EE502 Second Midterm Exam 11 May 2016

First name:________________________

Last name: KEY

Student ID:________________________

Signature:________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	Q2	Q3	Q4	Total

Q1.

For $\alpha,\ \beta,\ \gamma \in [0,\ 2\pi)$ consider the second-order LTI system

$$\dot{x} = \underbrace{\left(\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}[\cos\beta\ \ \sin\beta]\right)}_{A} x + \underbrace{\begin{bmatrix}\cos\gamma\\ \sin\gamma\end{bmatrix}}_{B} u$$
$$y = Cx$$

(10) **(a)** Find the eigenvectors and eigenvalues of A^T.

(5) **(b)** Find conditions on the triple $(\alpha,\ \beta,\ \gamma)$ under which the system is controllable.

(5) **(c)** Find conditions on the triple $(\alpha,\ \beta,\ \gamma)$ under which the system is stabilizable.

(5) **(d)** Find conditions on the triple $(\alpha,\ \beta,\ \gamma)$ under which the system is BIBO stable for all C.

Recall: $\cos(\theta-\psi) = \cos\theta\cos\psi + \sin\theta\sin\psi$ & $\sin(\theta-\psi) = \sin\theta\cos\psi - \cos\theta\sin\psi$.

a) Since $A^T = xz^T$ $(x, z \in \mathbb{R}^2)$ $v_1 = x$ and $v_2 \perp z$

$$\Rightarrow \boxed{v_1 = \begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix}\ \&\ v_2 = \begin{bmatrix}\sin\alpha\\ -\cos\alpha\end{bmatrix}}$$

$$A^T v_1 = \begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix}\underbrace{[\cos\alpha\ \sin\alpha]\begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix}}_{\cos(\alpha-\beta)} = \underbrace{\cos(\alpha-\beta)}_{\lambda_1} v_1$$

$$A^T v_2 = \begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix}\underbrace{[\cos\alpha\ \sin\alpha]\begin{bmatrix}\sin\alpha\\ -\cos\alpha\end{bmatrix}}_{0} = \underbrace{0}_{\lambda_2}\cdot v_2$$

$$\Rightarrow \boxed{\lambda_1 = \cos(\alpha-\beta)\ \&\ \lambda_2 = 0}$$

b) Controllability $\Leftrightarrow$ $B^T v_1 \neq 0$ & $B^T v_2 \neq 0$

$$\left.\begin{aligned} B^T v_1 &= \cos(\beta-\gamma)\\ B^T v_2 &= \sin(\alpha-\gamma)\end{aligned}\right\} \Rightarrow \boxed{\cos(\beta-\gamma)\neq 0 \text{ AND } \sin(\alpha-\gamma)\neq 0}$$

c) If $\lambda_1 < 0$ then we need only $B^T v_2 \neq 0$.

If $\lambda_1 \geq 0$ then we need controllability.

Therefore:

$$\boxed{\{\sin(\alpha-\gamma)\neq 0\} \text{ AND } \{\cos(\alpha-\beta) < 0 \text{ OR } \cos(\beta-\gamma)\neq 0\}}$$

d) We need the cont. decomp. to be of the form:

$$\dot{z} = \begin{bmatrix}\lambda_1 & a_{12}\\ 0 & \lambda_2\end{bmatrix} z + \begin{bmatrix}b_c\\ 0\end{bmatrix} u \quad \text{with } \lambda_1 < 0$$

$$\Rightarrow \boxed{\sin(\alpha-\gamma) = 0 \text{ AND } \cos(\alpha-\beta) < 0}$$

Q2.

Consider the second-order discrete-time system below.

$$\begin{aligned} x_1^+ &= x_2^2 \\ x_2^+ &= x_1^2 + 2x_2 \,. \end{aligned}$$

(a) Find all the equilibrium points of this system.

(b) At each equilibrium obtain the linearization.

(c) Determine (if possible) the local stability of each equilibrium based on its linearization.

a) $f(x) = x \Rightarrow \begin{bmatrix} x_2^2 \\ x_1^2 + 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow (x_2^2)^2 + 2x_2 = x_2 \Rightarrow x_2(x_2^3 + 1) = 0$

$\Rightarrow x_2 = 0 \ (x_1 = 0)$ OR $x_2 = -1 \ (x_1 = 1)$

$\Rightarrow \boxed{x_{eq} \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}}$

b) $\dfrac{\partial f}{\partial x} = \begin{bmatrix} 0 & 2x_2 \\ 2x_1 & 2 \end{bmatrix}$

$x_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad A = \left.\dfrac{\partial f}{\partial x}\right|_{x = x_{eq}} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow \boxed{\xi^+ = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \xi}$

$x_{eq} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad A = \left.\dfrac{\partial f}{\partial x}\right|_{x = x_{eq}} = \begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix}$

$\Rightarrow \boxed{\xi^+ = \begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix} \xi}$

c) $x_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad$ eigenvalues of A: $\lambda_1 = 0, \lambda_2 = 2$

$|\lambda_2| > 1 \Rightarrow \boxed{\text{unstable}}$

$x_{eq} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$|sI - A| = \begin{vmatrix} s & 2 \\ -2 & s-2 \end{vmatrix} = s(s-2) + 4 = s^2 - 2s + 4$

$\Rightarrow \lambda_1 \lambda_2 = 4 \Rightarrow$ either $|\lambda_1| > 1$ or $|\lambda_2| > 1$

$\Rightarrow \boxed{\text{unstable}}$

Q3.

Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, let us define the set $\mathcal{S} := \{v \in \mathbb{C}^n : v^* e^{At} B = 0 \text{ for all } t \geq 0\}$. Prove *only one* of the following statements. [If you attempt both, only the ♠ will be graded.]

♠ If the pair (A, B) is controllable then $\mathcal{S} = \{0\}$.

proof Suppose not. That is, (A,B) cont. but $\exists\, v \neq 0$ such that $v^* e^{At} B \equiv 0$. We can write

$$\left.\begin{array}{l} v^* e^{At} B \equiv 0 \\ v^* A e^{At} B \equiv 0 \\ \vdots \\ v^* A^{n-1} e^{At} B \equiv 0 \end{array} \right\} \frac{d}{dt}\{\cdots\} \quad \vdots \quad \Bigg|_{t=0} \Rightarrow \left.\begin{array}{l} v^* B = 0 \\ v^* AB = 0 \\ \vdots \\ v^* A^{n-1} B = 0 \end{array}\right\} (*)$$

$$(*) \Rightarrow v^* \underbrace{\left[B \ AB \cdots A^{n-1}B\right]}_{C} = 0 \quad \Rightarrow \text{rank } C \neq n$$

$\Rightarrow (A,B)$ not cont.

$\Rightarrow$ contradiction.

♣ If $\mathcal{S} = \{0\}$ then the pair (A, B) is controllable.

proof Suppose not. That is, $v^* e^{At} B \equiv 0 \Rightarrow v = 0$ but (A,B) is not controllable. Then we can find an eigenvector $(x \neq 0)$ satisfying $A^T x = \lambda x$ & $B^T x = 0$

$$A^T x = \lambda x \Rightarrow e^{A^T t} x = e^{\lambda t} x$$

$$\Rightarrow B^T e^{A^T t} x \equiv e^{\lambda t} B^T x \equiv 0$$

$$\Rightarrow x^* e^{At} B \equiv 0$$

$$\Rightarrow x = 0$$

$\Rightarrow$ contradiction.

Q4.

Consider the system

$$\dot{x} = \underbrace{\begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u.$$

(5) (a) Is the above representation in controllable decomposition?

(5) (b) Can we find a matrix $P^T = P > 0$ satisfying $AP + PA^T = -BB^T$?

(10) (c) Find the feedback gain $K \in \mathbb{R}^{1\times 2}$ such that the control law $u = -Kx$ places the eigenvalues of the closed-loop system at $\lambda_{1,2} = -2, -3$.

(5) (d) Can we find a matrix $P^T = P > 0$ satisfying $[A - BK]P + P[A - BK]^T = -BB^T$, where K is the gain you found in part (c) ?

a) $C = [B \;\; AB] = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. rank $C = 2 = n$

$\Rightarrow$ sys. controllable $\Rightarrow$ $\boxed{\text{YES}}$

b) This requires (since (A,B) is cont.) that A is exp. stable.

$|sI - A| = \begin{vmatrix} s-3 & 1 \\ -5 & s+1 \end{vmatrix} = (s-3)(s+1) + 5 = s^2 - 2s + 2$

$= (s-1)^2 + 1 \Rightarrow \lambda_{1,2} = 1 \mp j$.

since $\mathrm{Re}\{\lambda_i\} > 0$ sys. unstable. $\Rightarrow$ $\boxed{\text{NO}}$

c) $\underbrace{A - BK}_{A_{cl}} = \begin{bmatrix} 3 & -1 \\ 5-k_1 & -1-k_2 \end{bmatrix}$ where $[k_1 \;\; k_2] = K$

$\Rightarrow |sI - A_{cl}| = (s-3)(s+1+k_2) + 5 - k_1$

$= s^2 + (1+k_2-3)s - 3(1+k_2) + 5 - k_1$

$= (s+2)(s+3) = s^2 + 5s + 6$

$\Rightarrow k_2 - 2 = 5 \Rightarrow k_2 = 7$

$\Rightarrow -3(1+k_2) + 5 - k_1 = 6 \Rightarrow k_1 = -25$

$\Rightarrow \boxed{K = [-25 \;\; 7]}$

d) Observe:

$\left.\begin{array}{l} \rightarrow (A_{cl}, B) \text{ is cont.} \\ \rightarrow A_{cl} \text{ is exp. stable} \end{array}\right\} \Rightarrow \boxed{\text{YES}}$