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Read before you start:

- There are FIVE QUESTIONS.
- The examination is OPEN-BOOK.
- Besides correctness, the CLARITY of your presentation will also be graded.
- NO COLLABORATION with others.

Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Total**

Q1.

Let the matrix $A \in \mathbb{R}^{4\times 4}$ be such that all of the following four systems are stable in the sense of Lyapunov:

$$\text{(i)}\ \ \dot{x} = Ax\,, \qquad \text{(ii)}\ \ \dot{x} = -Ax\,, \qquad \text{(iii)}\ \ x^+ = Ax\,, \qquad \text{(iv)}\ \ x^+ = A^{-1}x\,.$$

Let now $B \in \mathbb{R}^{4\times 1}$ be a nonzero matrix and consider the following single-input single-output system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= B^T x\,. \end{aligned} \tag{1}$$

(a) Show that the system (1) cannot be controllable.

(b) Show that the system (1) cannot be BIBO stable.

Sol'n. (a) Let λ denote an arbitrary eigenvalue of A. Stability of the first system implies $\text{Re}(\lambda) \leq 0$; second, $\text{Re}(\lambda) \geq 0$; third, $|\lambda| \leq 1$; fourth, $|\lambda| \geq 1$. The intersection of these four regions of the complex plane is the set $\{+j, -j\} \ni \lambda$. Since A is real, the eigenvalues appear as conjugate pairs. Furthermore, since $\text{Re}(\lambda) = 0$, the stability of the first system implies that all the Jordan blocks should be of size 1×1. Hence the Jordan form J of A reads

$$J = \begin{bmatrix} +j & 0 & 0 & 0 \\ 0 & +j & 0 & 0 \\ 0 & 0 & -j & 0 \\ 0 & 0 & 0 & -j \end{bmatrix}.$$

Let $V \in \mathbb{C}^{4\times 4}$ be such that $A = VJV^{-1}$. Since $J^2 = -I$ we have $A^2 = VJV^{-1}VJV^{-1} = VJ^2V^{-1} = -I$. Hence the controllability matrix reads

$$C = [B \ \ AB \ \ A^2B \ \ A^3B] = [B \ \ AB \ \ -B \ \ -AB]\,.$$

We now see that $\text{rank}(C) \leq 2$. Lack of controllability then follows from this rank deficiency.

(b) Recall that the set of poles of the transfer function $H(s)$ is a subset of the set of eigenvalues of A. This allows only three candidates.

$$\text{Either} \quad H(s) = 0\,, \quad \text{or} \quad H(s) = \frac{\cdots}{s^2+1}\,, \quad \text{or} \quad H(s) = \frac{\cdots}{(s^2+1)^2}\,.$$

Of these, only the first one $H(s) = 0$ is the transfer function of a BIBO stable system. Let us now show that $H(s) = 0$ is impossible. Suppose $H(s) = 0$. Then the impulse response $h(t) = B^T e^{At} B$ is zero for all $t \geq 0$. Since $e^{At}\big|_{t=0} = I$, we can write $0 = h(0) = B^T B = \|B\|^2$. This however contradicts the fact that B is nonzero.

Q2.

Consider the following third order system ("$*$" means "don't care")

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 3 & * \\ 2 & 2 & * \\ 1 & -1 & * \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{B} u\,.$$

This system is known to be uncontrollable.

(a) Find a basis for the controllable subspace.

(b) Determine whether this system is stabilizable.

Sol'n. (a) Recall that the range space of the controllability matrix $C = [B \;\; AB \;\; A^2B]$ coincides with the controllable subspace $\mathcal{C}$. Observe that we have

$$C = \begin{bmatrix} 1 & -2 & * \\ -1 & 0 & * \\ 0 & 2 & * \end{bmatrix}.$$

Since the system is uncontrollable, $\text{rank}(C) < 3$. We also see that the first two columns of C are linearly independent. Hence $\text{rank}(C)$ must equal 2. Consequently, $\text{range}(C)$ can be spanned by the first two columns. This allows us to write

$$\mathcal{C} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

(b) That the system is uncontrollable implies the existence of an eigenvector v and an eigenvalue λ satisfying $A^Tv = \lambda v$ and $B^Tv = 0$. Combining these we can write $B^TA^Tv = \lambda B^Tv = 0$. In other words,

$$v \in \text{null} \begin{bmatrix} B^T \\ B^TA^T \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & 2 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Since $A^Tv = \lambda v$ we have to have

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & -1 \\ * & * & * \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which yields $\lambda = 4$. Hence the system is not stabilizable because $\text{Re}(\lambda) \geq 0$.

Q3.

Let $P \in \mathbb{R}^{n\times n}$ be symmetric positive definite and $B \in \mathbb{R}^{n\times m}$. Construct the complex matrix

$$Z = BB^T + jP\,.$$

Consider now the following system

$$\dot{x} = Px + Bu\,.$$

(a) Show that the system is controllable if Z has no eigenvalue on the imaginary axis.

(b) Show that Z has no eigenvalue on the imaginary axis if the system is controllable.

Sol'n. (a) Let the system be uncontrollable. Then there exist an eigenvector v and an eigenvalue σ satisfying $\sigma v = P^T v = Pv$ and $B^T v = 0$. Since P is symmetric positive definite, σ is real and positive. Then $Zv = (BB^T + jP)v = B(B^T v) + j(Pv) = j\sigma v$. That is, Z has a purely imaginary eigenvalue.

(b) To show the other direction suppose Z has a purely imaginary eigenvalue $\lambda = j\omega$ (where ω is real). Let η be the corresponding eigenvector. Without loss of generality let η be a unit vector. That is, $\eta^*\eta = 1$. We can write

$$j\omega = \eta^*(j\omega\eta) = \eta^* Z\eta = \eta^*(BB^T + jP)\eta = \|B^T\eta\|^2 + j(\eta^* P\eta)$$

which allows us to see $B^T\eta = 0$ because $\eta^* P\eta$ is a real number. Now, we can proceed as

$$j\omega = Z\eta = (BB^T + jP)\eta = B(B^T\eta) + jP\eta = jP\eta$$

which tells us at once that $\omega\eta = P\eta = P^T\eta$. Hence the system has to be uncontrollable because this eigenvector η of P^T belongs to null(B^T).

Q4.

Consider the following system

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \\ 0 & 0 & -3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} u$$

$$y = \underbrace{[0 \;\; 1 \;\; 1]}_{C} x\,.$$

(a) Find a feedback gain $K = [k_1 \;\; k_2 \;\; k_3]$ such that under the feedback law $u = -Kx$ the eigenvalues of the closed-loop system are $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$.

(b) Consider now the control law of the form $u = -Kx + \gamma\bar{r}$ where K is found in part (a), $\gamma \in \mathbb{R}$ is your design parameter, and $\bar{r} \in \mathbb{R}$ denotes some constant reference value (unknown to the designer). Find γ so that, under the proposed control law, the output of the system converges to the reference value $y(t) \to \bar{r}$ for all initial conditions $x(0)$. (Your answer should be independent of $\bar{r}$.)

Sol'n. (a) Let $\lambda(A - BK)$ mean the set of eigenvalues of $A - BK$. The system is in controllable decomposition form. This allows us to see

$$\lambda(A - BK) = \lambda\left(\begin{bmatrix} 1 - k_1 & 2 - k_2 \\ 3 & 4 \end{bmatrix}\right) \cup \{-3\}$$

Letting $k_1 = 8$ and $k_2 = 12$ we obtain

$$\lambda\left(\begin{bmatrix} 1 - k_1 & 2 - k_2 \\ 3 & 4 \end{bmatrix}\right) = \lambda\left(\begin{bmatrix} -7 & -10 \\ 3 & 4 \end{bmatrix}\right) = \{-1, \, -2\}\,.$$

Hence $K = [8 \;\; 12 \;\; k_3]$ would work for any k_3.

(b) The closed-loop dynamics read

$$\begin{aligned} \dot{x} &= (A - BK)x + \gamma B\bar{r} \\ y &= Cx\,. \end{aligned}$$

Since $A - BK$ is a Hurwitz matrix, the steady state exists. That is, $x(t) \to \bar{x}$ where $\bar{x}$ is constant. In the steady state $\dot{x} = 0$. Therefore $0 = (A - BK)\bar{x} + \gamma B\bar{r}$ whence $\bar{x} = -\gamma(A - BK)^{-1}B\bar{r}$. Now, what we want is, clearly, $\bar{r} = C\bar{x} = -\gamma C(A - BK)^{-1}B\bar{r}$. Hence

$$\gamma = -\frac{1}{C(A - BK)^{-1}B} = \frac{2}{3}\,.$$

Q5.

Given a controllable pair (A, B) suppose there exists $P = P^T > 0$ such that $A^TP + PA \leq 0$. Consider now the matrix $H = A - BB^TP$.

(a) Show that the pair (H, B) is controllable.

(b) Show that the system $\dot{x} = Hx$ is exponentially stable.

Sol'n. (a) Suppose not. Then we can find an eigenvector v and an eigenvalue λ such that $H^Tv = \lambda v$ and $B^Tv = 0$. Now, we can write $A^Tv = (H^T + PBB^T)v = H^Tv + PB(B^Tv) = \lambda v$. But this implies (A, B) is uncontrollable.

(b) Note that $A^TP + PA \leq 0$ implies $Q(A^TP + PA)Q \leq 0$ for any symmetric positive definite matrix Q. Let us choose $Q = P^{-1}$. Since P is symmetric positive definite, so is Q. We can write

$$\begin{aligned}
QH^T + HQ &= P^{-1}(A^T - PBB^T) + (A - BB^TP)P^{-1} \\
&= P^{-1}A^T + AP^{-1} - 2BB^T \\
&= Q(A^TP + PA)Q - 2BB^T \\
&\leq -2BB^T .
\end{aligned}$$

Now, let λ be an eigenvalue of H^T with the corresponding eigenvector η, i.e., $H^T\eta = \lambda\eta$. Since the pair (H, B) is controllable we have that $B^T\eta \neq 0$. We can proceed as

$$\begin{aligned}
0 &> -2\|B^T\eta\|^2 \\
&= -2\eta^*BB^T\eta \\
&\geq \eta^*(QH^T + HQ)\eta \\
&= \eta^*Q(H^T\eta) + (H^T\eta)^*Q\eta \\
&= (\lambda + \lambda^*)\eta^*Q\eta \\
&= 2\mathrm{Re}(\lambda)\eta^*Q\eta
\end{aligned}$$

which implies $\mathrm{Re}(\lambda) < 0$ since $\eta^*Q\eta > 0$. Recall that H and H^T share the same eigenvalues. As a result, all the eigenvalues of H are on the open left half-plane. Exponential stability then follows.