EE502 Second Midterm Exam 1 June 2022

First name:________________________

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Read before you start:

- There are four questions.
- The duration of the examination is 100 minutes.
- The examination is closed-book.
- Please explain all your answers.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the following system

A_c $\quad$ B_c

$$\dot{x} = \begin{bmatrix} \boxed{\begin{matrix} 5 & -3 \\ 6 & -4 \end{matrix}} & \begin{matrix} 3 & 1 \\ 1 & 4 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} -2 & 1 \\ -5 & 0 \end{matrix}} \end{bmatrix} x + \begin{bmatrix} \boxed{\begin{matrix} 0 \\ -1 \end{matrix}} \\ \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix} u$$

C_c $\quad$ A_u

$$y = [\boxed{2 \ -1}\ 2 \ 7]\, x.$$

(6) **(a)** Is this system in controllable decomposition form?

(6) **(b)** Is this system stabilizable?

(6) **(c)** Is this system BIBO stable?

(7) **(d)** Determine, if possible, the range of $k \in \mathbb{R}$ for which the output feedback law $u = -ky$ renders the closed-loop system asymptotically stable.

a) check whether the pair (A_c, B_c) is controllable.

$$\text{rank}\begin{bmatrix} B_c & A_cB_c \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix} = 2$$

$\Rightarrow$ $\boxed{\text{YES}}$ the system is in Con. decomp. form.

b) Check the eigenvalues of A_u.

$$|sI - A_u| = \begin{vmatrix} s+2 & -1 \\ 5 & s \end{vmatrix} = s^2 + 2s + 5$$

$$\Rightarrow \lambda_{1,2}(A_u) = -1 \mp j2 \Rightarrow \text{Re}\{\lambda_{1,2}(A_u)\} < 0$$

$\Rightarrow$ $\boxed{\text{YES}}$ the system is stabilizable.

c) Check the poles of the TF.

$$H(s) = C_c[sI - A_c]^{-1}B_c$$

$$= [2 \ -1]\begin{bmatrix} s-5 & 3 \\ -6 & s+4 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \frac{1}{s-2}$$

$$\Rightarrow \lambda = 2 > 0$$

$\Rightarrow$ $\boxed{\text{NO}}$ the system is not BIBO stable.

d) Check the eigenvalues of $M = A_c - kB_cC_c$.

$$M = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} - k\begin{bmatrix} 0 \\ -1 \end{bmatrix}[2 \ -1]$$

$$= \begin{bmatrix} 5 & -3 \\ 6+2k & -4-k \end{bmatrix}$$

$$\Rightarrow |sI - M| = (s-5)(s+k+4) + 3(6+2k)$$

$$= s^2 + (k-1)s + k - 2$$

$$= (s+1)(s+k-2)$$

$$\Rightarrow \lambda_{1,2}(M) = -1, \ 2-k$$

For $\text{Re}\{\lambda_{1,2}(M)\} < 0$ we need $\boxed{k > 2}$

Q2.

Given

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

determine (without direct computation) whether a symmetric positive definite solution $P \in \mathbb{R}^{2\times 2}$ exists or not for each of the following matrix (in)equalities.

(a) $AP + PA^T + BB^T = 0$.

(b) $A^TP + PA + BB^T = 0$.

(c) $APA^T - P - BB^T < 0$.

(d) $A^TPA - P - BB^T < 0$.

$$|sI - A| = \begin{vmatrix} s & -1 \\ \frac{1}{2} & s+\frac{3}{2} \end{vmatrix} = s\left(s+\frac{3}{2}\right) + \frac{1}{2}$$

$$= (s+1)\left(s+\frac{1}{2}\right)$$

$$\Rightarrow \lambda(A) = \lambda(A^T) = \left\{-1, -\frac{1}{2}\right\}$$

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$$\det[B \;\; AB] = \det\begin{bmatrix} 2 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} = 0$$

$\Rightarrow$ (A, B) not controllable

Let $v_1 \in \mathbb{R}^2$ be the eigenvector of A^T for $\lambda_1 = -1$. Then

$$[sI - A^T]_{s=-1} = \begin{bmatrix} -1 & 1/2 \\ -1 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1/2 \\ -1 & 1/2 \end{bmatrix} v_1 = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad B^Tv_1 = [2 \;\; -1]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$$

$\Rightarrow$ (A, B) not stabilizable in the DT sense.

$$\det[B \;\; A^TB] = \det\begin{bmatrix} 2 & 1/2 \\ -1 & 7/2 \end{bmatrix} \neq 0$$

$\Rightarrow$ (A^T, B) controllable

$\Rightarrow$ (A^T, B) stabilizable

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a) Since A is exp. stable (in the CT sense) this equation has a solution if and only if (A, B) is controllable. But (A, B) isn't controllable.

$\Rightarrow$ Solution $\boxed{\text{DOESN'T EXIST.}}$

b) A^T is exp. stable and (A^T, B) controllable.

$\Rightarrow$ Solution $\boxed{\text{EXISTS.}}$

c) (A, B) is not stabilizable (in the DT sense)

$\Rightarrow$ solution $\boxed{\text{DOESN'T EXIST.}}$

d) (A^T, B) is stabilizable.

$\Rightarrow$ solution $\boxed{\text{EXISTS.}}$

Q3.

Consider the system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u \,.$$

(a) Is this system stable?

(b) Is this system controllable?

(c) Is this system stabilizable?

(d) Find, if possible, a feedback gain $K \in \mathbb{R}^{1\times 2}$ such that the feedback law $u = -Kx$ renders the closed-loop system asymptotically stable.

a) $|sI - A| = \begin{vmatrix} s & -4 \\ -2 & s-2 \end{vmatrix} = s^2 - 2s - 8 = (s-4)(s+2)$

$\lambda_1 = 4 \,,\ \lambda_2 = -2 \quad \Rightarrow \quad \lambda_1 > 0$

$\Rightarrow$ $\boxed{\text{NO}}$ the system is not stable.

b) $\det\begin{bmatrix} B & AB \end{bmatrix} = \det\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix} = 0$

$\Rightarrow$ $\boxed{\text{NO}}$ the system is not controllable.

c) Check the eigenvector v_1 of A^T for the unstable eigenvalue $\lambda_1 = 4$.

$$\left[sI - A^T\right]_{s=4} = \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix} v_1 = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$B^T v_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq 0$$

$\Rightarrow$ $\boxed{\text{YES}}$ the system is stabilizable.

d) Let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$

$$A - BK = \begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 4-k_2 \\ 2-k_1 & 2-k_2 \end{bmatrix}$$

$$\Rightarrow |sI - (A - BK)| = \begin{vmatrix} s+k_1 & k_2-4 \\ k_1-2 & s+k_2-2 \end{vmatrix}$$

$$= (s+k_1)(s+k_2-2) - (k_1-2)(k_2-4)$$

$$= (s+2)(s+k_1+k_2-4)$$

$$\Rightarrow \lambda_{cl} = \{-2, \ -(k_1+k_2-4)\}$$

$\Rightarrow k_1 + k_2 > 4$ achieves stabilization.

Choose, for instance, $\boxed{K = \begin{bmatrix} 2 & 3 \end{bmatrix}}$

Q4.

Definition. Let $\beta_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ be given. The functions $\beta_1(t), \beta_2(t), \ldots, \beta_n(t)$ are said to be *linearly independent* if the equation

$$\alpha_1\beta_1(t) + \alpha_2\beta_2(t) + \cdots + \alpha_n\beta_n(t) = 0 \qquad \forall t \in \mathbb{R}$$

(where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$) has no solution other than the trivial solution: $\alpha_i = 0$ for all i.

Now, prove the following claim.

Claim. Let the system $\dot{x} = Ax + Bu$ (with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$) be controllable. Then the entries of the vector $e^{At}B$ must be linearly independent.

Proof Suppose not. That is, (A, B) is cont. but the entries of $e^{At}B$ are linearly dep.

Then we can find a nonzero $\eta \in \mathbb{R}^n$ such that

$$\eta^T e^{At} B = 0 \quad \text{for all } t. \qquad (1)$$

Using the properties of matrix exponential

$$\rightarrow \; e^{At}\Big|_{t=0} = I$$

$$\rightarrow \; \frac{d}{dt}\{e^{At}\} = Ae^{At}$$

we can obtain from (1):

$$\left.\begin{array}{l} \eta^T B = 0 \\ \eta^T AB = 0 \\ \vdots \\ \eta^T A^{n-1}B = 0 \end{array}\right\} (2)$$

The set of equations (2) can be written as

$$\eta^T \underbrace{[B \;\; AB \; \cdots \; A^{n-1}B]}_{C} = 0$$

which implies rank $C \neq n$. Hence (A, B) is not controllable. The result follows by contradiction. ▨

Alternative proof - Suppose not. Then we can find a nonzero $\eta \in \mathbb{R}^n$ such that

$$\eta^T e^{At} B = 0 \quad \text{for all } t$$

Construct the controllability gramian

$$W = \int_0^1 e^{At} B B^T e^{A^T t}\, dt$$

We can write

$$\eta^T W = \eta^T \int_0^1 e^{At} B B^T e^{A^T t}\, dt$$

$$= \int_0^1 \underbrace{[\eta^T e^{At} B]}_{0} B^T e^{A^T t}\, dt$$

$$= 0$$

which implies rank $W \neq n$. Hence (A, B) is not controllable. The result follows by contradiction. ▨