First name:___________________________

Last name: KEY

Student ID:___________________________

Signature:___________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 150 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1. 25%

Consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$V(x) = x_1^4 + (x_2^2 - x_1)^2 + x_2^2$$

whose gradient vanishes only at the origin. For each of the below systems determine:

(7) • Whether the origin is stable or not. (If stable, check also asymptotic stability.)

(3) • Whether finite escape times occur or not.

(3) • Whether (nontrivial) periodic solutions exist or not.

$$\text{a)} \begin{cases} \dot{x}_1 = -\dfrac{\partial V}{\partial x_1} \\ \dot{x}_2 = -\dfrac{\partial V}{\partial x_2} \end{cases} \qquad \text{b)} \begin{cases} \dot{x}_1 = \dfrac{\partial V}{\partial x_2} \\ \dot{x}_2 = -\dfrac{\partial V}{\partial x_1} \end{cases}$$

V is pos. def. & radially unbounded.

a) $\dot{x} = -\nabla V(x) =: f(x)$

$\Rightarrow \dot{V} = \langle \nabla V, f \rangle = \langle \nabla V, -\nabla V \rangle = -\|\nabla V(x)\|^2$

$-\|\nabla V(x)\|^2$ is neg. def. because $\nabla V(x) = 0 \Leftrightarrow x = 0$

$\Rightarrow$ the origin is GAS.

- GAS $\Rightarrow$ no finite escape times
- GAS $\Rightarrow$ no periodic solutions

b) $\dot{V} = \langle \nabla V, f \rangle$

$$= \left\langle \begin{bmatrix} \partial V/\partial x_1 \\ \partial V/\partial x_2 \end{bmatrix}, \begin{bmatrix} \partial V/\partial x_2 \\ -\partial V/\partial x_1 \end{bmatrix} \right\rangle$$

$$= \frac{\partial V}{\partial x_1} \cdot \frac{\partial V}{\partial x_2} + \frac{\partial V}{\partial x_2} \cdot \left(-\frac{\partial V}{\partial x_1}\right) = 0$$

$\Rightarrow \dot{V} = 0 \Rightarrow$ the origin is stable but not asy. stable.

- Stability $\Rightarrow$ no finite escape times
- $\dot{V} = 0 \Rightarrow V(x(t)) = V(x(0))$

  $\Rightarrow$ every solution (except $x(t) \equiv 0$) is a nontrivial periodic solution.

Q2. 25%

Consider the Duffing map

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1 + 3x_2 - x_2^3 \\ y &= x_2 . \end{aligned}$$

(5) a) Find the equilibrium point(s) of this system.

(5) b) For the initial condition $x(0) = [1 \;\; 0]^T$ find $x(3)$.

(10) c) For this system find the observer dynamics $\hat{x}^+ = g(\hat{x}, y)$ by applying Glad's method.

(5) d) For the initial conditions $x(0) = [1 \;\; 0]^T$ and $\hat{x}(0) = [-7 \;\; 3]^T$ find $\hat{x}(3)$.

a) $f(x_{eq}) = x_{eq}$

$$\Rightarrow \begin{bmatrix} x_2 \\ -x_1 + 3x_2 - x_2^3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow x_2 = x_1 \quad \& \quad -x_1 + 3x_2 - x_2^3 = x_2$$

$$\Rightarrow -x_1 + 3x_1 - x_1^3 = x_1$$

$$\Rightarrow x_1^3 - x_1 = 0 \Rightarrow x_1 = -1, 0, 1$$

$$\Rightarrow \boxed{x_{eq} \in \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$$

b) $$x(1) = \begin{bmatrix} x_2 \\ -x_1 + 3x_2 - x_2^3 \end{bmatrix}_{x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$x(2) = \begin{bmatrix} x_2 \\ -x_1 + 3x_2 - x_2^3 \end{bmatrix}_{x = \begin{bmatrix} 0 \\ -1 \end{bmatrix}} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$x(3) = \begin{bmatrix} x_2 \\ -x_1 + 3x_2 - x_2^3 \end{bmatrix}_{x = \begin{bmatrix} -1 \\ -2 \end{bmatrix}} = \boxed{\begin{bmatrix} -2 \\ 3 \end{bmatrix}}$$

c) First find $f^{-1}(x) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$$f f^{-1}(x) = x \Rightarrow f \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \beta \\ -\alpha + 3\beta - \beta^3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \beta = x_1 \quad \& \quad \alpha = 3\beta - \beta^3 - x_2$$
$$= 3x_1 - x_1^3 - x_2$$

$$\Rightarrow f^{-1}(x) = \begin{bmatrix} 3x_1 - x_1^3 - x_2 \\ x_1 \end{bmatrix}$$

Glad's method: $h(\eta) = y$

$$h(f^{-1}(\eta)) = h(f^{-1}(\hat{x}))$$

$$\Rightarrow \left. \begin{aligned} \eta_2 &= y \\ \eta_1 &= \hat{x}_1 \end{aligned} \right\} \Rightarrow \eta = \begin{bmatrix} \hat{x}_1 \\ y \end{bmatrix}$$

$$\Rightarrow g(\hat{x}, y) = f(\eta) = \begin{bmatrix} y \\ -\hat{x}_1 + 3y - y^3 \end{bmatrix}$$

$$\Rightarrow \boxed{\hat{x}^+ = \begin{bmatrix} y \\ -\hat{x}_1 + 3y - y^3 \end{bmatrix}}$$

d) Glad's observer is dead beat. Hence

$$\hat{x}(3) = x(3) = \boxed{\begin{bmatrix} -2 \\ 3 \end{bmatrix}}$$

Q3. 25%

Part I. For each of the below systems determine whether the system is lossless, output strictly passive, or strictly passive.

(5) a) $H_1 : \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + u_1 \\ y_1 &= x_2 \end{cases}$ *Hint: Consider* $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

(5) b) $H_2 : \begin{cases} \dot{x}_3 &= -x_3 + u_2 \\ y_2 &= x_3^3 \end{cases}$ *Hint: Choose a suitable storage function* $V_2(x_3)$.

a) $\dot{V}_1 = x_1^3 \dot{x}_1 + x_2 \dot{x}_2 = x_1^3 x_2 + x_2(-x_1^3 + u_1)$

$= x_2 u_1$

$\Rightarrow \dot{V}_1 = y_1 u_1 \Rightarrow$ $\boxed{\text{lossless}}$

b) $\dot{V}_2 = \frac{\partial V_2}{\partial x_3} \cdot (-x_3 + u_2)$

$= -x_3 \cdot \frac{\partial V_2}{\partial x_3} + \frac{\partial V_2}{\partial x_3} \cdot u_2$

$= -x_3 \cdot \frac{\partial V_2}{\partial x_3} + y_2 \cdot u_2$

$\Rightarrow \frac{\partial V_2}{\partial x_3} = y_2 = x_3^3 \Rightarrow V_2(x_3) = \frac{1}{4} x_3^4$

$\Rightarrow \dot{V}_2 = -x_3^4 + y_2 u_2$

$\Rightarrow$ $\boxed{\text{both strictly passive \& output strictly passive}}$

Q3. Cont'd

Part II. Consider now the following feedback connection of the systems H_1 and H_2.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1^3 + ky_2\\ \dot{x}_3 &= -x_3 - ky_1\,.\end{aligned}$$

For each of the below cases determine whether the origin is stable or unstable. (If stable, check also asymptotic stability.)

(5) **a)** When $k = 1$.

(5) **b)** When $k = -1$.

(5) **c)** When $k = 0$.

Choose $V(x) = V_1(x_1, x_2) + V_2(x_3)$

$\Rightarrow \dot{V} = \dot{V}_1 + \dot{V}_2 = u_1 y_1 - x_3^4 + u_2 y_2$

$= (ky_2) y_1 - x_3^4 + (-ky_1) y_2$

$= -x_3^4$

a) $\dot{V} \leq 0 \Rightarrow$ stable

$\dot{V} \equiv 0 \Rightarrow x_3 \equiv 0 \Rightarrow \dot{x}_3 \equiv 0 \Rightarrow y_1 \equiv 0$

$\Rightarrow x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$

$\boxed{\text{asy. stable}}$ (by invariance principle)

b) Similar to part (a)

$\boxed{\text{asy. stable}}$

c) $\dot{V} \leq 0 \Rightarrow \boxed{\text{stable}}$

$k = 0 \Rightarrow \dot{V}_2 = 0 \Rightarrow V_2(x_1(t), x_2(t)) = V_2(x_1(0), x_2(0))$

$\Rightarrow V_2 \not\to 0 \Rightarrow (x_1(t), x_2(t)) \not\to 0$

$\Rightarrow \boxed{\text{NOT asy. stable}}$

Q4. 25%

Consider the system $x^+ = f(x)$ with $x \in \mathbb{R}^n$. Assume that we can write $f(x) = Ax + g(x)$ for some $A \in \mathbb{R}^{n\times n}$ and $g : \mathbb{R}^n \to \mathbb{R}^n$. Further assume that the following hold.

- $\|g(x)\| \le \gamma\|x\|$ for some $\gamma > 0$.
- $A^T P A - P = -I$ for some symmetric positive definite $P \in \mathbb{R}^{n\times n}$.

(5) **a)** What can be said about the eigenvalues of the matrix A?

(15) **b)** Using the Lyapunov function $V(x) = x^T P x$ find a range for γ for which the origin is asymptotically stable.

(5) **c)** Suppose γ belongs to the range you found in part (b). What can be said about the set of (real) solutions of the equation $f(z) = z$?

a) $|\lambda_i| < 1$

b) $V(x^+) - V(x) = f(x)^T P f(x) - x^T P x$

$$= (Ax+g)^T P (Ax+g) - x^T P x$$

$$= x^T A^T P A x + 2g^T P A x + g^T P g - x^T P x$$

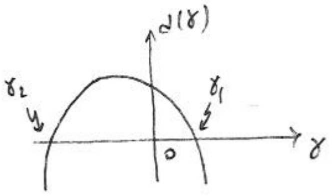
$$= x^T(A^T P A - P)x + 2g^T P A x + g^T P g$$

$$= -\|x\|^2 + 2g^T P A x + g^T P g$$

$$\le -\|x\|^2 + 2\gamma\|P\|\cdot\|A\|\cdot\|x\|^2 + \gamma^2\|P\|\,\|x\|^2$$

$$= -\|x\|^2 \underbrace{\left\{ -\|P\|\gamma^2 - 2\|P\|\cdot\|A\|\gamma + 1 \right\}}_{d(\gamma)}$$

$d(\gamma) > 0 \Rightarrow V(x^+) - V(x) < 0 \Rightarrow$ asy. stable



$$\gamma^2 + 2\|A\|\gamma - \|P\|^{-1} = 0$$

$$\Rightarrow (\gamma + \|A\|)^2 - \|A\|^2 - \|P\|^{-1} = 0$$

$$\Rightarrow \gamma_{1,2} = -\|A\| \mp \sqrt{\|A\|^2 + \|P\|^{-1}}$$

$$\Rightarrow \gamma_1 = -\|A\| + \sqrt{\|A\|^2 + \|P\|^{-1}}$$

$\Rightarrow$ range $= \boxed{0 < \gamma < \gamma_1}$

c) V is rad. unbounded $\Rightarrow$ GAS

GAS $\Rightarrow$ $x = 0$ is the only equilibrium.

$\Rightarrow$ set of solutions to $f(z) = z$ $= \boxed{\{0\}}$