**First name:**____________________________

**Last name:**____________________________

**Student ID:**____________________________

**Signature:**____________________________

**Read before you start:**

- There are FIVE QUESTIONS.
- The examination is OPEN-BOOK.
- Besides correctness, the CLARITY of your presentation will also be graded.
- NO COLLABORATION with others.

| **Q1** | **Q2** | **Q3** | **Q4** | **Q5** | **Total** |
|---|---|---|---|---|---|
| | | | | | |

## Q1.

Let the matrix $A \in \mathbb{R}^{4\times 4}$ be such that all of the following four systems are stable in the sense of Lyapunov:

$$\text{(i)}\ \ \dot{x} = Ax\,, \qquad \text{(ii)}\ \ \dot{x} = -Ax\,, \qquad \text{(iii)}\ \ x^+ = Ax\,, \qquad \text{(iv)}\ \ x^+ = A^{-1}x\,.$$

Let now $B \in \mathbb{R}^{4\times 1}$ be a nonzero matrix and consider the following single-input single-output system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= B^T x\,. \end{aligned} \tag{1}$$

**(a)** Show that the system (1) cannot be controllable.

**(b)** Show that the system (1) cannot be BIBO stable.

**Q2.**

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Consider the following third order system ("$*$" means "don't care")

$$\dot{x} = \begin{bmatrix} 1 & 3 & * \\ 2 & 2 & * \\ 1 & -1 & * \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u \,.$$

This system is known to be uncontrollable.

**(a)** Find a basis for the controllable subspace.

**(b)** Determine whether this system is stabilizable.

## Q3.

Let $P \in \mathbb{R}^{n \times n}$ be symmetric positive definite and $B \in \mathbb{R}^{n \times m}$. Construct the complex matrix

$$Z = BB^T + jP\,.$$

Consider now the following system

$$\dot{x} = Px + Bu\,.$$

**(a)** Show that the system is controllable if $Z$ has no eigenvalue on the imaginary axis.

**(b)** Show that $Z$ has no eigenvalue on the imaginary axis if the system is controllable.

## Q4.

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Consider the following system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \\ y &= [0 \;\; 1 \;\; 1] x \,. \end{aligned}$$

**(a)** Find a feedback gain $K = [k_1 \;\; k_2 \;\; k_3]$ such that under the feedback law $u = -Kx$ the eigenvalues of the closed-loop system are $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$.

**(b)** Consider now the control law of the form $u = -Kx + \gamma\bar{r}$ where $K$ is found in part (a), $\gamma \in \mathbb{R}$ is your design parameter, and $\bar{r} \in \mathbb{R}$ denotes some constant reference value (unknown to the designer). Find $\gamma$ so that, under the proposed control law, the output of the system converges to the reference value $y(t) \to \bar{r}$ for all initial conditions $x(0)$. (Your answer should be independent of $\bar{r}$.)

## Q5.

Given a controllable pair $(A, B)$ suppose there exists $P = P^T > 0$ such that $A^T P + PA \leq 0$. Consider now the matrix $H = A - BB^T P$.

**(a)** Show that the pair $(H, B)$ is controllable.

**(b)** Show that the system $\dot{x} = Hx$ is exponentially stable.