First name:____________________________

Last name:____________________________

Student ID:____________________________

Signature:____________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 120 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Consider the second-order system below.

$$\begin{aligned}\dot{x}_1 &= x_2 - a x_1 \\ \dot{x}_2 &= -x_1^3 - b x_2 + u \\ y &= x_2\end{aligned}$$

For each of the following cases using the storage function candidate $V(x) = (x_1^4 + 2x_2^2)/4$ determine whether the system is lossless, output strictly passive, or strictly passive. Moreover, investigate the stability properties of the origin of the unforced ($u = 0$) system.

a) When $a = 0$ and $b = 0$.

b) When $a = 0$ and $b > 0$.

c) When $a > 0$ and $b > 0$.

Q2.

Consider the below second-order systems.

$$\text{System (1)}: \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + u_1 \\ y_1 &= x_2 \end{cases} \qquad \text{System (2)}: \begin{cases} \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_3^3 + u_2 \\ y_2 &= x_4 \end{cases}$$

Let $u = [u_1\ u_2]^T$ and $y = [y_1\ y_2]^T$ and $x = [x_1\ x_2\ x_3\ x_4]^T$. Suppose we couple these systems by letting $u = -Qy + v$, where Q is a symmetric positive definite matrix.

a) Show that the map from v to y is finite-gain $\mathcal{L}_2$ stable. What is the gain?

b) Show that when $v = 0$ the origin $x = 0$ is globally asymptotically stable.

Q3.

Consider the Duffing map

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1 + 3x_2 - x_2^3 \\ y &= x_2\,. \end{aligned}$$

a) Find the equilibrium point(s) of this system.

b) For the initial condition $x(0) = [1 \;\; 0]^T$ find $x(3)$.

c) For this system find the observer dynamics $\hat{x}^+ = g(\hat{x},\, y)$ by applying Glad's method.

d) For the initial conditions $x(0) = [1 \;\; 0]^T$ and $\hat{x}(0) = [-7 \;\; 3]^T$ find $\hat{x}(3)$.

Q4.

Let

$$A = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$$

a) Determine the stability properties of the origin of the system $x^+ = Ax$.

b) Find a symmetric positive definite matrix P satisfying the discrete-time Lyapunov equation $A^T PA - P + I = 0$. *Hint: What is* A^2*?*

c) Show that the origin of the system $x^+ = Ax + \varepsilon\|x\|B$, where $B \in \mathbb{R}^{2\times 1}$ is a constant matrix and $\varepsilon > 0$, is asymptotically stable for small enough ε. *Hint: Try* $V(x) = x^T Px$.