First name:________________________

Last name: Sol'n

Student ID:________________________

Signature:________________________

Read before you start:

- There are four questions.
- The examination is open-book.
- No computer/calculator is allowed.
- The duration of the examination is 150 minutes.
- Besides correctness, the CLARITY of your presentation will also be graded.

Q1	Q2	Q3	Q4	Total

Q1. 25%

Consider the second-order system below.

$$\begin{aligned}\dot{x}_1 &= x_2 - ax_1\\ \dot{x}_2 &= -x_1^3 - bx_2 + u\\ y &= x_2\end{aligned}$$

For each of the following cases using the storage function candidate $V(x) = (x_1^4 + 2x_2^2)/4$ determine whether the system is lossless, output strictly passive, or strictly passive. Moreover, investigate the stability properties of the origin of the unforced ($u = 0$) system.

a) When $a = 0$ and $b = 0$.

b) When $a = 0$ and $b > 0$.

c) When $a > 0$ and $b > 0$.

$V = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ (V is pos. def.)

$\Rightarrow \dot{V} = x_1^3\dot{x}_1 + x_2\dot{x}_2$

$= x_1^3(x_2 - ax_1) + x_2(-x_1^3 - bx_2 + u)$

$= -ax_1^4 - bx_2^2 + x_2 u$

a) $a = 0$ & $b = 0$

$\dot{V} = x_2 u = uy \Rightarrow \boxed{\text{lossless}}$

$u = 0 \Rightarrow \dot{V} = 0$

$\Rightarrow$ the origin is stable but not asy. stable.

b) $a = 0$ & $b > 0$

$\dot{V} = -by^2 + uy \Rightarrow \boxed{\text{output strictly passive}}$

$u = 0 \Rightarrow \dot{V} = -bx_2^2$

$\dot{V}$ is neg. semi def. $\Rightarrow$ the origin is stable. How about asy. stability?

$\dot{V} = 0 \Rightarrow x_2 = 0$

$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$

$\Rightarrow$ the only solution that can stay identically in the set $\{\dot{V} = 0\}$ is the trivial solution $x(t) \equiv 0$. By invariance principle the origin is asy. stable. Also, V is rad. unbounded. Hence the origin is GAS.

c) $a > 0$ & $b > 0$

$\dot{V} = -ax_1^4 - bx_2^2 + uy \quad \psi(x) := ax_1^4 + bx_2^2$

$= -\psi(x) + uy$

ψ is pos. def. $\Rightarrow \boxed{\text{strictly passive}}$

$u = 0 \Rightarrow \dot{V} = -\psi(x)$

$\left.\begin{array}{l}\dot{V} \text{ is neg. def.}\\ V \text{ is rad. unbounded}\end{array}\right\} \Rightarrow$ the origin is GAS.

Q2. 25%

Consider the below second-order systems.

$$\text{System (1)}: \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + u_1 \\ y_1 &= x_2 \end{cases} \qquad \text{System (2)}: \begin{cases} \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_3^3 + u_2 \\ y_2 &= x_4 \end{cases}$$

Let $u = [u_1\ u_2]^T$ and $y = [y_1\ y_2]^T$ and $x = [x_1\ x_2\ x_3\ x_4]^T$. Suppose we couple these systems by letting $u = -Qy + v$, where Q is a symmetric positive definite matrix.

a) Show that the map from v to y is finite-gain $\mathcal{L}_2$ stable. What is the gain?

b) Show that when $v = 0$ the origin $x = 0$ is globally asymptotically stable.

let $V_1(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ & $V_2(x) = \frac{1}{4}x_3^4 + \frac{1}{2}x_4^2$

$\Rightarrow \dot{V}_1 = u_1 y_1$ & $\dot{V}_2 = u_2 y_2$

Let $V(x) = V_1(x) + V_2(x)$ (V is pos. def.)

$\Rightarrow \dot{V} = \dot{V}_1 + \dot{V}_2 = u_1 y_1 + u_2 y_2 = u^T y$

a) $\dot{V} = u^T y \Big|_{u = -Qy + v} = -y^T Q y + v^T y$

$\Rightarrow \dot{V} \le -\lambda_{min}(Q) y^T y + v^T y$

$\Rightarrow$ By Lemma 6.5 the map from v to y is finite-gain $\mathcal{L}_2$ stable with gain $\le \frac{1}{\lambda_{min}(Q)}$

b) when $v = 0$ we have

$\dot{V} \le -\lambda_{min}(Q) \|y\|^2$ ($\dot{V}$ neg. semi def.)

$\dot{V} = 0 \Rightarrow y = 0 \Rightarrow u = 0$

$$y(t) \equiv 0 \Rightarrow \begin{cases} x_2(t) \equiv 0 \\ x_4(t) \equiv 0 \end{cases} \Rightarrow \begin{cases} \dot{x}_2(t) \equiv 0 \\ \dot{x}_4(t) \equiv 0 \end{cases} \quad (1)$$

$$u(t) \equiv 0 \Rightarrow \begin{cases} u_1(t) \equiv 0 \\ u_2(t) \equiv 0 \end{cases} \quad (2)$$

(1) & (2) $\Rightarrow x_1(t) \equiv 0$ & $x_3(t) \equiv 0$

That is, the only solution that can stay identically in the set $\{\dot{V} = 0\}$ is the trivial solution $x(t) \equiv 0$. By invariance principle & the fact that V is radially unbounded the origin is GAS. ∎

Q3.

25%

Consider the Henon map

$$
\begin{aligned}
x_1^+ &= x_2 + 1 - a x_1^2 \\
x_2^+ &= b x_1 \\
y &= h(x)
\end{aligned}
$$

where a and b are nonzero constants. For this system find the observer dynamics $\hat{x}^+ = g(\hat{x}, y)$ by applying Glad's method for each of the below cases.

a) When $h(x) = x_1$.

b) When $h(x) = x_2$.

let $f^{-1}(x) = \begin{bmatrix} \xi_1(x_1, x_2) \\ \xi_2(x_1, x_2) \end{bmatrix}$

$f(f^{-1}(x)) = x \Rightarrow \begin{bmatrix} \xi_2 + 1 - a\xi_1^2 \\ b\xi_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow \xi_1 = \frac{x_2}{b}$ & $\xi_2 = x_1 + a\xi_1^2 - 1$

$= x_1 + \frac{a}{b^2} x_2^2 - 1$

$\Rightarrow f^{-1}(x) = \begin{bmatrix} \frac{x_2}{b} \\ x_1 + \frac{a}{b^2} x_2^2 - 1 \end{bmatrix}$

a) $h(x) = x_1$

let $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ satisfy

$$\left.\begin{aligned} h(\eta) &= h(x) \\ h(f^{-1}(\eta)) &= h(f^{-1}(\hat{x})) \end{aligned}\right\} (*)$$

$$\left.\begin{aligned} \Rightarrow \eta_1 &= x_1 \\ \frac{\eta_2}{b} &= \frac{\hat{x}_2}{b} \end{aligned}\right\} \eta = \begin{bmatrix} x_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} y \\ \hat{x}_2 \end{bmatrix}$$

$\Rightarrow g(\hat{x}, y) = f(\eta) = \begin{bmatrix} \hat{x}_2 + 1 - a y^2 \\ b y \end{bmatrix}$

$$\boxed{\Rightarrow \hat{x}^+ = \begin{bmatrix} \hat{x}_2 + 1 - a y^2 \\ b y \end{bmatrix}}$$

b) $h(x) = x_2$

$(*) \Rightarrow \quad \eta_2 = x_2$

$\eta_1 + \frac{a}{b^2}\eta_2^2 \cancel{-1} = \hat{x}_1 + \frac{a}{b^2}\hat{x}_2^2 \cancel{-1}$

$\Rightarrow \eta_1 = \hat{x}_1 + \frac{a}{b^2}\hat{x}_2^2 - \frac{a}{b^2} x_2^2$

$\Rightarrow \eta = \begin{bmatrix} \hat{x}_1 + \frac{a}{b^2}\hat{x}_2^2 - \frac{a}{b^2} y^2 \\ y \end{bmatrix}$

$g(\hat{x}, y) = f(\eta) = \begin{bmatrix} y + 1 - a\left[\hat{x}_1 + \frac{a}{b^2}\hat{x}_2^2 - \frac{a}{b^2} y^2\right]^2 \\ b\left(\hat{x}_1 + \frac{a}{b^2}\hat{x}_2^2 - \frac{a}{b^2} y^2\right) \end{bmatrix}$

$$\boxed{\Rightarrow \hat{x}^+ = \begin{bmatrix} y + 1 - a\left(\hat{x}_1 + \frac{a}{b^2}\hat{x}_2^2 - \frac{a}{b^2} y^2\right)^2 \\ b\left(\hat{x}_1 + \frac{a}{b^2}\hat{x}_2^2 - \frac{a}{b^2} y^2\right) \end{bmatrix}}$$

Q4. 25%

Let

$$A = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$$

a) Determine the stability properties of the origin of the system $x^+ = Ax$.

b) Find a symmetric positive definite matrix P satisfying the discrete-time Lyapunov equation $A^T P A - P + I = 0$. *Hint: What is A^2?*

c) Show that the origin of the system $x^+ = Ax + \varepsilon\|x\|B$, where $B \in \mathbb{R}^{2\times 1}$ is a constant matrix and $\varepsilon > 0$, is asymptotically stable for small enough ε. *Hint: Try $V(x) = x^T P x$.*

a) $|sI - A| = \begin{vmatrix} s+2 & 4 \\ -1 & s-2 \end{vmatrix} = s^2 - 4 + 4 = s^2$

$\Rightarrow \lambda_{1,2} = 0 \Rightarrow |\lambda_i| < 1 \Rightarrow \boxed{\text{GAS}}$

b) ch. poly: $s^2 \Rightarrow A^2 = 0$

$P = \sum_{k=0}^{\infty} A^{kT} I A^k = I + A^T A + \underbrace{A^{2T} A^2 + \cdots}_{=0 \text{ since } A^2 = 0}$

$\Rightarrow P = I + A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \underbrace{\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}}_{\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}}$

$$\boxed{\Rightarrow P = \begin{bmatrix} 6 & 10 \\ 10 & 21 \end{bmatrix}}$$

c) $\Delta V(x) = V(x^+) - V(x)$

$= (\varepsilon\|x\|B^T + x^T A^T) P (Ax + \varepsilon\|x\|B) - x^T P x$

$= \varepsilon^2 \|x\|^2 B^T P B + 2\varepsilon\|x\| B^T P A x + x^T \underbrace{(A^T P A - P)}_{-I} x$

$\Rightarrow \Delta V(x) \leq -\|x\|^2 + (\varepsilon^2 \|B^T P B\| + 2\varepsilon \|B^T P A\|)\|x\|^2$

let $\varepsilon < 1$

$\Rightarrow \Delta V(x) \leq -\frac{1}{2}\|x\|^2 - \|x\|^2 \left\{ \frac{1}{2} - \left[\|B^T P B\| + 2\|B^T P A\|\right]\varepsilon \right\}$

Choose $\varepsilon \leq \min\left\{1, \dfrac{1}{2(\|B^T P B\| + 2\|B^T P A\|)}\right\}$

Then $\Delta V(x) \leq -\frac{1}{2}\|x\|^2$ (ΔV neg. def.)

ΔV neg. def. $\Rightarrow$ the origin is GAS. ☒