find conditions on the triple $(P,\ Q,\ R)$ under which the system is passive.

(b) Find conditions on the triple $(P,\ Q,\ R)$ under which the system is output strictly passive.

(c) Find conditions on the triple $(P,\ Q,\ R)$ under which the system is input-to-state stable.

(d) Suppose both Q and R are positive definite. Find an upper bound on the $\mathcal{L}_2$ gain of the system.

Q3.

Consider the following system

$$\dot{x} = Ax - B\rho(Cx)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and the function $\rho : \mathbb{R}^m \to \mathbb{R}^m$ is continuous. Suppose there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying $PB = C^T$. Further assume that ρ satisfies $y^T\rho(y) > 0$ for all nonzero $y \in \mathbb{R}^m$. Show that this system cannot display finite escape times.

Q4.

Consider the following linear system

$$\begin{aligned} x^+ &= Fx \\ y &= Hx \end{aligned}$$

where $F \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{1\times n}$. Suppose F is nonsingular and the pair (H, F) is observable. Show that applying Glad's observer construction to this system leads to the observer dynamics $\hat{x}^+ = F\hat{x} + L(y - H\hat{x})$ for some constant matrix $L \in \mathbb{R}^{n\times 1}$. Find a closed-form expression for this L (in terms of F and H).

Q5.

Consider the following system

$$x^+ = f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(0) = 0$. Suppose the origin is globally exponentially stable. Let the function $V : \mathbb{R}^n \to \mathbb{R}$ be constructed as

$$V(x) = \|x\|^2 + \|f(x)\|^2 + \|f(f(x))\|^2 + \|f(f(f(x)))\|^2 + \cdots$$

Show that there exist positive constants c_1, c_2, c_3 such that V satisfies

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2 \tag{1}$$

$$V(f(x)) - V(x) \leq -c_3\|x\|^2 \tag{2}$$

for all $x \in \mathbb{R}^n$.