First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No computer/calculator is allowed.
- The duration of the examination is 150 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1. 25%

Consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$V(x) = x_1^4 + (x_2^2 - x_1)^2 + x_2^2$$

whose gradient vanishes only at the origin. For each of the below systems determine:

- Whether the origin is stable or not. (If stable, check also asymptotic stability.)
- Whether finite escape times occur or not.
- Whether (nontrivial) periodic solutions exist or not.

a)
$$\begin{cases} \dot{x}_1 &= -\dfrac{\partial V}{\partial x_1} \\ \dot{x}_2 &= -\dfrac{\partial V}{\partial x_2} \end{cases}$$

b)
$$\begin{cases} \dot{x}_1 &= \dfrac{\partial V}{\partial x_2} \\ \dot{x}_2 &= -\dfrac{\partial V}{\partial x_1} \end{cases}$$

Q2. 25%

Consider the Duffing map

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1 + 3x_2 - x_2^3 \\ y &= x_2\,. \end{aligned}$$

a) Find the equilibrium point(s) of this system.

b) For the initial condition $x(0) = [1 \;\; 0]^T$ find $x(3)$.

c) For this system find the observer dynamics $\hat{x}^+ = g(\hat{x},\, y)$ by applying Glad's method.

d) For the initial conditions $x(0) = [1 \;\; 0]^T$ and $\hat{x}(0) = [-7 \;\; 3]^T$ find $\hat{x}(3)$.

Q3. 25%

Part I. For each of the below systems determine whether the system is lossless, output strictly passive, or strictly passive.

a) $H_1 : \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + u_1 \\ y_1 &= x_2 \end{cases}$ *Hint: Consider* $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

b) $H_2 : \begin{cases} \dot{x}_3 &= -x_3 + u_2 \\ y_2 &= x_3^3 \end{cases}$ *Hint: Choose a suitable storage function* $V_2(x_3)$.

Part II. Consider now the following feedback connection of the systems H_1 and H_2.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + k y_2 \\ \dot{x}_3 &= -x_3 - k y_1 \,. \end{aligned}$$

For each of the below cases determine whether the origin is stable or unstable. (If stable, check also asymptotic stability.)

a) When $k = 1$.

b) When $k = -1$.

c) When $k = 0$.

Q4. 25%

Consider the system $x^+ = f(x)$ with $x \in \mathbb{R}^n$. Assume that we can write $f(x) = Ax + g(x)$ for some $A \in \mathbb{R}^{n\times n}$ and $g : \mathbb{R}^n \to \mathbb{R}^n$. Further assume that the following hold.

- $\|g(x)\| \leq \gamma\|x\|$ for some $\gamma > 0$.
- $A^T PA - P = -I$ for some symmetric positive definite $P \in \mathbb{R}^{n\times n}$.

a) What can be said about the eigenvalues of the matrix A?

b) Using the Lyapunov function $V(x) = x^T Px$ find a range for γ for which the origin is asymptotically stable.

c) Suppose γ belongs to the range you found in part (b). What can be said about the set of (real) solutions of the equation $f(z) = z$?