First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

For $\alpha,\ \beta,\ \gamma \in [0,\ 2\pi)$ consider the second-order LTI system

$$\begin{aligned} \dot{x} &= \underbrace{\left(\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} [\cos\beta \ \ \sin\beta] \right)}_{A} x + \underbrace{\begin{bmatrix} \cos\gamma \\ \sin\gamma \end{bmatrix}}_{B} u \\ y &= Cx \end{aligned}$$

(a) Find the eigenvectors and eigenvalues of A^T.

(b) Find conditions on the triple $(\alpha,\ \beta,\ \gamma)$ under which the system is controllable.

(c) Find conditions on the triple $(\alpha,\ \beta,\ \gamma)$ under which the system is stabilizable.

(d) Find conditions on the triple $(\alpha,\ \beta,\ \gamma)$ under which the system is BIBO stable for all C.

Recall: $\cos(\theta - \psi) = \cos\theta\cos\psi + \sin\theta\sin\psi$ *&* $\sin(\theta - \psi) = \sin\theta\cos\psi - \cos\theta\sin\psi$.

Q2.

Consider the second-order discrete-time system below.

$$\begin{aligned} x_1^+ &= x_2^2 \\ x_2^+ &= x_1^2 + 2x_2\,. \end{aligned}$$

(a) Find all the equilibrium points of this system.

(b) At each equilibrium obtain the linearization.

(c) Determine (if possible) the local stability of each equilibrium based on its linearization.

Q3.

Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, let us define the set $\mathcal{S} := \{v \in \mathbb{C}^n : v^* e^{At} B = 0 \text{ for all } t \geq 0\}$. Prove *only one* of the following statements. [If you attempt both, only the ♠ will be graded.]

♠ If the pair (A, B) is controllable then $\mathcal{S} = \{0\}$.

♣ If $\mathcal{S} = \{0\}$ then the pair (A, B) is controllable.

Q4.

Consider the system

$$\dot{x} = \underbrace{\begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u\,.$$

(a) Is the above representation in controllable decomposition?

(b) Can we find a matrix $P^T = P > 0$ satisfying $AP + PA^T = -BB^T$?

(c) Find the feedback gain $K \in \mathbb{R}^{1\times 2}$ such that the control law $u = -Kx$ places the eigenvalues of the closed-loop system at $\lambda_{1,2} = -2, -3$.

(d) Can we find a matrix $P^T = P > 0$ satisfying $[A - BK]P + P[A - BK]^T = -BB^T$, where K is the gain you found in part (c) ?